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« Etude théorique et numérique de la stabilité de certains systèmes distribués avec contrôle frontière de type dynamique »

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Etude théorique et numérique de la stabilité de certains systèmes distribués avec contrôle frontière de type dynamique

Résumé

Cette thèse est consacrée à l'étude de la stabilisation de certains systèmes distribués avec contrôle frontière de type dynamique. Nous considérons, d'abord, la stabilisation de l'équation de la poutre de Rayleigh avec un seul contrôle frontière dynamique moment ou force. Nous montrons que le système n'est pas uniformément (autrement dit exponentiellement) stable; mais par une méthode spectrale, nous établissons le taux polynomial optimal de décroissance de l'énergie du système. Ensuite, nous étudions la stabilisation indirecte de l'équation des ondes avec un amortissement frontière de type dynamique fractionnel. Nous montrons que le taux de décroissance de l'énergie dépend de la nature géométrique du domaine. En utilisant la méthode fréquentielle et une méthode spectrale, nous montrons la non stabilité exponentielle et nous établissons, plusieurs résultats de stabilité polynomiale. Enfin, nous considérons l'approximation de l'équation des ondes mono-dimensionnelle avec un seul amortissement frontière de type dynamique par un schéma de différence finie. Par une méthode spectrale, nous montrons que l'énergie discrétisée ne décroit pas uniformément (par rapport au pas du maillage) polynomialement vers zéro comme l'énergie du système continu. Nous introduisons, alors, un terme de viscosité numérique et nous montrons la décroissance polynomiale uniforme de l'énergie de notre schéma discret avec ce terme de viscosité.

Mots-clés

Contrôle frontière dynamique, non stabilité exponentielle, stabilité polynomiale, optimalité, étude spectrale, méthode fréquentielle, base de Riesz, méthode des multiplicateurs, inegalité d'observabilité, comportement asymptotique, fonction de transfère, semi discrétisation, terme de viscosité.

Theoretical and numerical study of the stability of some distributed systems with dynamic boundary control

Abstract

This thesis is devoted to the study of the stabilization of some distributed systems with dynamic boundary control. First, we consider the stabilization of the Rayleigh beam equation with only one dynamic boundary control moment or force. We show that the system is not uniformly (exponentially) stable. However, using a spectral method, we establish the optimal polynomial decay rate of the energy of the system. Next, we study the indirect stability of the wave equation with a fractional dynamic boundary control. We show that the decay rate of the energy depends on the nature of the geometry of the domain. Using a frequency approach and a spectral method, we show the non exponential stability of the system and we establish, different polynomial stability results. Finally, we consider the finite difference space discretization of the 1-d wave equation with dynamic boundary control. First, using a spectral approach, we show that the polynomial decay of the discretized energy is not uniform with respect to the mesh size, as the energy of the continuous system. Next, we introduce a viscosity term and we establish the uniform (with respect to the mesh size) polynomial energy decay of our discrete scheme.

Keywords

Dynamic boundary control, non exponential stability, polynomial stability, optimality, spectral analysis, frequency domain method, Riesz basis, multiplier method, observability inequality, asymptotic behavior, transfer function, semi-discretization, viscosity terms.

A mon père..
Hassan Kamel SAMMOURY

Né le Vendredi 2 Mai 1941, décédé le jeudi 3 Dècembre 2015.

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Avant-propos

La théorie du contrôle et de la stabilisation d'un système physique gouverné par des équations mathématiques, en particulier par des EDP, peut être décrit comme étant le processus qui consiste à influer le comportement asymptotique du système pour atteindre un but désiré, principalement par l'utilisation d'un contrôle qui modifie son état final. Cette théorie est appliquée dans un large éventail de disciplines scientifiques et techniques comme la réduction du bruit, la vibration de structures, les vagues et les tremblements de terre sismiques, la régulation des systèmes biologiques comme le système cardiovasculaire humain, la conception des systèmes robotiques, le contrôle laser mécanique quantique, les systèmes moléculaires, etc.

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Introduction

Cette thèse est consacrée à l'étude de la stabilisation de certains systèmes distribués avec un contrôle frontière de type dynamique. La notion de contrôle dynamique ainsi que le contrôle indirect ont été introduite par Russell dans [85] et depuis lors, elle a attiré l'attention de beaucoup d'auteurs. En particulier, voir [1, 3, 5, 6, 7, 17, 80, 81, 91, 92].

La thèse est divisée en trois parties. Dans la première partie, nous considérons la stabilisation de l'équation de la poutre de Rayleigh avec un seul contrôle frontière dynamique moment ou force. D'abord, en utilisant la théorie de la décomposition spectrale, nous montrons que le système est fortement stable et par une méthode de perturbation compacte de Russell, nous prouvons que la décroissance de l'énergie du système vers zéro n'est pas exponentielle. Nous passons alors à une décroissance de type polynomiale. Dans le cas d'un seul contrôle frontière dynamique moment, nous faisons une étude spectrale très fine du système non amorti qui nous conduit à un résultat d'observabilité. Ensuite, nous appliquons une méthodologie introduite dans [12] et nous établissons le taux optimal de décroissance polynomiale de l'énergie de type $\frac{1}{t}$. Dans le cas d'un seul contrôle frontière dynamique force, nous donnons le développement asymptotique des valeurs propres et des fonctions propres des systèmes amorti et non amorti. Nous montrons après que le système de vecteurs propres du problème amorti forme une base de Riesz. Finalement, en appliquant une méthode introduite dans [63], nous établissons le taux optimal de décroissance polynomiale de l'énergie de type $\frac{1}{\sqrt{t}}$.

La deuxième partie est consacrée à l'étude de la stabilisation indirecte de l'équation des ondes avec un amortissement frontière de type dynamique fractionnel dans un domaine borné de $\mathbb{R}^N$, $N \geq 2$. D'abord, en utilisant un critère général d'Arendt et Batty dans [90], nous montrons la stabilité forte du système. Ensuite, nous prouvons que le système n'est pas exponentiellement stable dans le cas où le domaine est un disque de $\mathbb{R}^2$. Alors, nous cherchons à établir une décroissance de l'énergie de type polynomiale pour des données initiales régulières en employant une méthode fréquentielle combinée avec une méthode de multiplicateur par morceaux. Nous constatons alors que le taux de décroissance polynomiale de l'énergie depend de la nature géométrique du domaine. Plus précisément, si le domaine est Lipschitzien et vérifie la condition d'optique géométrique, nous établissons un taux de type $\frac{1}{t^{\frac{1}{4}}}$. De plus, si le domaine est presque étoilé et de classe $C^{1,1}$, nous établissons un taux de type $\frac{1}{t}$ et nous conjecturons que ce taux est optimal. Plus tard, nous nous intéressons à démontrer qu'une telle décroissance polynomiale semble être aussi établie même si les conditions géométriques précédentes ne sont pas satisfaites. Pour cela, nous considérons le système dans un carré de $\mathbb{R}^2$. Nous montrons d'abord que l'énergie ne décroit pas exponentiellement vers zéro. Finalement, en appliquant une méthode basée sur l'analyse de Fourier, une inégalité d'Ingham et une méthode d'interpolation, nous établissons un taux de décroissance polynomiale de l'énergie de type $\frac{1}{t}$ pour des donnés initiales assez régulières. Nous conjecturons que ce taux de décroissance est optimal.

Dans la troisième partie, nous passons à un autre sujet qui traite la stabilisation de l'approximation de l'équation des ondes mono-dimensionnelle avec un seul amortissement frontière de type dynamique par un schéma de différence finie. Premièrement, nous montrons que l'énergie discrétisée ne décroit pas uniformément (par rapport au pas du maillage) polynomialement vers zéro comme celle du système continu. Deuxièment, nous introduisons un terme du viscosité numérique dans le schéma d'approximation qui nous conduit à une décroissance uniforme (par rapport au pas du maillage) polynomiale de l'énergie comme celle du système continu. Finalement, nous montrons la convergence du schéma discrétisé vers l'équation des ondes initiale.

Notations: Dans toute la thèse, la notation $A \lesssim B$ (respectivement $A \gtrsim B$) signifie l'existence d'une constante positive C_1 (respectivement C_2), indépendante de A et B tel que $A \leq C_1 B$ (respectivement $A \geq C_2 B$). La notation $A \sim B$ désigne que $A \lesssim B$ et $A \gtrsim B$ sont satisfaites simultanément.

Aperçu de la thèse

Cette thèse est divisée en cinq chapitres.
Dans le premier chapitre, nous rappelons quelques définitions et théorèmes concernant la théorie de semigroupe et l'analyse spectrale. Ainsi, nous présentons et discutons les méthodes utilisées dans cette thèse pour obténir notre résultats de la stabilité.

Le deuxième chapitre est consacré à la stabilisation de l'équation de la poutre de Rayleigh amortie par un seul contrôle frontière dynamique moment:

$$\begin{cases} y_{tt} - \gamma y_{xxtt} + y_{xxxx} & = 0, \quad 0 < x < 1, \quad t > 0, \\ y(0,t) = y_x(0,t) & = 0, \qquad\qquad\qquad t > 0, \\ y_{xx}(1,t) + \eta(t) & = 0, \qquad\qquad\qquad t > 0, \\ y_{xxx}(1,t) - \gamma y_{xtt}(1,t) & = 0, \qquad\qquad\qquad t > 0, \\ \eta_t(t) - y_{xt}(1,t) + \alpha\eta(t) & = 0, \qquad\qquad\qquad t > 0, \end{cases} \tag{0.0.1}$$

où γ est le coefficient de moment d'inertie et $\alpha > 0$ est le coefficient de contrôle dynamique moment. L'amortissement est appliqué indirectement, via une équation différentielle ordinaire en η, à l'extrémité droite de la poutre. Ce type de contrôle indirect à été introduit par Russell dans [85] et depuis lors, il a retenu l'attention de plusieurs auteurs. Dans le cas d'un amortissement statique, quand $\eta(t)$ est constante, la stabilisation du système (0.0.1) a été largement étudiée par des approches différentes (voir [55, 79]). Cependant, dans le cas des contrôles dynamiques, Wehbe dans [92], a considéré l'équation de la poutre de Rayleigh avec deux contrôles dynamiques frontières. D'abord, par une méthode de perturbation compacte, il a prouvé que l'équation de la poutre de Rayleigh n'est pas uniformément stable. Ensuite, par une méthode spectrale, il a établi le taux de décroissance optimal de l'énergie pour des données initiales régulières. Le fait de la présence de deux contrôles frontière dynamique ensemble dans la démonstration, montre que le cas général, quand la poutre de Rayleigh est amortie par un seul contrôle dynamique frontière, reste un problème ouvert. Alors, dans ce chapitre, nous considérons l'équation de poutre de Rayleigh avec un seul contrôle frontière dynamique moment. D'abord, nous écrivons le système (0.0.1) sous forme d'une équation d'évolution du premier ordre

$$\begin{cases} U_t(t) + \mathcal{A}_\alpha U(t) & = & 0, \qquad t > 0, \\ U(0) & = & U_0 \in \mathcal{H}, \end{cases} \tag{0.0.2}$$

où $U = (y, y_t, \eta)$, $\mathcal{H}$ est un espace de Hilbert convenable, $\mathcal{A}_\alpha = \mathcal{A}_0 + \alpha\mathcal{B}$, $\mathcal{A}_0$ est un opérateur non borné maximal monotone du domaine $D(\mathcal{A}_0) \subset$

$\mathcal{H}$ et $\mathcal{B}$ est un opérateur borné monotone. Nous montrons après que le problème (0.0.2) est fortement stable dans l'espace de Hilbert $\mathcal{H}$ en utilisant la théorie de décomposition spectrale, et par une méthode de perturbation compacte de Russell, nous prouvons que la décroissance de l'énergie $\mathcal{E}$ du problème (0.0.2) vers zéro n'est pas exponentielle. Alors, une décroissance de type polynomiale est espérée. Pour cela, nous faisons une étude très fine du spectre $\sigma(\mathcal{A}_\alpha)$ de l'opérateur $\mathcal{A}_\alpha$ pour $\alpha \geq 0$. Plus précisément, nous montrons que pour $\alpha \geq 0$, il existe $k_\alpha \in \mathbb{N}^*$ suffisamment large tel que le spectre $\sigma(\mathcal{A}_\alpha)$ de l'opérateur $\mathcal{A}_\alpha$ est donné comme suit:

$$\sigma(\mathcal{A}_\alpha) = \sigma_{\alpha,0} \cup \sigma_{\alpha,1},$$

avec

$$\sigma_{\alpha,0} = \{\kappa_{\alpha,j}\}_{j\in J}\,, \quad \sigma_{\alpha,1} = \{\lambda_{\alpha,k}\}_{\substack{k\in\mathbb{Z}\\ |k|\geq k_0}}\,, \quad \sigma_{\alpha,0} \cap \sigma_{\alpha,1} = \emptyset$$

et J est un ensemble fini. De plus, $\lambda_{\alpha,k}$ est simple et elle satisfait le développement asymptotique suivant:

$$\lambda_{\alpha,k} = i\left(\frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} + \frac{D}{k} + \frac{E}{k^2}\right) + \frac{\alpha}{\pi^2 k^2} + o(\frac{1}{k^2}),$$

avec

$$D = \frac{2\gamma - 1 - 2\sqrt{\gamma}\tanh(\gamma^{-\frac{1}{2}})}{2\gamma^{\frac{3}{2}}\pi}$$

et

$$E = \frac{2(-1)^k}{\gamma^{\frac{3}{2}}\cosh(\gamma^{-\frac{1}{2}})\pi^2} + \frac{4\gamma - 2 - \sqrt{\gamma}\tanh(\gamma^{-\frac{1}{2}})}{2\gamma^{\frac{3}{2}}\pi}.$$

Nous déduisons alors, qu'il existe $T > 0$ et $C_T > 0$ telle que la solution U du problème non amorti associé à (0.0.2) satisfait

$$\int_0^T \|\mathcal{B}^* U(t)\|^2_{\mathcal{H}} dt \geq C_T \|U_0\|_{(D(\mathcal{A}_0))'}, \tag{0.0.3}$$

où $\mathcal{B}^*$ représente l'opérateur adjoint associé à $\mathcal{B}$ et $(D(\mathcal{A}_0))'$ est le dual de l'espace $D(\mathcal{A}_0)$ par rapport au produit scalaire de l'espace de Hilbert $\mathcal{H}$. De plus, nous montrons que la fonction de transfert définit par

$$H : \mathbb{C}_+ = \{\lambda \in \mathbb{C};\, \Re(\lambda) > 0\} \longmapsto \mathcal{L}(\mathcal{H}),$$

$$H(\lambda) = -\alpha\mathcal{B}^*(\lambda + \mathcal{A}_0)^{-1}\mathcal{B}$$

est bornée dans un sous espace de $\mathbb{C}_+$. En combinant l'inégalité d'observabilité (0.0.3) et la bornitude de la fonction de transfert H (voir [12]), nous déduisons qu'il existe une constante $c > 0$ telle que pour toute donnée initiale $U_0 \in D(\mathcal{A}_0)$, l'énergie $\mathcal{E}$ associée au problème amorti (0.0.2) satisfait:

$$\mathcal{E}(t) \leq \frac{c}{1+t} \|U_0\|^2_{D(\mathcal{A}_0)}, \quad \forall t > 0. \tag{0.0.4}$$

Finalement, en utilisant l'étude spectrale précédente de $\sigma(\mathcal{A}_\alpha)$ et un théorème de Borichev et Tomilov dans [20], nous montrons que le taux de décroissance obtenu dans (0.0.4) est optimal dans le sens que pour tout $\epsilon > 0$, la décroissance de l'énergie $\mathcal{E}$ ne peut pas atteindre un taux de type $\frac{1}{t^{1+\epsilon}}$.

Dans le troisième chapitre, nous continuons l'étude effectuée dans le deuxième chapitre en considérant l'équation de la poutre de Rayleigh amortie par un seul contrôle frontière dynamique force:

$$\begin{cases} y_{tt} - \gamma y_{xxtt} + y_{xxxx} = 0, & 0 < x < 1,\ t > 0, \\ y(0,t) = y_x(0,t) = y_{xx}(1,t) = 0, & t > 0, \\ y_{xxx}(1,t) - \gamma y_{xtt}(1,t) - \xi(t) = 0, & t > 0, \\ \xi_t(t) - y_t(1,t) + \beta \xi(t) = 0, & t > 0, \end{cases} \tag{0.0.5}$$

où γ est le coefficient de moment d'inertie et $\beta > 0$ est le coefficient de contrôle dynamique force. L'amortissement est appliqué indirectement, via une équation différentielle ordinaire en ξ, à l'extrémité droite de la poutre. D'abord, nous commençons par la formulation du système (0.0.5) sous forme d'une équation d'évolution du premier ordre

$$\begin{cases} U_t(t) + \tilde{\mathcal{A}}_\beta U(t) = 0, & t > 0, \\ U(0) = U_0 \in \mathcal{H}, \end{cases} \tag{0.0.6}$$

où $U = (y, y_t, \xi)$, $\mathcal{H}$ est un espace de Hilbert convenable, $\tilde{\mathcal{A}}_\beta = \tilde{\mathcal{A}}_0 + \beta \tilde{\mathcal{B}}$, $\tilde{\mathcal{A}}_0$ est un opérateur non borné maximal monotone du domaine $D(\tilde{\mathcal{A}}_0) \subset \mathcal{H}$ et $\tilde{\mathcal{B}}$ est un opérateur borné monotone. Ensuite, en appliquant la théorie de décomposition spectrale dans [19], nous montrons, comme dans [79], que la poutre de Rayleigh est fortement stable pour toute donnée initiale si et seulement si $\gamma > \gamma_0$ où γ_0 est la solution de l'équation $\sqrt{\gamma_0}\sinh^{-1}(\sqrt{\gamma_0}\pi) = 1$. En utilisant Mathematica, nous obtenons l'estimation $\gamma_0 \simeq 0.45001246517627713$. Nous savons que la poutre de Rayleigh n'est pas exponentiellement stable ni avec un seul contrôle force direct (voir [79]) ni avec deux contrôles dynamiques (voir [92]). Nous nous intéressons donc à la décroissance polynomiale optimale de l'énergie du système pour des données initiales régulières dans $D(\tilde{\mathcal{A}}_0)$. Pour cela, en utilisant une approximation explicite, nous donnons d'abord le développement asymptotique des valeurs propres et des fonctions propres des systèmes amorti et non amorti. Ensuite, nous appliquons le théorème 1.2.10 dans [2] (qui est une version modifiée du théorème de Bari voir [43, Chaptire 6, théorème 2.3]) et nous prouvons que les vecteurs propres normalisés $\widetilde{U}_k$ du système amorti forment une base de Riesz dans $\mathcal{H}$. Plus précisément, nous démontrons que ces vecteurs propres sont quadratiquement liés avec les vecteurs propres normalisés $\widetilde{U}_k^0$ de l'opérateur $\tilde{\mathcal{A}}_0$ par l'inégalité suivante:

$$\sum_{k=\max\{k_0,k_\beta\}}^{+\infty} ||\widetilde{U}_k - \widetilde{U}_k^0||_{\mathcal{H}} < +\infty.$$

Finalement, en appliquant le théorème 2.4 donné dans [63], nous déduisons qu'il existe une constante $c > 0$ telle que pour toute donnée initiale $U_0 \in D(\tilde{\mathcal{A}}_0)$, l'énergie $\widetilde{\mathcal{E}}$ associée au problème (0.0.6) satisfait

$$\widetilde{\mathcal{E}}(t) \leq \frac{c}{\sqrt{t}} \|U_0\|^2_{D(\widetilde{\mathcal{A}}_0)} \tag{0.0.7}$$

et le taux obtenu ci-dessus est optimal dans le sens que pour tout $\epsilon > 0$, la décroissance de l'énergie $\widetilde{\mathcal{E}}$ ne peut pas atteindre un taux $\frac{1}{t^{\frac{1}{2}+\epsilon}}$.

Le quatrième chapitre est consacré à l'étude de la stabilité de l'équation des ondes avec un amortissement frontière de type dynamique fractionnel. D'abord, soit Ω un domaine borné dans $\mathbb{R}^d$, $d \geq 2$, avec frontière lipschitzienne $\Gamma = \Gamma_0 \cup \Gamma_1$; Γ_0 et Γ_1 sont deux sous ensembles de Γ tel que $\Gamma_0 \cap \Gamma_1 = \emptyset$ et $\Gamma_1 \neq \emptyset$. Dans [30, 31, 41], N. Fourier, I. Lasiecka et P. Graber ont étudié la stabilité du problème suivant (avec $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$):

$$\begin{cases}
u_{tt} - \Delta u - k_\Omega \Delta u_t + c_\Omega u_t & = 0, \quad \text{in} \quad \Omega \times \mathbb{R}_+^*, \\
u & = 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+^*, \\
u - w & = 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+^*, \\
w_{tt} - k_\Gamma \Delta_T(\alpha w_t + w) + \partial_\nu(u + k_\Omega u_t) + c_\Gamma w_t & = 0, \quad \text{in} \quad \Gamma_1 \times \mathbb{R}_+^*, \\
w & = 0, \quad \text{on} \quad \partial\Gamma_1 \times \mathbb{R}_+^*, \\
u(\cdot,\cdot,0) = u_0, \quad u_t(\cdot,\cdot,0) = u_1, & \qquad\ \ \text{in} \quad \Omega, \\
w(\cdot,0) = w_0, \quad w_t(\cdot,0) = w_1, & \qquad\ \ \text{on} \quad \Gamma_1,
\end{cases}$$

où ∂_ν désigne la dérivée normale sur Γ_1, ν est le vecteur unitaire normal dérigé vers l'extérieur de la frontière et Δ_T représente l'opérateur de Laplace-Beltrami sur Γ. Dans le système précédent, deux types de dissipation apparaissent: interne (si $c_\Omega > 0$) et frontière (si $k_\Gamma > 0$), interne de type fractionnel (si $k_\Omega > 0$) et frontière de type fractionnel visco-élastique (si $k_\Gamma \alpha > 0$). La première description physique de ce modèle est donnée dans [66]. Dans [30, 31], Fourier et Lasiecka ont démontré que le système précédent est exponentiellement stable si un de ces trois conditions suivants est satisfait: si $k_\Omega > 0$ (un amortissement interne de type visco-élastique), ou $c_\Omega > 0$ et $c_\Gamma > 0$ (deux amortissement interne et frontière de type fractionnel) ou $c_\Omega > 0$ et $k_\Gamma \alpha > 0$ (un amortissement interne de type fractionnel et un amortissement frontière de type visco-élastique). Le premier cas correspond à un amortissement direct, tandis que les autres cas correspondent à un phénomène d'un sur-amortissement. Alors, le cas d'un seul amortissement frontière de type dynamique fractionnel reste un problème ouvert. Dans ce chapitre, nous nous intéressons à ce cas, *i.e.* lorsque $k_\Omega = c_\Omega = \alpha = 0$ et $k_\Gamma = c_\Gamma = 1$. Plus précisément, nous considérons le système suivant:

$$\begin{cases} u_{tt} - \Delta u & = 0, & \text{in} & \Omega \times \mathbb{R}_+^*, \\ u & = 0, & \text{on} & \Gamma_0 \times \mathbb{R}_+^*, \\ u - w & = 0, & \text{on} & \Gamma_1 \times \mathbb{R}_+^*, \\ w_{tt} - \Delta_T w + \partial_\nu u + w_t & = 0, & \text{in} & \Gamma_1 \times \mathbb{R}_+^*, \\ w & = 0, & \text{on} & \partial\Gamma_1 \times \mathbb{R}_+^*, \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, & & \text{in} & \Omega, \\ w(\cdot,0) = w_0, \qquad w_t(\cdot,0) = w_1, & & \text{in} & \Gamma_1. \end{cases} \tag{0.0.8}$$

Nous montrons que la stabilité du système (0.0.8) dépend de la nature géométrique du domaine Ω. Nous commençons par la formulation du système (0.0.8) sous la forme d'une équation d'évolution du premier ordre. Si $\Gamma_0 \neq \emptyset$, nous définissons d'abord l'espace

$$H^1_{\Gamma_0}(\Omega) = \left\{ u \in H^1(\Omega);\ u = 0 \quad \text{sur} \quad \Gamma_0 \right\}$$

et nous introduisons l'espace de Hilbert $\mathcal{H}$

$$\mathcal{H} = \{(u, v, w, z) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times H^1_0(\Gamma_1) \times L^2(\Gamma_1) : \gamma u = w \text{ sur } \Gamma_1\},$$

où γ désigne l'opérateur du trace définit de $H^1(\Omega)$ dans $H^{\frac{1}{2}}(\Gamma)$, muni du produit scalaire

$$\begin{aligned} \Big((u^1,\, v^1,\, w^1,\, z^1), (u^2,\, v^2,\, w^2,\, z^2)\Big)_{\mathcal{H}} =& (\nabla u^1, \nabla u^2)_{L^2(\Omega)} + (v^1,\, v^2)_{L^2(\Omega)} \\ &+ (\nabla_T w^1,\, \nabla_T w^2)_{L^2(\Gamma_1)} \\ &+ (z^1,\, z^2)_{L^2(\Gamma_1)}, \end{aligned}$$

$$\forall (u^1,\, v^1,\, w^1,\, z^1),\ (u^2,\, v^2,\, w^2,\, z^2) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times H^1_0(\Gamma_1) \times L^2(\Gamma_1),$$

et muni de la norme $\|\cdot\|_{\mathcal{H}} = (\cdot,\cdot)^{\frac{1}{2}}_{\mathcal{H}}$. Si $\Gamma_0 = \emptyset$, nous définissons $\mathcal{H}$ de la même manière mais muni de la norme usuelle $\|(u, v, w, z)\|^2 := \|(u, v, w, z)\|^2_{\mathcal{H}} + \|u\|^2_\Omega + \|w\|^2_\Gamma$. Nous introduisons aussi l'opérateur non borné maximal dissipatif $\mathcal{A}$ qui engendre un C_0-semigroupe de contraction $(e^{t\mathcal{A}})_{t\geq 0}$ par

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (u,\, v,\, w,\, z) \in \mathcal{H};\\ \Delta_T w - \partial_\nu u \in L^2(\Gamma_1)\\ \quad\quad v \in H^1_{\Gamma_0}(\Omega), \quad\quad \Delta u \in L^2(\Omega),\\ \quad\quad z \in H^1_0(\Gamma_1), \quad\quad \gamma v = z \text{ sur } \Gamma_1 \end{array} \right\},$$

$$\mathcal{A}U = \begin{pmatrix} v \\ \Delta u \\ z \\ \Delta_T w - \partial_\nu u - z \end{pmatrix}, \forall U = \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} \in D(\mathcal{A}).$$

Nous écrivons alors notre système (0.0.8) sous la forme

$$\begin{cases} U_t(t) &= \mathcal{A}U(t), \quad t > 0, \\ U(0) &= U_0 \in \mathcal{H}. \end{cases} \tag{0.0.9}$$

De plus, nous caractérisons le domaine $D(\mathcal{A})$ de l'opérateur $\mathcal{A}$ lorsque le bord du domaine Ω est suffisamment régulier ou dans le cas où Ω est le carré unité de $\mathbb{R}^2$. Ensuite, nous étudions la stabilité forte du problème (0.0.9) en appliquant un théorème d'Arendt et Batty dans [90]. Nous distinguons deux cas:

• Si $\Gamma_0 \neq \emptyset$, nous montrons que le C_0-semigoupe de contraction $(e^{t\mathcal{A}})_{t\geq 0}$ est fortement stable dans l'espace de Hilbert $\mathcal{H}$.

• Si $\Gamma_0 = \emptyset$, nous montrons que le C_0-semigroupe de contraction $(e^{t\mathcal{A}})_{t\geq 0}$ est fortement stable dans l'espace de Hilbert $\mathcal{H}_0$ définit par

$$\mathcal{H}_0 = \left\{ (u, v, w, z) \in \mathcal{H} : \int_\Omega v dx + \int_{\Gamma_1} z d\Gamma + \int_{\Gamma_1} w d\Gamma = 0 \right\}.$$

En outre, nous montrons que la décroissance de l'énergie E associée au problème (0.0.9) vers zéro n'est pas exponentielle dans le cas général. Pour ce but, nous considérons notre système (0.0.9) dans le disque unité de $\mathbb{R}^2$ avec $\Gamma_0 = \emptyset$. Puis, nous faisons une étude spectrale de l'opérateur $\mathcal{A}$ et nous trouvons une famille de valeurs propres qui s'approche de l'axe imaginaire. Nous passons alors à une stabilité de type polynomiale. En appliquant une méthode fréquentielle (voir [20]), nous établissons deux taux de décroissances polynomiales. Dans un premier temps, en supposant que la frontière Γ de notre domaine Ω est Lipschitzienne, $\Gamma_0 \neq \emptyset$, $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ et en utilisant des résultats de la stabilité exponentielle de l'équation des ondes avec l'amortissement

$$\frac{\partial y}{\partial \nu} = -y_t, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^*_+,$$

nous établissons un taux de décroissance polynomiale de l'énergie de type

$\frac{1}{t^{\frac{1}{4}}}$. Dans un deuxième temps, en supposant que le domaine Ω est presque étoilé, la frontière Γ de Ω est de classe $C^{1,1}$, et que $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, nous établissons un taux de décroissance polynomiale de l'énergie de type $\frac{1}{t}$. Plus tard, nous voulons montrer que la stabilité polynomiale de l'équation des ondes avec un amortissement frontière de type dynamique fractionnel, reste valable même si les conditions géométriques précédentes ne sont pas satisfaits. Dans ce but, nous considérons le problème (0.0.8) dans le carré unité $\Omega = (0,1)^2$ avec frontière $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_1 = \{(0,y),\ y \in (0,1)\}$, $\Gamma_0 = \Gamma \backslash \overline{\Gamma_1}$ et $\Gamma_0 \cap \Gamma_1 = \emptyset$. Plus précisément, nous considérons le système suivant:

$$\begin{cases} u_{tt} - \Delta u & = 0, & \text{in} & \Omega \times \mathbb{R}_+^*, \\ u & = 0, & \text{on} & \Gamma_0 \times \mathbb{R}_+^*, \\ u & = w, & \text{on} & \Gamma_1 \times \mathbb{R}_+^*, \\ w_{tt} - w_{yy} - u_x + w_t & = 0, & \text{on} & \Gamma_1 \times \mathbb{R}_+^*, \\ w(0) = w(1) & = 0, & \text{on} & \mathbb{R}_+^*, \\ u(\cdot,\cdot,0) = u_0, \quad u_t(\cdot,\cdot,0) = u_1, & & \text{in} & \Omega, \\ w(\cdot,0) = w_0, \quad w_t(\cdot,0) = w_1, & & \text{on} & \Gamma_1. \end{cases} \tag{0.0.10}$$

Dans ce cas, nous démontrons d'abord, que l'énergie ne décroit pas exponentiellement vers zéro. Plus précisément, en utilisant la méthode de séparation de variable, nous étudions le spectre de l'opérateur $\mathcal{A}$ et nous trouvons une branche des valeurs propres qui s'approche de l'axe imaginaire. Nous montrons qu'il existe $k_1 \in \mathbb{N}^*$ suffisamment large tel que le spectre $\sigma(\mathcal{A})$ de l'opérateur $\mathcal{A}$ est sous la forme:

$$\sigma(\mathcal{A}) = \widetilde{\sigma}_0 \cup \widetilde{\sigma}_1 \tag{0.0.11}$$

où

$$\widetilde{\sigma}_0 = \{\kappa_{l,j}\}_{j \in J}, \quad \widetilde{\sigma}_1 = \{\lambda_{l,k}\}_{\substack{k \in \mathbb{Z} \\ |k| \geq k_1}}, \quad \widetilde{\sigma}_0 \cap \widetilde{\sigma}_1 = \emptyset, \tag{0.0.12}$$

J est un ensemble fini, $l \in \mathbb{N}^*$ et $\lambda_{l,k}$ est simple et elle satisfait le comportement asymptotique suivant:

$$\lambda_{l,k} = i\left(k\pi + \frac{l^2\pi}{2k}\right) - \frac{1}{\pi^2 k^2} + o\left(\frac{1}{k^2}\right). \tag{0.0.13}$$

Après, par une étude spectrale et une inégalité d'Ingham, nous établissons un taux de décroissance polynomiale de type $\dfrac{1}{t}$ de l'énergie du système mono-dimensionnel avec paramètre associé au système (0.0.10) en gérant la constante de la décroissance. Finalement, en utilisant l'analyse de Fourier et le taux de décroissance obtenu dans le cas mono-dimensionnel, nous montrons un taux de décroissance polynomiale de l'énergie de type $\dfrac{1}{t}$ pour des données initiales assez régulières.

Dans le cinquième chapitre, nous passons à un autre sujet qui traite la stabilisation de l'approximation de l'équation des ondes mono-dimensionnelle avec un seul amortissement frontière de type dynamique par un schéma de différence finie. Plus précisément, nous considérons l'approximation de système suivant:

$$\begin{cases} y''(x,t) - y_{xx}(x,t) &= 0, & (x,t) \in]0,1[\times \mathbb{R}_*^+, \\ y(0,t) &= 0, & t \in \mathbb{R}_*^+, \\ y_x(1,t) + \eta(t) &= 0, & t \in \mathbb{R}^+, \\ \eta'(t) - y'(1,t) + \beta\eta(t) &= 0, & t \in \mathbb{R}_*^+, \\ y(x,0) &= y^0(x), & x \in]0,1[, \\ y'(x,0) &= y^1(x), & x \in]0,1[, \\ \eta(0) &= \eta^0, \end{cases} \tag{0.0.14}$$

où $(y^0, y^1, \eta^0) \in \mathcal{H} = H_L^1(0,1) \times L^2(0,1) \times \mathbb{C}$ avec

$$H_L^1(0,1) = \left\{ y \in H^1(0,1);\ y(0) = 0 \right\},$$

β est une constante positive et $'$ désigne la dérivée par rapport au temps t. Le système (0.0.14) se présente dans de nombreux domaines de la mécanique et de l'ingénierie. Ce modèle peut être considéré comme un modèle qui décrit la description des vibrations de structures, de la propagation des ondes acoustiques ou sismiques, etc.
Dans [91], Wehbe a démontré la décroissance polynomiale de type $\frac{1}{t}$ de l'énergie E associée au système (0.0.14). Dans ce chapitre, nous voulons tester si l'énergie du schéma discrétisé admet la même propriété uniformément par rapport au pas du maillage. Dans de nombreuses applications (voir [3, 28, 64, 87, 96]), bien que le système continu est exponentiellement ou polynomialement stable, les systèmes discretisés associés n'héritent pas la même propriété uniformément par rapport au pas du maillage. Dans [87], Tebou et Zuazua ont considéré l'approximation par un schéma de différence finie de l'équation des ondes mono-dimensionnelle avec un contrôle frontière de type statique. D'abord, en utilisant une étude spectrale, ils ont démontré que l'énergie du schéma discrétisé ne décroit pas uniformément (par rapport au pas du maillage) exponentiellement vers zéro comme le système continu. Ensuite, en ajoutant un terme de type viscosité numérique dans le schéma discrétisé et en utilisant une méthode basée sur des inégalités d'observabilités, Tebou et Zuazua ont montré la décroissance uniforme (par rapport au pas du maillage) exponentielle de l'énergie vers zéro. Finalement, ils ont démontré la convergence du schéma discrétisé avec le terme viscosité numérique vers l'équation des ondes d'origine. A cause de la présence du terme dynamique, la méth-

ode utilisée dans [87] ne fonctionne pas pour notre système. Dans [3], Abdallah *et al.*, ont considéré l'approximation de l'équation d'évolution du deuxième ordre avec un contrôle borné. D'abord, en introduisant un terme visco numérique dans le schema d'approximation, ils ont démontré la décroissance uniforme (par rapport au pas du maillage) exponentielle ou polynomiale de l'énergie vers zéro. Ensuite, ils ont utilisé le théoreme de Trotter-Kato dans [49] pour démontrer la convergence du schéma numériques avec le terme de viscosité vers le problème d'origine. Notons que notre système ne rentre pas dans le cadre de celle de [3]. Alors, l'étude de la stabilité de l'approximation de l'équation des ondes mono-dimensionnelle avec un contrôle fontière de type dynamique reste un problème ouvert.

Dans un premier temps, par une méthode spectrale, nous montrons que l'énergie du système discrétisé ne décroit pas uniformément (par rapport au pas du maillage) polynomialement vers zéro. Ce résultat est une conséquence de l'existence d'une valeur propre à haute fréquence qui ne satisfait pas une condition suffisante pour la décroissance uniforme (par rapport au pas du maillage) polynomiale de l'énergie discrétisée. Plusieurs remèdes ont été proposés pour surmonter cette difficulté comme la régularisation de Tychonoff [38, 39, 78, 87], un algorithme bi-grille [36, 70], une méthode mixte d'éléments finis [15, 22, 23, 37, 68], ou le filtrage des valeurs propres à hautes fréquences [47, 57]. Dans un deuxième temps, comme dans [87], nous ajoutons un terme de viscosité numérique dans le schéma d'approximation, et en utilisant une méthode de multiplicateur inspiré de [91], nous montrons que l'énergie $\tilde{E}$ discrétisé décroit uniformément (par rapport au pas du maillage) polynomialement vers zéro. Plus précisément, nous montrons qu'il existe une constante M uniformément bornée par rapport au pas du maillage tel que l'énergie $\tilde{E}$ satisfait

$$\int_0^T \tilde{E}^2(t)dt \leq M\tilde{E}^2(0), \qquad \forall T > 0.$$

Dès lors, en appliquant le théorème 9.1 donné dans [52], nous déduisons que l'énergie $\tilde{E}$ de système discrétisé satisfait

$$\tilde{E}(t) \leq \frac{M}{M+t}\tilde{E}(0), \quad \forall t > 0.$$

Finalement, nous démontrons la convergence du schéma discrétisé avec le terme de viscosité vers l'équation des ondes d'origine, en appliquant la même stratégie utilisée dans [87].

1

Preliminaries

Since the analysis in this thesis depends on the semigroup theory, otherworldly investigation hypotheses (keeping in mind the end goal to present the principle topic of our study), let's review a portion of the central definitions and hypotheses in this chapter which will be used to prove our main results in the next chapters.
The vast majority of the evolution equations can be reduced to the form

$$\begin{cases} U_t(t) &= \mathcal{A}U(t), \quad t > 0, \\ U(0) &= U_0 \in \mathcal{H}, \end{cases} \tag{1.0.1}$$

where $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $(T_{\mathcal{A}}(t))_{t\geq 0}$ in a Hilbert space $\mathcal{H}$. Therefore, we begin by presenting a few essential ideas concerning the semigroups which involve some results regarding the existence, uniqueness and regularity of the solution of system (1.0.1). Next, we exhibit and talk about many recent results on the strong, exponential and polynomial stability and Riesz basis in several sections. For more details we refer to [12, 19, 20, 29, 43, 46, 62, 69, 74, 75, 76, 83].

1.1 Semigroups, Existence and uniqueness of solution

We begin this section by the definition of semigroup.

Definition 1.1.1. *Let X be a Banach space and let $I : X \to X$ its identity operator.*

1) *A one parameter family $(T(t))_{t\geq 0}$, of bounded linear operators from X into X is a semigroup of bounded linear operators on X if*

   (i) $T(0) = I$*;*

   (ii) $T(t+s) = T(t)T(s)$ *for every* $s, t \geq 0$.

2) *A semigroup of bounded linear operators, $(T(t))_{t\geq 0}$, is uniformly continuous if*
$$\lim_{t\to 0}\|T(t) - I\| = 0.$$

3) *A semigroup $(T(t))_{t\geq 0}$ of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators or a C_0-semigroup if*
$$\lim_{t\to 0}T(t)x = x.$$

4) *The linear operator $\mathcal{A}$ defined by*
$$\mathcal{A}x = \lim_{t\to 0}\frac{T(t)x - x}{t}, \quad \forall x \in D(\mathcal{A}),$$
where
$$D(\mathcal{A}) = \left\{x \in X;\ \lim_{t\to 0}\frac{T(t)x - x}{t}\ \textit{exists}\right\}$$
is the infinitesimal generator of the semigroup $(T_{\mathcal{A}}(t))_{t\geq 0}$. ■

Next, we recall the definitions of the resolvent and the spectrum of an operator.

Definition 1.1.2. *Let $\mathcal{A}$ be a linear unbounded operator in a Banach space X.*

1) *The resolvent set of $\mathcal{A}$ denoted by $\rho(\mathcal{A})$ contained all the complex number $\lambda \in \mathbb{C}$ such that $(\lambda I - \mathcal{A})^{-1}$ exists as an inverse operator in X.*

2) *The spectrum of $\mathcal{A}$ denoted by $\sigma(\mathcal{A})$ is the set $\mathbb{C}\backslash\rho(\mathcal{A})$.*

■

Remark 1.1.3. *From the above definitions, we can split the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ into three disjoint sets, the ponctuel spectrum denoted by $\sigma_p(\mathcal{A})$, the continuous spectrum denoted by $\sigma_c(\mathcal{A})$ and the residual spectrum denoted by $\sigma_r(\mathcal{A})$ where these sets are defined as follows:*

- $\lambda \in \sigma_p(\mathcal{A})$ *if* $\ker(\lambda I - \mathcal{A}) \neq \{0\}$ *and in this case* λ *is called an eigenvalue of* $\mathcal{A}$*;*
- $\lambda \in \sigma_c(\mathcal{A})$ *if* $\ker(\lambda I - \mathcal{A}) = \{0\}$ *and* $Im(\lambda I - \mathcal{A})$ *is dense in* X *but* $(\lambda I - \mathcal{A})^{-1}$ *is not a bounded operator;*
- $\lambda \in \sigma_r(\mathcal{A})$ *if* $\ker(\lambda I - \mathcal{A}) = \{0\}$ *but* $Im(\lambda I - \mathcal{A})$ *is not dense in* X*.*

■

Some properties of semigroup and its generator operator $\mathcal{A}$ are given in the following theorems:

Theorem 1.1.4. *(Pazy [75]) Let $\mathcal{A}$ be the infinitesimal generator of a C_0- semigroup of contractions $(T_{\mathcal{A}}(t))_{t\geq 0}$. Then, the resolvent $(\lambda I - \mathcal{A})^{-1}$ of $\mathcal{A}$ contains the open right half-plane,* i.e., $\rho(\mathcal{A}) \subset \{\lambda : \Re(\lambda) > 0\}$ *and for such λ we have*

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{1}{\Re(\lambda)}.$$

■

Theorem 1.1.5. *(Kato [50]) Let $\mathcal{A}$ be a closed operator in a Banach space X such that the resolvent $(I - \mathcal{A})^{-1}$ of $\mathcal{A}$ exists and is compact. Then the spectrum $\sigma(A)$ of $\mathcal{A}$ consists entirely of isolated eigenvalues with finite multiplicities.* ■

Theorem 1.1.6. *(Pazy [75]) Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Hilbert space $\mathcal{H}$. Then there exist two constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{\omega t}, \quad \forall t \geq 0.$$

■

If $\omega = 0$, the semigroup $(T(t))_{t\geq 0}$ is called uniformly bounded and if moreover $M = 1$, then it is called a C_0-semigroup of contractions. For the existence of solution of problem (1.0.1), we typically use the following Lumer-Phillips and Hille-Yosida theorems from [75]:

Theorem 1.1.7. *(Lumer-Phillips) Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in a Hilbert space $\mathcal{H}$. If*

(i) $\mathcal{A}$ is dissipative, i.e., $\Re\left(< \mathcal{A}x, x >_{\mathcal{H}}\right) \leq 0, \ \forall x \in D(\mathcal{A})$

and if

(ii) there exists a $\lambda_0 > 0$ such that the range $\mathcal{R}(\lambda_0 I - \mathcal{A}) = \mathcal{H}$,

then $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$. ■

Theorem 1.1.8. *(Hille-Yosida) Let $\mathcal{A}$ be a linear operator on a Banach space X and let $\omega \in \mathbb{R}$, $M \geq 1$ be two constants. Then the following properties are equivalent*

(i) $\mathcal{A}$ generates a C_0-semigroup $(T_{\mathcal{A}}(t))_{t\geq 0}$, satisfying

$$\|T_{\mathcal{A}}(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0;$$

(ii) $\mathcal{A}$ is closed, densely defined, and for every $\lambda > \omega$ one has $\lambda \in \rho(\mathcal{A})$ and

$$\|(\lambda - \omega)^n (\lambda - \mathcal{A})^{-n}\| \leq M, \quad \forall n \in \mathbb{N};$$

(iii) $\mathcal{A}$ is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with $\Re(\lambda) > \omega$, one has $\lambda \in \rho(\mathcal{A})$ and

$$\|(\lambda - \mathcal{A})^{-n}\| \leq \frac{M}{(\Re(\lambda) - \omega)^n}, \quad \forall n \in \mathbb{N}.$$

■

Consequently, $\mathcal{A}$ is maximal dissipative operator on a Hilbert space $\mathcal{H}$ if and only if it generates a C_0-semigroup of contractions $(T_{\mathcal{A}}(t))_{t\geq 0}$ on $\mathcal{H}$. Thus, the existence of solution is justified by the following corollary which follows from Lumer-Phillips theorem.

Corollary 1.1.9. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{A}$ be a linear operator defined from $D(\mathcal{A}) \subset \mathcal{H}$ into $\mathcal{H}$. If $\mathcal{A}$ is maximal dissipative operator then the initial value problem (1.0.1) has a unique solution $U(t) = T_{\mathcal{A}}(t)U_0$ such that $U \in C([0, +\infty), \mathcal{H})$, for each initial datum $U_0 \in \mathcal{H}$. Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C([0, +\infty), D(\mathcal{A})) \cap C^1([0, +\infty), \mathcal{H}).$$

■

Finally, we also recall the following theorem concerning a perturbations by a bounded linear operators (see Theorem 1.1 in Chapter 3 of [75]):

Theorem 1.1.10. *Let X be a Banach space and let $\mathcal{A}$ be the infinitesimal generator of a C_0-semigroup $(T_{\mathcal{A}}(t))_{t\geq 0}$ on X, satisfying $\|T_{\mathcal{A}}(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{\omega t}$ for all $t \geq 0$. If $\mathcal{B}$ is a bounded linear operator on X, then the operator $\mathcal{A}+\mathcal{B}$ becomes the infinitesimal generator of a C_0-semigroup $(T_{\mathcal{A}+\mathcal{B}}(t))_{t\geq 0}$ on X, satisfying $\|T_{\mathcal{A}+\mathcal{B}}(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{(\omega+M\|\mathcal{B}\|)t}$ for all $t \geq 0$.*

■

1.2 Strong and exponential stability

First, we start by introducing the notion of the strong and the exponential stabilities.

Definition 1.2.1. *Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $(T_{\mathcal{A}}(t))_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. We say that the semigroup $(T_{\mathcal{A}}(t))_{t\geq 0}$ is*

(i) Strongly (asymptotically) stable if for all initial data $U_0 \in \mathcal{H}$ we have

$$\|T_{\mathcal{A}}(t)U_0\|_{\mathcal{H}} \xrightarrow[t\to+\infty]{} 0. \tag{1.2.1}$$

(ii) Exponentially stable if there exist two positive constants C and w such that

$$\|T_{\mathcal{A}}(t)U_0\|_{\mathcal{H}} \leq Ce^{-wt}\|U_0\|_{\mathcal{H}}, \quad \forall t > 0,\ \forall U_0 \in \mathcal{H}. \tag{1.2.2}$$

■

Next, in order to show the strong stability of our system, we apply the next theorem due to Arendt and Batty in [90].

Theorem 1.2.2. *Let $\mathcal{A}$ be the generated operator of a bounded semigroup $(T_{\mathcal{A}}(t))_{t\geq 0}$ on a Banach space X. Assume that no eigenvalues of $\mathcal{A}$ lies on the imaginary axis. If $\sigma(\mathcal{A})\cap i\mathbb{R}$ is countable, then $(T_{\mathcal{A}}(t))_{t\geq 0}$ is stable.*

■

Remark 1.2.3. *If the resolvent $(I-\mathcal{A})^{-1}$ of $\mathcal{A}$ is compact, then its spectrum is absolutely framed of eigenvalues. Thus, the state of Theorem*

1.2.2 lessens to $\sigma_d(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. We also refer to the decomposition theory of Sz.-Nagy-Foias [69, pp. 9-10] and Foguel [29]. ■

Now, we recall two results which gives necessary and sufficient conditions for which a semigroup is exponentially stable.

Theorem 1.2.4. *(Huang-Prüss [46, 76]) Let $(T_{\mathcal{A}}(t))_{t\geq 0}$ be a C_0-semigroup on a Hilbert space $\mathcal{H}$ and $\mathcal{A}$ be its infinitesimal generator. Then, the C_0-semigroup of contractions $(T_{\mathcal{A}}(t))_{t\geq 0}$ is exponentially stable if and only if*

(i) $i\mathbb{R} \subseteq \rho(\mathcal{A})$,

(ii) $\sup\limits_{\omega\in\mathbb{R}} \|(i\omega - \mathcal{A})^{-1}\| < \infty$. ■

Theorem 1.2.5. *Let $(T_{\mathcal{A}}(t))_{t\geq 0}$ be a C_0-semigroup on a Hilbert space $\mathcal{H}$ and $\mathcal{A}$ be its infinitesimal generator. Then $(T_{\mathcal{A}}(t))_{t\geq 0}$ is exponentially stable if and only if there exists $t_0 > 0$ such that*

$$\|T_{\mathcal{A}}(t_0)\|_{\mathcal{L}(\mathcal{H})} < 1. \tag{1.2.3}$$

■

Since the studies systems in this thesis do not achieve the exponential stability, we present the used methodologies used to prove this objective. The first one is based on the following compact perturbation theory of Russell in [83]:

Theorem 1.2.6. *Assume that $\mathcal{A}$ is skew-adjoint. Then, it does not exist two compacts operators $\mathcal{B}$, $\mathcal{C}$, and $t_0 > 0$ such that*

$$\|T_{\mathcal{A}+\mathcal{B}}(t_0)\|_{\mathcal{L}(\mathcal{H})} < 1 \quad \text{and} \quad \|T_{\mathcal{A}+\mathcal{C}}(-t_0)\|_{\mathcal{L}(\mathcal{H})} < 1,$$

where $(T_{\mathcal{A}+\mathcal{B}}(t))_{t\geq 0}$ (resp. $(T_{\mathcal{A}+\mathcal{C}}(t))_{t\geq 0}$) designate the C_0-semigroup generated by $\mathcal{A}+\mathcal{B}$ (resp. $\mathcal{A}+\mathcal{C}$). ■

Our strategy is to split our operator $\mathcal{A}$ as the sum of two operators $\mathcal{A}_0$ and $\mathcal{B}$ where $\mathcal{A}_0$ (respectively $\mathcal{B}$) is a skew-adjoint operator (respectively compact operator). Next, we show that for all $t > 0$ we have

$$\|T_{\mathcal{A}_0+\mathcal{B}}(t)\|_{\mathcal{L}(\mathcal{H})} = \|T_{\mathcal{A}_0-\mathcal{B}}(-t)\|_{\mathcal{L}(\mathcal{H})}.$$

Therefore, by combining the result of Theorem 1.2.6 with the one of 1.2.5, we deduce that the C_0-semigroup of contraction $(T_{\mathcal{A}}(t))_{t\geq 0}$ cannot be exponentially stable.

The second one, is a classical method based on the spectrum analysis. Indeed, we show that the spectrum of the operator $\mathcal{A}$ approaches asymptotically the imaginary axis, *i.e.* we show the existence of a sequence of eigenvalues of $\mathcal{A}$ whose real part is close to the imaginary axis. Then, using the eigenvectors associated to these eigenvalues, we show that the resolvent of $\mathcal{A}$ is not bounded on the imaginary axis. Thus, from Theorem 1.2.4, we deduce that decay of the energy of system (1.0.1) to zero is not exponential.

1.3 Polynomial stability

As we have already said in the previous section, the energies of our systems in this thesis have no uniform (exponentially) decay rate, therefore we look for a polynomial one. In general, polynomial stability results are obtained using different methods like: multipliers method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them (see [52, 58, 60]). In this section, we discuss the used methods in our work to established a polynomial energy decay of system (1.0.1). First, we say that the C_0-semigroup $(T_{\mathcal{A}}(t))_{t\geq 0}$ generated by $\mathcal{A}$ is polynomially stable if there exists two positive constants β and C such that

$$\|T_{\mathcal{A}}(t)\|_{\mathcal{L}(\mathcal{H})} \leq Ct^{-\beta}, \quad t > 0. \tag{1.3.1}$$

We start by a methodology introduced in [12] and applied at the first order Cauchy problem. This requires, on one hand, to establish an observability inequality of solution of the undamped system associated to (1.0.1), and on the other hand to verify the boundedness property of the transfer function. First, we decompose the operator $\mathcal{A}$ as $\mathcal{A}_0 + BB^*$ where $\mathcal{A}_0 : D(\mathcal{A}) \longrightarrow \mathcal{H}$ is an unbounded operator, $B \in \mathcal{L}(\mathcal{H})$ and where B^* designates the adjoint operator associated to B. We rewrite problem (1.0.1) as

$$\begin{cases} U_t(t) &= \mathcal{A}_0 U(t) + BB^* U(t), \quad t > 0, \\ U(0) &= U_0 \in \mathcal{H}. \end{cases}$$

Next, we introduce the transfer function H by

$$H : \mathbb{C}_+ = \{\lambda \in \mathbb{C};\ \Re(\lambda) > 0\} \longrightarrow \mathcal{L}(\mathbb{C}) \tag{1.3.2}$$

$$H(\lambda) = B^*(\lambda + \mathcal{A}_0)^{-1}B.$$

Moreover, we define the set $C_\omega = \{\lambda \in \mathbb{C};\ \Re(\lambda) = \omega\}$ where $\omega > 0$ and we denote by (P) the following proposition:

(P) : The transfer function H defined in (1.3.2) is bounded on C_ω.

Furthermore, we introduce the Banach's spaces X_1 and Y_1 such that

$$D(\mathcal{A}) \subset Y_1 \subset \mathcal{H} \subset X_1,$$

$$\forall U \in D(\mathcal{A}), \quad \|U\|_{D(\mathcal{A})} \sim \|U\|_{Y_1},$$

and such that

$$[Y_1, X_1]_\theta = \mathcal{H}$$

where $\theta \in]0,1[$. Now, we are ready to present the theorem which gives a polynomial decay of energy of system (1.0.1).

Theorem 1.3.1. *Assume that the proposition (P) holds. If for all $U_0 \in X_1$, there exists $T > 0$ such that*

$$\int_0^T \|B^*\Phi(t)\|^2_{\mathcal{H}} \geq C\|U_0\|^2_{X_1}$$

for some constant C, where Φ is the solution of the conservative system associated to (1.0.1), then there exists a constant $\widetilde{C}$ such that for all $t > 0$ and for all $U_0 \in D(\mathcal{A})$, the energy E of (1.0.1) satisfies

$$E(t) \leq \frac{\widetilde{C}}{(1+t)^{\frac{\theta}{1-\theta}}}\|U_0\|^2_{D(\mathcal{A})}.$$

■

The second method is a frequency domain approach method given in [20, Theorem 2.4]. It is based on the boundedness of the resolvent of $\mathcal{A}$ on the imaginary axis.

Theorem 1.3.2. *Let $(T_{\mathcal{A}}(t))_{t\geq 0}$ be a bounded C_0-semigroup of contractions on a Hilbert space $\mathcal{H}$ generated by $\mathcal{A}$ such that the following condi-*

tion (H_2) *holds:*

$$(H_2): \quad i\mathbb{R} \subset \rho(\mathcal{A}).$$

Then, the following conditions are equivalent:

$(H_3): \sup_{|\beta|\geq 1} \frac{1}{|\beta|^l} \left\| (i\beta I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < +\infty.$

(H_4) : *there exists a constant* $C > 0$ *such that for all* $U_0 \in D(\mathcal{A})$ *we have*

$$E(t) \leq \frac{c}{t^{\frac{2}{l}}} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0,$$

where E *is the energy of system* (1.0.1). ■

The third one is based on a multiplier method and it consists to determine an integral inequality for the energy of our system. One of these methods is given by the following theorem from [52]:

Theorem 1.3.3. *Let* $E: \mathbb{R}_+ \to \mathbb{R}_+$ *be a non-increasing function and assume that there are two constants* $\alpha > 0$ *and* $T > 0$ *such that*

$$\int_t^{\infty} E^{\alpha+1}(s)ds \leq TE^{\alpha}(0)E(t), \quad \forall t \in \mathbb{R}_+.$$

Then we have

$$E(t) \leq E(0) \left(\frac{T + \alpha t}{T + \alpha T} \right)^{-\frac{1}{\alpha}}, \quad \forall t \geq T.$$

■

In this thesis, we use the following corollary deduced from the above theorem:

Corollary 1.3.4. *Let* $E: \mathbb{R}_+ \to \mathbb{R}_+$ *be a non-increasing function and assume that there are two constants* $\alpha > 0$ *and* $M > 0$ *such that*

$$\int_S^T E^{\alpha+1}(s)ds \leq ME^{\alpha}(0)E(T), \quad \forall 0 \leq S \leq T < +\infty.$$

Then we have

$$E(t) \leq E(0) \left(\frac{(\alpha + 1)M}{M + \alpha t} \right)^{-\frac{1}{\alpha}}, \quad \forall t \geq 0.$$

■

The fourth method is based on the spectral analysis of the operator $\mathcal{A}$. Indeed, it requires firstly to determine the asymptotic behavior of the eigenvalues associated to $\mathcal{A}$ and secondly to prove that the set of the generalized eigenvectors of $\mathcal{A}$ form a Riesz basis in the Hilbert space $\mathcal{H}$. This method is given in [63]. We also refer to [59] and [91, Lemma 3.1, Remark 3.1].

Theorem 1.3.5. *Let $(T_{\mathcal{A}}(t))_{t\geq 0}$ be a C_0-semigroup of contractions generated by the operator $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. Let $(\lambda_{k,n})_{1\leq k\leq K,\ n\geq 1}$ denotes the kth branch of eigenvalues of $\mathcal{A}$ and $\{e_{k,n}\}_{1\leq k\leq K,\ n\geq 1}$ the system of eigenvectors which forms a Riesz basis in $\mathcal{H}$. Assume that for each $1\leq k\leq K$ there exist a positive sequence $(\mu_{k,n})_{1\leq k\leq K, n\geq 1}$; $\mu_{k,n} \xrightarrow[n\to+\infty]{} +\infty$ and two positive constants $\alpha_k \geq 0$, $\beta_k > 0$ such that*

$$\Re(\lambda_{k,n}) \leq -\frac{\beta_k}{\mu_{k,n}^{\alpha_k}} \quad \textit{and} \quad |\Im(\lambda_{k,n})| \geq \mu_{k,n} \quad \forall n \geq 1.$$

Then, for any $U_0 \in D(\mathcal{A}^\theta)$ with $\theta > 0$, there exists a constant $M > 0$ independent of U_0 such that

$$\|T_{\mathcal{A}}(t)U_0\|^2 \leq \|\mathcal{A}^\theta U_0\|_{\mathcal{H}}^2 \frac{M}{t^{2\theta\delta}} \quad \forall t > 0,$$

where the decay rate δ is given by

$$\delta := \min_{1\leq k\leq K} \frac{1}{\alpha_k} = \frac{1}{\alpha_l}. \tag{1.3.3}$$

Moreover, if there exists two constants $c_1 > 0$, $c_2 > 0$ such that

$$\Re(\lambda_{l,n}) \geq -\frac{c_1}{\mu_{l,n}^{\alpha_l}} \quad \textit{and} \quad |\Im(\lambda_{l,n})| \leq c_2\mu_{l,n} \quad \forall n \geq 1,$$

then the decay rate δ given in (1.3.3) is optimal. ■

Remark 1.3.6. *The benefit of the last method given in Theorem 1.3.5 is the optimality of decay rate δ in the sense that for any $\epsilon > 0$, we cannot expect a decay rate of type $\frac{1}{t^{2\theta\delta+\epsilon}}$. But, because of the essential difficulty intervening from the determination of the spectrum of the system, this method is obviously limited to one-dimensional problem. However, the polynomial decay rate of energy of the first three methods cannot probably be optimal, but these methods are well applied in the multidimensional problem.*

1.4 Riesz basis

Since Theorem 1.3.5 consists that a family of eigenvectors of $\mathcal{A}$ must form a Riesz basis in the Hilbert space $\mathcal{H}$, in this section, we give the basic definitions and theorems needed for Riesz basis generation. We refer to [14, 40, 43].

Definition 1.4.1. *(i) A non-zero element φ in a Hilbert space $\mathcal{H}$ is called a generalized eigenvector of a closed linear operator $\mathcal{A}$, corresponding to an eigenvalue λ of $\mathcal{A}$, if there exists $n \in \mathbb{N}^*$ such that*

$$(\lambda I - \mathcal{A})^n \varphi = 0 \quad \textit{and} \quad (\lambda I - \mathcal{A})^{n-1} \varphi \neq 0.$$

If $n = 1$, then φ is an eigenvector.

(ii) The root subspace of $\mathcal{A}$ corresponding to an eigenvalue λ is defined by

$$\mathcal{N}_\lambda(\mathcal{A}) = \bigcup_{n=1}^{\infty} \ker\left((\lambda I - \mathcal{A})^n\right).$$

(iii) The closed subspace spanned by all the generalized eigenvectors of $\mathcal{A}$ is called the root subspace of $\mathcal{A}$. ■

Definition 1.4.2. *Let $\Phi = \{\varphi_n\}_{n\in\mathbb{N}}$ be an arbitrary family of vectors in a Hilbert space $\mathcal{H}$.*

(i) The family Φ is said to be a Riesz basis in the closure of its linear span if Φ is an image by an isomorphic mapping of some orthonormal family. Φ is said to be a Riesz basis if Φ is a Riesz basis in the closure of its linear span and Φ is a complete family; i.e., $\overline{Span\{\varphi_n;\ n \in \mathbb{N}\}} = \mathcal{H}$.

(ii) The family Φ is said to be $\omega-$linearly independent if whenever $\sum_{n\in\mathbb{N}} a_n\varphi_n = 0$ for $\sum_{n\in\mathbb{N}} |a_n|^2 < \infty$ then $a_n = 0$ for every $n \in \mathbb{N}$. ■

Proposition 1.4.3. *(Bari's Theorem, Bari 1951; Gokhberg and krein 1988; Nikolski 1980)*
Let $\Phi = \{\varphi_n\}_{n\in\mathbb{N}}$ be an arbitrary family of vectors in a Hilbert space $\mathcal{H}$. Φ is said to be a Riesz basis in the closure of its linear span if and only if there exists positive constants C_1 and C_2 such that for any sequence

$\{\alpha_n\}_{n\in\mathbb{N}}$, we have

$$C_1 \sum_{n\in\mathbb{N}} |\alpha_n|^2 \le \left\| \sum_{n\in\mathbb{N}} \alpha_n \varphi_n \right\|^2 \le C_2 \sum_{n\in\mathbb{N}} |\alpha_n|^2.$$

In this case, each element $f \in \overline{Span\{\varphi_n,\ n\in\mathbb{N}\}}$ is written as

$$f = \sum_{n\in\mathbb{N}} < f, \psi_n >_{\mathcal{H}} \varphi_n,$$

where $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ is biorthogonal to Φ. ■

The following two theorems give the necessary and the sufficient conditions so that a family $\{\phi_n\}_{n\in\mathbb{N}}$ forms a Riesz basis.

Theorem 1.4.4. *(Theorem 2.1 of Chapter VI in [40])*
An arbitrary family $\{\phi_n\}_{n\in\mathbb{N}}$ of vectors forms a Riesz basis of a Hilbert space $\mathcal{H}$ if and only if $\{\phi_n\}_{n\in\mathbb{N}}$ is complete in $\mathcal{H}$ and there corresponds to it a complete biorthogonal sequence $\{\psi_n\}_{n\in\mathbb{N}}$ such that for any $f \in \mathcal{H}$ one has

$$\sum_{n\in\mathbb{N}} |< \phi_n, f >|^2 < \infty, \quad \sum_{n\in\mathbb{N}} |< \psi_n, f >|^2 < \infty. \tag{1.4.1}$$

■

Theorem 1.4.5. *(Classical Bari's Theorem)*
Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a Riesz basis of a Hilbert space $\mathcal{H}$ and another $\omega-$ linearly independent family $\{\psi_n\}_{n\in\mathbb{N}}$ is quadratically close to $\{\varphi_n\}_{n\in\mathbb{N}}$ in the sense that

$$\sum_{n=1}^{\infty} \|\varphi_n - \psi_n\|^2 < \infty.$$

Then $\{\psi_n\}_{n\in\mathbb{N}}$ also forms a Riesz basis of $\mathcal{H}$. ■

Typically, the comprehension of the number of generalized eigenfunctions corresponding to low eigenvalues seems difficult. The application of the above classical Bari's theorem seems also difficult even if the behavior of the high eigenvalues and their corresponding multiplicities are clearly known. Consequently, in case the behavior of low eigenvalues is vague, we suggest using Theorem 6.3 of [43] which is a new form of Bari's theorem (see Theorem 2.3 of Chapter VI in [40]).

Theorem 1.4.6. *Let $\mathcal{A}$ be a densely defined operator in a Hilbert space $\mathcal{H}$ with a compact resolvent. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a Riesz basis of $\mathcal{H}$. If there are* *an integer* $N \geq 0$ *and a sequence of generalized eigenvectors* $\{\phi_n\}_{n=N+1}^{\infty}$ *of* $\mathcal{A}$ *such that*

$$\sum_{n=N+1}^{\infty} \| \varphi_n - \phi_n \|^2 < \infty,$$

then the set of generalized eigenvectors of $\mathcal{A}$*,* $\{\phi_n\}_{n=1}^{\infty}$*, forms a Riesz basis of* $\mathcal{H}$*.*

■

In this thesis, we use the following theorem from [2, Theorem 1.4.10] which clarifies the results of Theorem 1.4.6:

Theorem 1.4.7. *Let* $\mathcal{A}$ *be a densely defined operator in a Hilbert space* $\mathcal{H}$ *with a compact resolvent. Let* $\{\varphi_n\}_{n=1}^{\infty}$ *be a Riesz basis of* $\mathcal{H}$*. If there are two integers* $N_1, N_2 \geq 0$ *and a sequence of generalized eigenvectors* $\{\phi_n\}_{n=N+1}^{\infty}$ *of* $\mathcal{A}$ *such that*

$$\sum_{n=1}^{\infty} \| \varphi_{n+N_2} - \phi_{n+N_1} \|^2 < \infty, \tag{1.4.2}$$

then the set of generalized eigenvectors (or root vectors) of $\mathcal{A}$*,* $\{\phi_n\}_{n=1}^{\infty}$ *forms a Riesz basis of* $\mathcal{H}$*.* ■

2

Rayleigh beam equation with only one dynamical boundary control moment

Abstract: In [92], Wehbe considered a Rayleigh beam equation with two dynamical boundary controls and established the optimal polynomial energy decay rate of type $\frac{1}{t}$. The proof exploits in an explicit way the presence of two boundary controls, hence the case of the Rayleigh beam damped by only one dynamical boundary control remained open. In this chapter, we fill this gap by considering a clamped Rayleigh beam equation subject to only one dynamical boundary control moment. First, we prove a polynomial decay in $\frac{1}{t}$ of the energy by using an observability inequality. For that purpose, we give the asymptotic expansion of eigenvalues and eigenfunctions of the undamped underling system. Next, using the real part of the asymptotic expansion of eigenvalues of the damped system, we prove that the obtained energy decay rate is optimal.

2.1 Introduction

In [92], Wehbe considered a Rayleigh beam clamped at one end and subjected to two dynamical boundary controls at the other end, namely

$$y_{tt} - \gamma y_{xxtt} + y_{xxxx} = 0, \qquad 0 < x < 1, \quad t > 0, \tag{2.1.1}$$
$$y(0,t) = y_x(0,t) = 0, \qquad t > 0, \tag{2.1.2}$$
$$y_{xx}(1,t) + a\eta(t) = 0, \qquad t > 0, \tag{2.1.3}$$
$$y_{xxx}(1,t) - \gamma y_{xtt}(1,t) - b\xi(t) = 0, \qquad t > 0, \tag{2.1.4}$$

where $\gamma > 0$ is the coefficient of moment of inertia, $a > 0$ and $b > 0$ are constants, η and ξ denote respectively the dynamical boundary control moment and force. The damping of the system is made via the indirect damping mechanism at the right extremity of the beam that involves the following two first order differential equations:

$$\eta_t(t) - y_{xt}(1,t) + \alpha\eta(t) = 0, \qquad t > 0, \tag{2.1.5}$$
$$\xi_t(t) - y_t(1,t) + \beta\xi(t) = 0, \qquad t > 0, \tag{2.1.6}$$

where $\alpha > 0$ and $\beta > 0$ are constants. The notion of indirect damping mechanisms has been introduced by Russell in [85] and since that time, it retains the attention of many authors. In [92], Wehbe considered the Rayleigh beam equation with two dynamical boundary controls moment and force, *i.e.* under the conditions $a > 0$ and $b > 0$. The lack of uniform stability was proved by a compact perturbation argument of Gibson [35] and a polynomial energy decay rate of type $\frac{1}{t}$ is obtained by a multiplier method usually used for nonlinear problems. Finally, using a spectral method, he proved that the obtained energy decay is optimal in the sense that for any $\varepsilon > 0$, we cannot expect a decay rate of type $\frac{1}{t^{1+\varepsilon}}$. But in [92] the effect of each control separately on the stability of the Rayleigh beam equation is not investigated. Indeed, the multiplier method exploits in an explicit way the presence of the two boundary controls. Furthermore, the lack of one of this two controls yield this method ineffective. Then, the important and interesting case when the Rayleigh beam equation is damped by only one dynamical boundary control ($a = 0$ and $b > 0$ or $a > 0$ and $b = 0$) remained open. The aim of this chapter is to fill this gap by considering a clamped Rayleigh beam equation subject to only one dynamical boundary control moment.

The stabilization of the Rayleigh beam equation retains the attention of many authors. Rao [79] studied the stabilization of Rayleigh beam equation subject to a positive internal viscous damping. Using a constructive approximation, he established the optimal exponential energy decay rate. In [55], Lagnese studied the stabilization of system (2.1.1)-(2.1.4) with two static boundary controls (the case $a > 0$, $b > 0$, $\eta(t) = y_{xt}(1,t)$ and $\xi(t) = y_t(1,t)$). He proved that the energy decays exponentially to zero for all initial data. Rao in [79] extended the results of [55] to the case of one boundary feedback. In the case of one control moment (the case $a > 0$, $b = 0$ and $\eta(t) = y_{xt}(1,t)$), using a compact perturbation theory due to Gibson [35], he established an exponential stability of system (2.1.1)-(2.1.4).

In this chapter, we consider the Rayleigh beam equation (2.1.1)-(2.1.4) with only one dynamical boundary control moment η, *i.e.* when $a = 1$, $b = 0$ and η is solution of (2.1.5). Using an explicit approximation of the characteristic equation, we give the asymptotic behavior of eigenvalues and eigenfunctions of the associated undamped system with the help of Rouché's theorem. Then to prove the polynomial energy decay, we apply the methodology given in [12]. This requires, on one hand, to establish an observability inequality of solution of the undamped system and on the other hand, to verify the boundedness property of the transfer function. This attend to establish a polynomial energy decay rate of type $\dfrac{1}{t}$ for smooth initial data. Finally, using the frequency domain approach given by Theorem 2.4 in [20], we prove that the obtained energy decay rate is optimal in the sense that for any $\varepsilon > 0$, we cannot expect a decay rate of type $\dfrac{1}{t^{1+\varepsilon}}$.

Let us briefly outline the content of this chapter. Section 2.2 considers the well-posedness property and the strong stability of the problem by the semigroup approach (see [75], [79] and [92]). Section 2.3 is divided into two subsections. In subsection 2.3.1, we propose an explicit approximation of the characteristic equation determining the eigenvalues of the corresponding undamped system. In subsection 2.3.2, we give an asymptotic expansion of eigenvalues and eigenfunctions of the corresponding operator. Then, we establish a polynomial energy decay rate for smooth initial data. In section 2.4, we prove that the obtained energy decay rate is optimal. In section 2.5, we give some open problems.

2.2 Well-posedness and strong stability

In this section, we study the existence, uniqueness and the asymptotic behavior of the solution of Rayleigh beam equation with only one dynamical boundary control moment:

$$\begin{cases} y_{tt} - \gamma y_{xxtt} + y_{xxxx} & = 0, \quad 0 < x < 1, \quad t > 0, \\ y(0,t) = y_x(0,t) & = 0, \qquad\qquad\qquad t > 0, \\ y_{xx}(1,t) + \eta(t) & = 0, \qquad\qquad\qquad t > 0, \\ y_{xxx}(1,t) - \gamma y_{xtt}(1,t) & = 0, \qquad\qquad\qquad t > 0, \\ \eta_t(t) - y_{xt}(1,t) + \alpha\eta(t) & = 0, \qquad\qquad\qquad t > 0. \end{cases} \tag{2.2.1}$$

Let y and η be smooth solutions of system (2.2.1), we define their associated energy by

$$E(t) = \frac{1}{2}\left(\int_0^1 (|y_t|^2 + \gamma|y_{xt}|^2 + |y_{xx}|^2)dx + |\eta(t)|^2\right), \quad t \geq 0. \tag{2.2.2}$$

A direct computation gives

$$\frac{d}{dt}E(t) = -\alpha|\eta(t)|^2 \leq 0, \quad t \geq 0. \tag{2.2.3}$$

Thus the system (2.2.1) is dissipative in the sense that the energy E is a nonincreasing function of the time variable t. We start our study by formulating the problem in an appropriate Hilbert space. We first introduce the following spaces:

$$V = \left\{y \in H^1(0,1);\ y(0) = 0\right\}, \quad \|y\|_V^2 = \int_0^1 (|y|^2 + \gamma|y_x|^2)dx, \tag{2.2.4}$$

$$W = \left\{y \in H^2(0,1);\ y(0) = y_x(0) = 0\right\}, \quad \|y\|_W^2 = \int_0^1 |y_{xx}|^2 dx \tag{2.2.5}$$

and the energy space

$$\mathcal{H} = W \times V \times \mathbb{C}, \tag{2.2.6}$$

endowed with the usual inner product

$$((y_1,\ z_1,\ \eta_1), (y_2,\ z_2,\ \eta_2))_{\mathcal{H}} = (y_1, y_2)_W + (z_1, z_2)_V + \eta_1\overline{\eta_2},$$

$$\forall (y_1,\, z_1,\, \eta_1), (y_2,\, z_2,\, \eta_2) \in \mathcal{H}.$$

Identify $L^2(0,1)$ with its dual so that we have the following continuous embedding:

$$W \subset V \subset L^2(0,1) \subset V' \subset W'. \tag{2.2.7}$$

Multiplying the first equation of the system (2.2.1) by $\overline{\Phi} \in W$ and integrating by parts yields

$$\int_0^1 (y_{tt}\overline{\Phi} + \gamma y_{xtt}\overline{\Phi_x})dx + \int_0^1 y_{xx}\overline{\Phi_{xx}}dx + \eta\overline{\Phi_x(1)} = 0. \tag{2.2.8}$$

Now, we define the following linear operators $A \in \mathcal{L}(W, W')$, $B \in \mathcal{L}(\mathbb{C}, W')$ and $C \in \mathcal{L}(V, V')$ by

$$< Ay, \Phi >_{W' \times W} = (y, \Phi)_W, \quad \forall y, \Phi \in W, \tag{2.2.9}$$

$$< B\eta, \Phi >_{W' \times W} = \eta\overline{\Phi_x(1)}, \quad \forall \eta \in \mathbb{C}, \forall \Phi \in W \tag{2.2.10}$$

and

$$< Cy, \Phi >_{V' \times V} = (y, \Phi)_V, \quad \forall y, \Phi \in V. \tag{2.2.11}$$

Then, by means of Lax-Milgram's theorem (see [21]), we see that A (respectively C) is the canonical isomorphism from W into W' (respectively from V into V'). On the other hand, using the usual trace theorems and Poincaré's inequality, we easily check that the operator B is continuous for the corresponding topology. Therefore, using the operators A, B and C and the continuous embedding (2.2.7), we formulate the variational equation (2.2.8) as

$$Cy_{tt} + Ay + B\eta = 0 \quad \text{ in } W'.$$

Assume that $Ay + B\eta \in V'$, then we obtain

$$y_{tt} + C^{-1}(Ay + B\eta) = 0 \qquad \text{in } V. \tag{2.2.12}$$

Next, we introduce the linear unbounded operator $\mathcal{A}_0$ by

$$D(\mathcal{A}_0) = \left\{(y, z, \eta) \in \mathcal{H};\ z \in W \ \text{ and } Ay + B\eta \in V'\right\}, \tag{2.2.13}$$

$$\mathcal{A}_0 U = \begin{pmatrix} -z \\ C^{-1}(Ay + B\eta) \\ -z_x(1) \end{pmatrix}, \quad \forall U = (y, z, \eta) \in D(\mathcal{A}_0) \tag{2.2.14}$$

and the linear bounded operator $\mathcal{B}$ by

$$\mathcal{B}U = \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix}, \quad \forall U = (y, z, \eta) \in \mathcal{H}. \tag{2.2.15}$$

Then, denoting $U = (y, y_t, \eta)$ the state of system (2.2.1) and defining $\mathcal{A}_\alpha = \mathcal{A}_0 + \alpha\mathcal{B}$ with $D(\mathcal{A}_\alpha) = D(\mathcal{A}_0)$, we can formulate the system (2.2.1) into a first-order evolution equation

$$\begin{cases} U_t(t) + \mathcal{A}_\alpha U(t) &= \quad 0, \qquad t > 0, \\ \qquad U(0) &= U_0 \in \mathcal{H}. \end{cases} \tag{2.2.16}$$

It is easy to show that $-\mathcal{A}_0$ is a maximal dissipative operator and $-\mathcal{B}$ is a dissipative operator in the energy space $\mathcal{H}$. Therefore, the operator $-\mathcal{A}_\alpha$ generates a C_0-semigroup $(e^{-t\mathcal{A}_\alpha})_{t\geq 0}$ of contractions in the energy space $\mathcal{H}$ following Lumer-Phillips' theorem (see [75]). Hence, we have the following results concerning the existence and uniqueness of the solution of the problem (2.2.16):

Theorem 2.2.1. *For any initial data $U_0 \in \mathcal{H}$, the problem (2.2.16) has a unique weak solution $U(t) = e^{-t\mathcal{A}_\alpha}U_0$ such that $U \in C^0\left([0,\infty[, \mathcal{H}\right)$. Moreover, if $U_0 \in D(\mathcal{A}_0)$, then the problem (2.2.16) has a strong solution $U(t) = e^{-t\mathcal{A}_\alpha}U_0$ such that $U \in C^1\left([0,\infty[, \mathcal{H}\right) \cap C^0\left([0,\infty[, D(\mathcal{A}_0)\right)$.* ■

Moreover, we characterize the space $D(\mathcal{A}_0)$ by the following proposition:

Proposition 2.2.2. *Let $U = (y, z, \eta) \in \mathcal{H}$. Then $U \in D(\mathcal{A}_0)$ if and only if the following conditions hold:*

$$\begin{cases} y \in W \cap H^3(0,1), \\ \qquad z \in W, \\ \ y_{xx}(1) + \eta = 0. \end{cases} \tag{2.2.17}$$

In particular, the resolvent $(I + \mathcal{A}_0)^{-1}$ of $-\mathcal{A}_0$ is compact on the energy space $\mathcal{H}$ and the solution of the system (2.2.1) satisfies

$$y \in C^0([0,\infty[, H^3(0,1) \cap W) \cap C^1([0,\infty[, W) \cap C^2([0,\infty[, V). \tag{2.2.18}$$

■

The proof is same as in Rao [79, Proposition 2.3] (see also Wehbe [92]) so we omit the details here. Moreover, since the resolvent of the bounded

operator $\mathcal{B}$ is compact, we deduce that the one of the unbounded operator -$\mathcal{A}_\alpha$ is also compact.
Now we investigate the strong stability of the problem (2.2.16) by the following theorem:

Theorem 2.2.3. *For any $\gamma > 0$, the semigroup of contractions $(e^{-t\mathcal{A}_\alpha})_{t\geq 0}$ is strongly asymptotically stable on the energy space $\mathcal{H}$, i.e. for any $U_0 \in \mathcal{H}$, we have*

$$\lim_{t\to+\infty} \|e^{-t\mathcal{A}_\alpha} U_0\|^2_{\mathcal{H}} = 0. \tag{2.2.19}$$

Proof: The proof is same as in Rao [79, Theorem 3.1], it is based on the spectral decomposition theory of Sz-Nagy-Foias [69], Foguel [29] and Benchimol [19]. In order to prove (2.2.19) and the fact that $-\mathcal{A}_\alpha$ has compact resolvent, it is sufficient to show that there is no spectrum in imaginary axis. We omit the details here. ■
Further, since $\mathcal{A}_0$ is skew adjoint operator and $\mathcal{B}$ is compact, then using a compact perturbation method of Russell [83] we deduce that the problem (2.2.16) is not uniformly stable (see also Rao [79], and Wehbe [92]).

2.3 Polynomial stability for smooth initial data

Our main result in this section is the following polynomial-type decay estimate:

Theorem 2.3.1. *(Polynomial energy decay rate)*
Let $\gamma > 0$. For all initial data $U_0 \in D(\mathcal{A}_0)$, there exists a constant $c > 0$ independent of U_0, such that the solution of the problem (2.2.16) satisfies the following estimate:

$$E(t) \leq \frac{c}{1+t}\|U_0\|^2_{D(\mathcal{A}_0)}, \quad \forall t > 0. \tag{2.3.1}$$

■

In order to prove (2.3.1), we need first to analyze the spectrum of the operator $\mathcal{A}_0$. Next, we will apply a method introduced by Ammari and Tucsnak in [12], where the polynomial stability for the damped problem is reduced to an observability inequality of the corresponding undamped problem (via the spectral analysis), combined with the boundedness property of the transfer function of the associated undamped system.

2.3.1 Spectral analysis of the conservative operator

First, since $\mathcal{A}_0$ is closed with a compact resolvent, its spectrum $\sigma(\mathcal{A}_0)$ consists entirely of isolated eigenvalues with finite multiplicities (see [50]). Moreover, as the coefficients of $\mathcal{A}_0$ are real then the eigenvalues appear by conjugate pairs. Further, the eigenvalues of $\mathcal{A}_0$ are on the imaginary axis.

Proposition 2.3.2. *Let λ be an eigenvalue of $\mathcal{A}_0$ and let $U = (y, z, \eta) \in D(\mathcal{A}_0), U \neq 0$, an associated eigenvector. Then λ is simple and we have $\eta \neq 0$.*

Proof: First, a straightforward computation shows that $0 \in \sigma(\mathcal{A}_0)$ and is simple. An associated eigenvector being $(-\dfrac{x^2}{2},\ 0,\ 1)$, thus its last component $\eta = 1$ does not vanish.
Next, let $\lambda = i\mu \in \sigma(\mathcal{A}_0),\ \mu \in \mathbb{R}^*$ and $U = (y, z, \eta)$ an associated eigenvector.
Assume that $\eta = 0$. Using equation (2.2.15), we get that $\mathcal{B}U = 0$. Thus, we obtain

$$\mathcal{A}_\alpha U = (\mathcal{A}_0 + \alpha\mathcal{B})U = \mathcal{A}_0 U = i\mu U. \tag{2.3.2}$$

Therefore $\lambda = i\mu$ is also an eigenvalue of $\mathcal{A}_\alpha$ and it is a contradiction with Theorem 2.2.3 since $\gamma > 0$.
Later, assume that there exists $\lambda \in \sigma(\mathcal{A}_0)$ such that λ is not simple. As $\mathcal{A}_0$ is a skew-adjoint operator, we deduce that there correspond at least two independent eigenvectors $U_1 = (y_1, z_1, \eta_1)$ and $U_2 = (y_2, z_2, \eta_2)$. Then, $U_3 = \eta_2 U_1 - \eta_1 U_2 = (y_3, z_3, \eta_3)$ is also an eigenvector associated to λ with $\eta_3 = 0$, hence the contradiction with the first part of the proof.

■

Now, in order to get a better knowledge of the spectrum we compute the characteristic equation. Let $\lambda = i\mu,\ \mu \in \mathbb{R}^*$, be an eigenvalue of $\mathcal{A}_0$ and $U = (y,\ z,\ \eta) \in D(\mathcal{A}_0)$ be an associated eigenfunction. Then we have

$$\begin{cases} z &= -i\mu y, \\ Ay + B\eta &= i\mu C z, \\ z_x(1) &= -i\mu\eta. \end{cases} \tag{2.3.3}$$

Using (2.2.9)-(2.2.11), we interpret (2.3.3) as the following variational equation:

$$\int_0^1 y_{xx}\overline{\Phi_{xx}}dx - \mu^2 \int_0^1 \left(y\overline{\Phi} + \gamma y_x \overline{\Phi_x}\right) dx + y_x(1)\overline{\Phi_x(1)} = 0, \qquad \forall \Phi \in W.$$

Equivalently, the function y is determined by the following system:

$$\begin{cases} y_{xxxx} + \gamma\mu^2 y_{xx} - \mu^2 y &= 0, \\ y(0) = y_x(0) &= 0, \\ y_{xx}(1) + y_x(1) &= 0, \\ y_{xxx}(1) + \gamma\mu^2 y_x(1) &= 0. \end{cases} \tag{2.3.4}$$

We have found that $\lambda = i\mu \neq 0$ is an eigenvalue of $\mathcal{A}_0$ if and only if there is a non trivial solution of (2.3.4). The general solution of the first equation of (2.3.4) is given by

$$y(x) = \sum_{i=1}^{4} c_i e^{r_i(\mu)x}, \tag{2.3.5}$$

where

$$r_1(\mu) = \sqrt{\frac{-\gamma\mu^2 + \mu\sqrt{\gamma^2\mu^2 + 4}}{2}}, r_2(\mu) = -r_1(\mu), \tag{2.3.6}$$

$$r_3(\mu) = \sqrt{\frac{-\gamma\mu^2 - \mu\sqrt{\gamma^2\mu^2 + 4}}{2}}, r_4(\mu) = -r_3(\mu).$$

Here and below, for simplicity we denote $r_i(\mu)$ by r_i. Thus the boundary conditions in (2.3.4) may be written as the following system:

$$M(\mu)C(\mu) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ r_1 & r_2 & r_3 & r_4 \\ g_1(\mu) & g_2(\mu) & g_3(\mu) & g_4(\mu) \\ h_1(\mu) & h_2(\mu) & h_3(\mu) & h_4(\mu) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0, \tag{2.3.7}$$

where $g_i(\mu) = r_i\left(r_i^2 + \gamma\mu^2\right)e^{r_i}$ and $h_i(\mu) = r_i\left(r_i + 1\right)e^{r_i}$ for $i = 1, 2, 3, 4$. Consequently (2.3.4) admits a non-trivial solution if and only if $f(\mu) :=$ $\det M(\mu) = 0$. Finally, we have found that $\lambda = i\mu$ is an eigenvalue of $\mathcal{A}_0$ if and only if μ satisfies the characteristic equation $f(\mu) = 0$.

Proposition 2.3.3. *(Spectrum of $\mathcal{A}_0$)*
There exists $k_0 \in \mathbb{N}^*$, sufficiently large, such that the spectrum $\sigma(\mathcal{A}_0)$ of

$\mathcal{A}_0$ is given by:

$$\sigma(\mathcal{A}_0) = \sigma_0 \cup \sigma_1, \tag{2.3.8}$$

where

$$\sigma_0 = \left\{ i\kappa_j^0 \right\}_{j \in J_0}, \quad \sigma_1 = \left\{ \lambda_k^0 = i\mu_k \right\}_{\substack{k \in \mathbb{Z} \\ |k| \geq k_0}}, \quad \sigma_0 \cap \sigma_1 = \emptyset, \tag{2.3.9}$$

J_0 is a finite set and κ_j^0, $\mu_k \in \mathbb{R}$. Moreover, μ_k satisfies the following asymptotic behavior:

$$\mu_k = \alpha_k - \frac{F_1}{F_0} \frac{1}{k\pi} + O(\frac{1}{k^2}), \quad |k| \to \infty, \tag{2.3.10}$$

where

$$\alpha_k = \frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}, \tag{2.3.11}$$

and where

$$\begin{cases} F_0 & = & 2\gamma^{\frac{3}{2}} \cosh(\frac{1}{\sqrt{\gamma}}), \\ F_1 & = & (1-2\gamma)\cosh(\frac{1}{\sqrt{\gamma}}) + 2\sqrt{\gamma}\sinh(\frac{1}{\sqrt{\gamma}}). \end{cases} \tag{2.3.12}$$

Proof: The proof is decomposed into two steps.
Step 1. First, we start by the expansion of r_1 and r_3 when $|\mu| \to \infty$. After some computations we find

$$r_1 = \frac{1}{\sqrt{\gamma}} + O(\frac{1}{\mu^2}) \tag{2.3.13}$$

and

$$r_3 = i\sqrt{\gamma}\mu + i\frac{1}{2\gamma^{\frac{3}{2}}\mu} + O(\frac{1}{\mu^3}). \tag{2.3.14}$$

This gives

$$r_1^2 e^{r_1} = \frac{e^{\frac{1}{\sqrt{\gamma}}}}{\sqrt{\gamma}} + O(\frac{1}{\mu^2}), \tag{2.3.15}$$

$$r_2^2 e^{r_2} = \frac{e^{-\frac{1}{\sqrt{\gamma}}}}{\sqrt{\gamma}} + O(\frac{1}{\mu^2}), \tag{2.3.16}$$

$$r_3^2 e^{r_3} = -\gamma e^{i\sqrt{\gamma}\mu}\mu^2 + O(1) \tag{2.3.17}$$

and

$$r_4^2 e^{r_4} = -\gamma e^{-i\sqrt{\gamma}\mu}\mu^2 + O(1). \tag{2.3.18}$$

Next, using (2.3.13)-(2.3.18), we find the asymptotic behavior of

$$g_1(\mu) = \sqrt{\gamma}e^{\frac{1}{\sqrt{\gamma}}}\mu^2 + O(1), \tag{2.3.19}$$

$$g_2(\mu) = -\sqrt{\gamma}e^{-\frac{1}{\sqrt{\gamma}}}\mu^2 + O(1), \tag{2.3.20}$$

$$g_3(\mu) = -\frac{ie^{i\sqrt{\gamma}\mu}\mu}{\sqrt{\gamma}} + O(\frac{1}{\mu}) \tag{2.3.21}$$

and

$$g_4(\mu) = \frac{ie^{-i\sqrt{\gamma}\mu}\mu}{\sqrt{\gamma}} + O(\frac{1}{\mu}). \tag{2.3.22}$$

Similarly, we get

$$h_1(\mu) = \left(\frac{1}{\sqrt{\gamma}} + 1\right)\frac{e^{\frac{1}{\sqrt{\gamma}}}}{\sqrt{\gamma}} + O(\frac{1}{\mu}), \tag{2.3.23}$$

$$h_2(\mu) = \left(\frac{1}{\sqrt{\gamma}} - 1\right)\frac{e^{-\frac{1}{\sqrt{\gamma}}}}{\sqrt{\gamma}} + O(\frac{1}{\mu}), \tag{2.3.24}$$

$$h_3(\mu) = \left(-\gamma\mu^2 + i(\sqrt{\gamma} - \frac{1}{2\sqrt{\gamma}})\mu\right)e^{i\sqrt{\gamma}\mu} + O(1) \tag{2.3.25}$$

and

$$h_4(\mu) = \left(-\gamma\mu^2 + i(\frac{1}{2\sqrt{\gamma}} - \sqrt{\gamma})\mu\right)e^{-i\sqrt{\gamma}\mu} + O(1). \tag{2.3.26}$$

Now, using (2.3.7) and (2.3.13)-(2.3.26), we can write $M(\mu)$ as follows

$$M(\mu) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ P_1^\mu & P_2^\mu & P_3^\mu & P_4^\mu \\ P_5^\mu & P_6^\mu & P_7^\mu & P_8^\mu \\ P_9^\mu & P_{10}^\mu & P_{11}^\mu & P_{12}^\mu \end{pmatrix}, \tag{2.3.27}$$

where

$$P_1^\mu = \frac{1}{\sqrt{\gamma}} + O(\frac{1}{\mu^2}), \quad P_2^\mu = -\frac{1}{\sqrt{\gamma}} + O(\frac{1}{\mu^2}), P_3^\mu = i\sqrt{\gamma}\mu + i\frac{1}{2\gamma^{\frac{3}{2}}\mu} + O(\frac{1}{\mu^2}),$$

$$P_4^\mu = -i\sqrt{\gamma}\mu - i\frac{1}{2\gamma^{\frac{3}{2}}\mu} + O(\frac{1}{\mu^3}), \quad P_5^\mu = \sqrt{\gamma}e^{\frac{1}{\sqrt{\gamma}}}\mu^2 + O(1),$$

$$P_6^\mu = -\sqrt{\gamma}e^{-\frac{1}{\sqrt{\gamma}}}\mu^2 + O(1),$$

$$P_7^\mu = -\frac{ie^{i\sqrt{\gamma}\mu}\mu}{\sqrt{\gamma}} + O(\frac{1}{\mu}), \quad P_8^\mu = \frac{ie^{-i\sqrt{\gamma}\mu}\mu}{\sqrt{\gamma}} + O(\frac{1}{\mu}),$$

$$P_9^\mu = \left(\frac{1}{\gamma} + \frac{1}{\sqrt{\gamma}}\right)e^{\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\mu}), \quad P_{10}^\mu = \left(\frac{1}{\gamma} - \frac{1}{\sqrt{\gamma}}\right)e^{-\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\mu}),$$

$$P_{11}^\mu = \left(-\gamma\mu^2 + i(\sqrt{\gamma} - \frac{1}{2\sqrt{\gamma}})\mu\right)e^{i\sqrt{\gamma}\mu} + O(1)$$

and

$$P_{12}^\mu = \left(-\gamma\mu^2 + i(\frac{1}{2\sqrt{\gamma}} - \sqrt{\gamma})\mu\right)e^{-i\sqrt{\gamma}\mu} + O(\mu).$$

Again after some computations, we find the following asymptotic development of $f(\mu) = det(M(\mu))$:

$$f(\mu) = \mu^5 f_0(\mu) + \mu^4 f_1(\mu) + O(\mu^3),$$

where

$$f_0(\mu) = -2iF_0\sqrt{\gamma}\cos(\sqrt{\gamma}\mu) \text{ and } f_1(\mu) = 2i\sqrt{\gamma}F_1\sin(\sqrt{\gamma}\mu), \tag{2.3.28}$$

with F_0 and F_1 are given by (2.3.12). For convenience we set

$$S(\mu) = \frac{f(\mu)}{\mu^5} = f_0(\mu) + \frac{f_1(\mu)}{\mu} + O(\frac{1}{\mu^2}), \tag{2.3.29}$$

that has the same root as f, except 0.

Step 2. We look at the roots of S. Is is easy to see that the roots of f_0 are given by

$$\alpha_k = \frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}, \quad k \in \mathbb{Z}.$$

Then, with the help of Rouché's theorem, there exists $k_0 \in \mathbb{N}^*$ large enough, such that for all $|k| \geq k_0$, the large roots of S (denoted by μ_k) are close to α_k. More precisely, there exists $k_0 \in \mathbb{N}^*$ large enough, such that the splitting of $\sigma(\mathcal{A}_0)$ given by (2.3.8)-(2.3.9) holds and we have

$$\mu_k = \alpha_k + o(1) = \frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} + o(1), \quad |k| \to \infty. \tag{2.3.30}$$

Equivalently, we can write

$$\mu_k = \frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} + l_k, \quad \lim_{|k|\to\infty} l_k = 0. \tag{2.3.31}$$

It follows that

$$\cos(\sqrt{\gamma}\mu_k) = -(-1)^k \sin(\sqrt{\gamma} l_k) = -(-1)^k \sqrt{\gamma} l_k + o(l_k^2) \tag{2.3.32}$$

and

$$\sin(\sqrt{\gamma}\mu_k) = (-1)^k \cos(\sqrt{\gamma} l_k) = (-1)^k (1 - \frac{\sqrt{\gamma} l_k^2}{2}) + o(l_k^2). \tag{2.3.33}$$

Using (2.3.31), (2.3.32) and (2.3.33) then from (2.3.29) we have

$$0 = S(\mu_k) = 2i\sqrt{\gamma}(-1)^k (F_0 \sqrt{\gamma} l_k + \frac{F_1}{k\pi}) + o(l_k^2) + O(\frac{1}{k^2}).$$

This implies that

$$l_k = -\frac{F_1}{F_0}\frac{1}{k\pi} + O(\frac{1}{k^2}). \tag{2.3.34}$$

Finally, inserting the previous identity in (2.3.31), we directly get (2.3.10). ■

Eigenvectors of $\mathcal{A}_0$. According to the decomposition of the spectrum $\sigma(\mathcal{A}_0)$ of $\mathcal{A}_0$, a set of eigenvectors associated with $\sigma(\mathcal{A}_0)$ is given as follows:

$$\{\Phi_j = (y_j, z_j, \eta_j)\}_{j\in J_0} \bigcup \{U_k = (y_k, z_k, \eta_k)\}_{\substack{k\in\mathbb{Z} \\ |k|\geq k_0}}, \tag{2.3.35}$$

$$\Phi_j \in D(\mathcal{A}_0),\ \forall j \in J_0, \quad U_k \in D(\mathcal{A}_0),\ \forall k \in \mathbb{Z},\ |k| \geq k_0,$$

where

$$\Phi_j = \begin{pmatrix} y_j \\ -i\kappa_j^0 y_j \\ y_{j,x}(1) \end{pmatrix} \quad \text{and} \quad U_k = \begin{pmatrix} y_k \\ -i\mu_k y_k \\ y_{k,x}(1) \end{pmatrix}. \tag{2.3.36}$$

Now, for $|k| \geq k_0$ and $\mu = \mu_k$, we give a solution up to a factor of problem (2.3.4) and some appropriated asymptotic behavior.

Proposition 2.3.4. *Let $|k| \geq k_0$. Then, a solution y_k of the undamped initial value problem (2.3.4) with $\mu = \mu_k$ satisfies the following estimations:*

$$\begin{cases} y_{k,x}(1) = -(-1)^k k\pi + O(1) \neq 0, \\ \|y_k\|_W \sim |k|^2 \text{ and } \|y_k\|_V \sim |k|, \quad |k| \to \infty. \end{cases} \tag{2.3.37}$$

Moreover, we deduce

$$\|U_k\|_{\mathcal{H}} \sim |k|^2, \quad |k| \to \infty. \tag{2.3.38}$$

Proof: For $\mu = \mu_k, |k| \geq k_0$, solving (2.3.4) amounts to find a solution $C(\mu_k) \neq 0$ of the system (2.3.7) of rank three. For clarity, we divide the proof into two steps.

Step 1. Estimate of $y_{k,x}(1)$. For simplicity of notation we write $C(\mu_k) = (c_1, c_2, c_3, c_4)$. Since we search $C(\mu_k)$ up to a factor we choose $c_3 = 1$, the possibility of this choice will be justify later. Therefore (2.3.7) becomes

$$\begin{cases} c_1 + c_2 + c_4 & = & -1, \\ r_1 c_1 + r_2 c_2 + r_4 c_4 & = & -r_3, \\ r_1(r_1+1)e^{r_1} c_1 + r_2(r_2+1)e^{r_2} c_2 + r_4(r_4+1)e^{r_4} c_4 & = & -r_3(r_3+1)e^{r_3}. \end{cases}$$

Next, using Cramer's rule we obtain

$$c_1 = \frac{\alpha_1}{\alpha_3}, \quad c_2 = \frac{\alpha_2}{\alpha_3}, \quad c_4 = \frac{\alpha_4}{\alpha_3}, \tag{2.3.39}$$

where

$$\begin{aligned} \alpha_1 =& 2r_1 r_3(1-r_1)e^{-r_1} + r_3(r_3^2 - r_1)(e^{r_3} + e^{-r_3}) \\ & + r_3^2(1-r_1)(e^{r_3} - e^{-r_3}), \end{aligned} \tag{2.3.40}$$

$$\begin{aligned} \alpha_2 =& 2r_1 r_3(1+r_1)e^{r_1} - r_3(r_3^2 + r_1)(e^{r_3} + e^{-r_3}) \\ & - r_3^2(1+r_1)(e^{r_3} - e^{-r_3}), \end{aligned} \tag{2.3.41}$$

$$\begin{aligned} \alpha_3 =& 2r_1 r_3(1-r_3)e^{-r_3} + r_1(r_1^2 - r_3)(e^{r_1} + e^{-r_1}) \\ & + r_1^2(1-r_3)(e^{r_1} - e^{-r_1}) \end{aligned} \tag{2.3.42}$$

and where

$$\begin{aligned} \alpha_4 =& 2r_1 r_3(1+r_3)e^{r_3} - r_1(r_1^2 + r_3)(e^{r_1} + e^{-r_1}) \\ & - r_1^2(1+r_3)(e^{r_1} - e^{-r_1}). \end{aligned} \tag{2.3.43}$$

First, we study the behavior of α_1. Inserting (2.3.13) and (2.3.14) (with

$\mu = \mu_k$) in (2.3.40) we find after some computations

$$\alpha_1 = -2i\gamma^{3/2}\cos(\sqrt{\gamma}\mu_k)\mu_k^3 + i(1+2\sqrt{\gamma}+2\gamma)\sin(\sqrt{\gamma}\mu_k)\mu_k^2 + O(\mu_k). \tag{2.3.44}$$

Now, inserting (2.3.34) in (2.3.32) and in (2.3.33) we obtain

$$\begin{cases} \cos(\sqrt{\gamma}\mu_k) &= (-1)^k \dfrac{F_1\sqrt{\gamma}}{F_0 k\pi} + O(\dfrac{1}{k^2}), \\ \sin(\sqrt{\gamma}\mu_k) &= (-1)^k + O(\dfrac{1}{k^2}), \end{cases} \tag{2.3.45}$$

where F_0 and F_1 are given by (2.3.12). Inserting (2.3.10) and (2.3.45) in (2.3.44), we find again after some computations

$$\begin{aligned} \alpha_1 &= -i(-1)^k \frac{\left(2F_1\gamma^{3/2} + F_0\left(-1-2\sqrt{\gamma}+2\gamma\right)\right)\pi^2}{F_0\gamma} k^2 + O(k) \\ &= -2i(-1)^k \frac{\pi^2\left(\tanh\left(\frac{1}{\sqrt{\gamma}}\right)-1\right)}{\sqrt{\gamma}} k^2 + O(k). \end{aligned} \tag{2.3.46}$$

Similarly, long computations left to the reader yields

$$\alpha_2 = 2i(-1)^k \frac{\pi^2\left(\tanh\left(\frac{1}{\sqrt{\gamma}}\right)+1\right)}{\sqrt{\gamma}} k^2 + O(k), \tag{2.3.47}$$

$$\alpha_3 = -2i(-1)^k \frac{\pi^2}{\sqrt{\gamma}} k^2 + O(k) \tag{2.3.48}$$

and

$$\alpha_4 = -2i(-1)^k \frac{\pi^2}{\sqrt{\gamma}} k^2 + O(k). \tag{2.3.49}$$

Remark that $\alpha_3 \neq 0$ provided we have chosen k_0 large enough; for this reason our choice $c_3 = 1$ is valid. Substituting (2.3.46)-(2.3.49) into (2.3.39), we obtain

$$\begin{cases} c_1 &= \tanh(\dfrac{1}{\sqrt{\gamma}}) - 1 + O(\dfrac{1}{k}), \\ c_2 &= -\tanh(\dfrac{1}{\sqrt{\gamma}}) - 1 + O(\dfrac{1}{k}), \\ c_3 &= 1, \\ c_4 &= 1 + O(\dfrac{1}{k}). \end{cases} \tag{2.3.50}$$

Finally, we have found that a solution (2.3.7) has the form

$$C(\mu_k) = C_0 + O(\frac{1}{|\mu_k|}), \tag{2.3.51}$$

where

$$C_0 = (-1 + \tanh(\frac{1}{\sqrt{\gamma}}),\ -1 - \tanh(\frac{1}{\sqrt{\gamma}}),\ 1,\ 1).$$

Note that the corresponding solution y_k of (2.3.4) is given by (2.3.5). From equation (2.3.5), we have

$$y_{k,x}(1) = r_1 c_1 e^{r_1} + r_2 c_2 e^{r_2} + r_3 c_3 e^{r_3} + r_4 c_4 e^{r_4}, \tag{2.3.52}$$

where we recall that for $i = 1, ..., 4, r_i = r_i(\mu_k)$ are given by (2.3.6) and c_i for $i = 1, ..., 4$, satisfy (2.3.50). Therefore using the series expansion (2.3.10), (2.3.13), (2.3.14) and (2.3.50) we easily find

$$y_{k,x}(1) = -(-1)^k 2k\pi + O(1) \neq 0. \tag{2.3.53}$$

Step 2. Estimates of $\|y_k\|_W$ and $\|y_k\|_V$. We start with

$$\begin{aligned} \|y_k\|_W^2 &= \int_0^1 |y_{k,xx}|^2 dx = \sum_{i=1}^{4}\sum_{j=1}^{4} c_i r_i^2 \left(\int_0^1 e^{r_i x}\overline{e^{r_j x}} dx\right) \overline{c_j r_j^2} \\ &= C_k G_k \overline{C_k}^T, \end{aligned} \tag{2.3.54}$$

where

$$G_k = \left(\int_0^1 e^{(r_i + \overline{r_j})x} dx\right)_{1 \le i,j \le 4} \quad \text{and} \quad C_k = (c_i r_i^2)_{i=1,\dots 4}.$$

First, since $r_2 = -r_1 \in \mathbb{R}$ (for $|k|$ large enough) and $r_3 = -r_4 \in i\mathbb{R}$, we directly find

$$\begin{aligned} \int_0^1 e^{(r_1+\overline{r_2})x} dx &= \int_0^1 e^{(r_2+\overline{r_1})x} dx \\ &= \int_0^1 e^{(r_3+\overline{r_3})x} dx \\ &= \int_0^1 e^{(r_4+\overline{r_4})x} dx \\ &= 1. \end{aligned} \tag{2.3.55}$$

In addition, using the identity $\int_0^1 e^{rx} dx = \frac{e^r}{r} - \frac{1}{r}$ for $r \neq 0$ and the asymptotic behavior (2.3.13)-(2.3.14) we find that

$$G_k = G_0 + O(\frac{1}{k}), \tag{2.3.56}$$

where

$$G_0 = \begin{pmatrix} \frac{\sqrt{\gamma}}{2}(e^{\frac{2}{\sqrt{\gamma}}}-1) & 1 & 0 & 0 \\ 1 & \frac{\sqrt{\gamma}}{2}(1-e^{-\frac{2}{\sqrt{\gamma}}}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.3.57}$$

and where $O(\frac{1}{k})$ is a matrix where all the entries are of order $\frac{1}{k}$. Next, using (2.3.13), (2.3.14) and (2.3.50), we obtain

$$C_k = (0, 0, -\gamma\mu_k^2, -\gamma\mu_k^2) + O(1). \tag{2.3.58}$$

Finally, inserting (2.3.56) and (2.3.58) in (2.3.54) we deduce that

$$\|y_k\|_W^2 = \gamma^2|\mu_k|^4 + O(|\mu_k|^3) \sim |k|^4, \quad |k| \to \infty. \tag{2.3.59}$$

Similarly, we easily prove that

$$\int_0^1 |y_k|^2 dx \sim 1, \qquad \int_0^1 |y_{k,x}|^2 dx \sim |\mu_k|^2 \sim |k|^2, \quad |k| \to \infty.$$

Therefore, we deduce that

$$\|y_k\|_V \sim |k|, \quad |k| \to \infty. \tag{2.3.60}$$

Moreover, using the estimations (2.3.53), (2.3.59) and (2.3.60) then from (2.3.36) we deduce

$$\|U_k\|_{\mathcal{H}} \sim |k|^2, \quad |k| \to \infty.$$

This completes the proof. ■

2.3.2 Observability inequality and boundedness of the transfer function

First, since $\mathcal{B}$ is a self-ajdoint operator and $\mathcal{B}\mathcal{B}^* = \mathcal{B}$, we rewrite the problem (2.2.16) as follows

$$\begin{cases} U_t(t) + (\mathcal{A}_0 + \alpha \mathcal{B}\mathcal{B}^*)\, U(t) &= \quad 0, \qquad t > 0, \\ \qquad U(0) &= U_0 \in \mathcal{H}. \end{cases} \tag{2.3.61}$$

We will establish an observability inequality for the undamped problem corresponding to (2.3.61) by the following lemma:

Lemma 2.3.5. *Let $\gamma > 0$. There exist $T > 0$ and $C_T > 0$ such that the solution U of the problem*

$$\begin{cases} U_t(t) + \mathcal{A}_0 U(t) &= \quad 0, \qquad t > 0, \\ \qquad U(0) &= U_0 \in \mathcal{H}, \end{cases} \tag{2.3.62}$$

satisfies the following observability inequality:

$$\int_0^T \|\mathcal{B}^* U(t)\|_{\mathcal{H}}^2 dt \geq C_T \|U_0\|^2_{(D(\mathcal{A}_0))'}, \tag{2.3.63}$$

where $(D(\mathcal{A}_0))'$ is the dual of $D(\mathcal{A}_0)$ with respect to the scalar product in $\mathcal{H}$.

Proof: Let $U_0 \in D(\mathcal{A}_0)$, then we can write

$$U_0 = \sum_{j \in J_0} U_0^j \widetilde{\Phi}_j + \sum_{|k| \geq k_0} U_0^k \widetilde{U}_k, \tag{2.3.64}$$

where $\left\{\widetilde{\Phi}_j\right\}_{j \in J_0} \bigcup \left\{\widetilde{U}_k\right\}_{\substack{k \in \mathbb{Z} \\ |k| \geq K_0}}$ denotes the set of normalized eigenvectors of $\mathcal{A}_0$ such that

$$\widetilde{\Phi}_j = (\widetilde{y}_j, \widetilde{z}_j, \widetilde{\eta}_j) = \frac{1}{\|\Phi_j\|_{\mathcal{H}}} \Phi_j, \quad \forall j \in J_0 \tag{2.3.65}$$

and

$$\widetilde{U}_k = (\widetilde{y}_k,\, \widetilde{z}_k,\, \widetilde{\eta}_k) = \frac{1}{\|U_k\|_{\mathcal{H}}} U_k, \quad \forall |k| \geq k_0. \tag{2.3.66}$$

From (2.3.64) we obtain

$$U(t) = \sum_{j \in J_0} U_0^j e^{i\kappa_j t} \widetilde{\Phi}_j + \sum_{|k| \geq k_0} U_0^k e^{i\mu_k t} \widetilde{U}_k. \tag{2.3.67}$$

Consequently, we have

$$\eta(t) = y_x(1,t) = -\sum_{j\in J_0} U_0^j e^{i\kappa_j t}\widetilde{y}_{j,x}(1) - \sum_{|k|\geq k_0} U_0^k e^{i\mu_k t}\widetilde{y}_{k,x}(1), \quad \forall t>0.$$

The spectral gap is satisfied by the eigenvalues of $\mathcal{A}_0$ because they are simple and for k large enough, we have $\mu_{k+1}-\mu_k \geq \dfrac{\pi}{4\sqrt{\gamma}}$, in other words, there exists $d>0$, such that

$$\min_{\substack{\lambda,\lambda'\in\sigma(\mathcal{A}_0)\\ \lambda\neq\lambda'}} |\lambda-\lambda'| \geq d > 0.$$

Thus, using Ingham's inequality (see [48]), we deduce that there exist $T>0$ and $c_T>0$ such that

$$\begin{aligned}\int_0^T \|B^*U(t)\|_{\mathcal{H}}^2 dt &= \int_0^T |\eta(t)|^2 dt \\ &= \int_0^T |y_x(1,t)|^2 dt \\ &\geq c_T\left(\sum_{j\in J_0} |U_0^j|^2 |\widetilde{y}_{j,x}(1)|^2 + \sum_{|k|\geq k_0} |U_0^k|^2|\widetilde{y}_{k,x}(1)|^2\right).\end{aligned} \tag{2.3.68}$$

On the other hand, using (2.3.37)-(2.3.38) and (2.3.66) we get

$$\sum_{|k|\geq k_0} |U_0^k|^2|\widetilde{y}_{k,x}(1)|^2 \sim \sum_{|k|\geq k_0} \frac{|U_0^k|^2}{|k|^2}. \tag{2.3.69}$$

Therefore, we deduce from (2.3.68), Proposition 2.3.2 and equation (2.3.10) that

$$\begin{aligned}\int_0^T \|\mathcal{B}^*U(t)\|_{\mathcal{H}}^2 dt &\geq c_T\left(\sum_{j\in J_0} |U_0^j|^2|\widetilde{y}_{j,x}(1)|^2 + \sum_{|k|\geq k_0} |U_0^k|^2 |\frac{1}{|k|^2}\right)\\ &\sim c_T\|U_0\|_{D(\mathcal{A}_0)'}^2.\end{aligned}$$

The proof of lemma is completed. ∎

Next, we introduce the transfer function H by

$$H:\mathbb{C}_+ = \{\lambda\in\mathbb{C};\ \Re(\lambda)>0\} \longrightarrow \mathcal{L}(\mathbb{C}) \tag{2.3.70}$$

$$\lambda \longrightarrow H(\lambda) = -\alpha\mathcal{B}^*(\lambda+\mathcal{A}_0)^{-1}\mathcal{B}.$$

Let $\omega>0$, we define the set $C_\omega = \{\lambda\in\mathbb{C};\ \Re(\lambda)=\omega\}$.

Lemma 2.3.6. *(Boundedness of H on C_ω)*
The transfer function H defined in (2.3.70) is bounded on C_ω.

Proof: First, since $\mathcal{A}_0$ generate a C_0-semigroup of contractions, we deduce (see Corollary I.3.6 in [75]) that there exists $c_\omega > 0$ such that

$$\|(\lambda + \mathcal{A}_0)^{-1}\|_{\mathcal{H}} \leq c_\omega, \quad \forall \lambda \in C_\omega.$$

Next, combining this estimate with the boundedness of the operators $\mathcal{B}$ and $\mathcal{B}^*$, we deduce the boundedness of the function $\mathcal{H}$ on C_ω.

■

Proof of the Theorem 2.3.1. The polynomial energy estimate (2.3.1) is obtained by application of Theorem 2.4 in [12] on the first order problem with $Y_1 = D(\mathcal{A}_0)$, $X_1 = (D(\mathcal{A}_0))'$ and $\theta = \dfrac{1}{2}$.

■

2.4 Optimal polynomial decay rate

The aim of this section is to prove the following optimality result:

Theorem 2.4.1. *(Optimal decay rate)*
The energy decay rate (2.3.1) is optimal in the sense that for any $\epsilon > 0$, we can not expect the decay rate $\dfrac{1}{t^{1+\epsilon}}$ for all initial data $U_0 \in D(\mathcal{A}_0)$.

■

To prove this theorem, we need the asymptotic behavior of the eigenvalues of the operator $\mathcal{A}_\alpha$. Let $\lambda \neq \alpha$ be an eigenvalue of $\mathcal{A}_\alpha$ and $U = (y,\, z,\, \eta)$ be an associated eigenfunction, then we obtain $\mathcal{A}_\alpha U = \lambda U$. Equivalently, we have the following system:

$$\begin{cases} y_{xxxx} - \gamma\lambda^2 y_{xx} + \lambda^2 y &= 0, \\ y(0) = y_x(0) &= 0, \\ y_{xxx}(1) - \gamma\lambda^2 y_x(1) &= 0, \\ y_{xx}(1) + \dfrac{\lambda}{\lambda - \alpha} y_x(1) &= 0. \end{cases} \tag{2.4.1}$$

The general solution of the first equation of (2.4.1) is given by

$$y(x) = \sum_{i=1}^{4} \tilde{c}_i e^{R_i(\lambda)x}, \tag{2.4.2}$$

where

$$R_1(\lambda) = \sqrt{\frac{\gamma\lambda^2 - \lambda\sqrt{\gamma^2\lambda^2 - 4}}{2}},\ R_2(\lambda) = -R_1(\lambda), \tag{2.4.3}$$

$$R_3(\lambda) = \sqrt{\frac{\gamma\lambda^2 + \lambda\sqrt{\gamma^2\lambda^2 - 4}}{2}},\ R_4(\lambda) = -R_3(\lambda).$$

Here and below, for simplicity we denote $R_i(\lambda)$ by R_i. Thus the boundary conditions in (2.4.1) may be written as the following system:

$$N(\lambda)\widetilde{C}(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ R_1 & R_2 & R_3 & R_4 \\ \widetilde{g}_1(\lambda) & \widetilde{g}_2(\lambda) & \widetilde{g}_3(\lambda) & \widetilde{g}_4(\lambda) \\ \widetilde{h}_1(\lambda) & \widetilde{h}_2(\lambda) & \widetilde{h}_3(\lambda) & \widetilde{h}_4(\lambda) \end{pmatrix} \begin{pmatrix} \widetilde{c}_1 \\ \widetilde{c}_2 \\ \widetilde{c}_3 \\ \widetilde{c}_4 \end{pmatrix} = 0, \tag{2.4.4}$$

where we have set $\widetilde{g}_i(\lambda) = R_i\left(R_i^2 - \gamma\lambda^2\right)e^{R_i}$ and $\widetilde{h}_i(\lambda) = R_i\left(R_i + \frac{\lambda}{\lambda-\alpha}\right)e^{R_i}$ for $i = 1, .., 4$. Since $\mathcal{A}_\alpha$ is closed with a compact resolvent, its spectrum consists entirely of isolated eigenvalues with finite multiplicities. Further as the coefficients of $\mathcal{A}_\alpha$ are real, the eigenvalues appear by conjugate pairs.

Proposition 2.4.2. *There exists a positive constant c such that any eigenvalue λ of $\mathcal{A}_\alpha$ satisfies*

$$0 < \Re(\lambda) \leq c.$$

Proof: Obviously, we already know that the real part of any eigenvalue of $\mathcal{A}_\alpha$ is positive, so we only have to prove that it is upper bounded.
Let $\lambda \neq \alpha$ be an eigenvalue of $\mathcal{A}_\alpha$ and $U = (y, -\lambda y, y_x(1))$ an associated eigenvector such that $\|U\|_{\mathcal{H}} = 1$. Multiplying the first equation of the system (2.4.1) by $\overline{y}$ and integrating by parts yields

$$\|y\|_W^2 + \lambda^2\|y\|_V^2 + \frac{\lambda}{\lambda - \alpha}|y_x(1)|^2 = 0. \tag{2.4.5}$$

Next, set $\lambda = u + iv$, $u \in \mathbb{R}_+^*$ and $v \in \mathbb{R}$. A straightforward computation gives

$$\frac{\lambda}{\lambda - \alpha} = \frac{u(u-\alpha) + v^2}{(u-\alpha)^2 + v^2} + i\frac{\alpha v}{(u-\alpha)^2 + v^2}, \tag{2.4.6}$$

then the imaginary part of the equation (2.4.5) gives

$$\left(2u\|y\|_V^2 - \frac{\alpha}{(u-\alpha)^2+v^2}|y_x(1)|^2\right)v = 0. \tag{2.4.7}$$

Assume that $v \neq 0$ then

$$|\lambda|^2\|y\|_V^2 = (u^2+v^2)\|y\|_V^2 = \frac{\alpha}{2u}\frac{u^2+v^2}{(u-\alpha)^2+v^2}|y_x(1)|^2.$$

If $u = \Re(\lambda)$ is not bounded and since $|y_x(1)|^2 \le \|U\|_{\mathcal{H}}^2 = 1$, it follows from the previous identity that for u large

$$|\lambda|^2\|y\|_V^2 = O(\frac{1}{u}).$$

Consequently (2.4.5) implies

$$\|y\|_W^2 + |y_x(1)|^2 = O(\frac{1}{u}),$$

then

$$\|U\|_{\mathcal{H}}^2 = \|y\|_W^2 + |\lambda|^2\|y\|_V^2 + |y_x(1)|^2 = O(\frac{1}{u}),$$

which is not possible. Therefore, for u large enough, we deduce from (2.4.7) that $\Im(\lambda) = v = 0$. Finally, taking the real part of the equation (2.4.5) with $v = 0$, we obtain

$$\|y\|_W^2 + u^2\|y\|_V^2 + \frac{u}{u-\alpha}|y_x(1)|^2 = 0.$$

Hence the contradiction with $\|U\|_{\mathcal{H}}^2 = 1$ if u is large enough. ■

In the following proposition we study the spectrum of $\mathcal{A}_\alpha$:

Proposition 2.4.3. *(Spectrum of $\mathcal{A}_\alpha$)*
There exists $k_1 \in \mathbb{N}^*$ sufficiently large such that the spectrum $\sigma(\mathcal{A}_\alpha)$ of $\mathcal{A}_\alpha$ is given by:

$$\sigma(\mathcal{A}_\alpha) = \widetilde{\sigma}_0 \cup \widetilde{\sigma}_1, \tag{2.4.8}$$

where

$$\widetilde{\sigma}_0 = \{\kappa_j\}_{j\in J}, \quad \widetilde{\sigma}_1 = \{\lambda_k\}_{\substack{k\in\mathbb{Z}\\ |k|\ge k_0}}, \quad \widetilde{\sigma}_0 \cap \widetilde{\sigma}_1 = \emptyset \tag{2.4.9}$$

and J is a finite set. Moreover, λ_k is simple and satisfies the following *asymptotic behavior*

$$\lambda_k = i\left(\frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} + \frac{D}{k} + \frac{E}{k^2}\right) + \frac{\alpha}{\pi^2 k^2} + o(\frac{1}{k^2}), \tag{2.4.10}$$

where

$$D = \frac{2\gamma - 1 - 2\sqrt{\gamma}\tanh(\gamma^{-\frac{1}{2}})}{2\gamma^{\frac{3}{2}}\pi} \tag{2.4.11}$$

and

$$E = \frac{2(-1)^k}{\gamma^{\frac{3}{2}}\cosh(\gamma^{-\frac{1}{2}})\pi^2} + \frac{4\gamma - 2 - \sqrt{\gamma}\tanh(\gamma^{-\frac{1}{2}})}{2\gamma^{\frac{3}{2}}\pi}. \tag{2.4.12}$$

Proof: The proof is divided into three steps. Step 1 furnishes an asymptotic development of the characteristic equation for large λ. Step 2 uses Rouché's theorem to localize high frequency eigenvalues. In step 3, we perform a limited development stopped when a non zero real part appear.

Step 1. First, we start by the expansion of R_1 and R_3 when $|\lambda| \to \infty$

$$R_1 = \frac{1}{\sqrt{\gamma}} + \frac{1}{2\gamma^{\frac{5}{2}}\lambda^2} + O(\frac{1}{\lambda^4}) \tag{2.4.13}$$

and

$$R_3 = \lambda\sqrt{\gamma} - \frac{1}{2\lambda\gamma^{\frac{3}{2}}} + O(\frac{1}{\lambda^3}). \tag{2.4.14}$$

Next, using (2.4.13) and (2.4.14), we find the asymptotic behavior of

$$\widetilde{g}_1(\lambda) = \left(-\sqrt{\gamma}\lambda^2 - \frac{1}{2\gamma^2} + \frac{1}{2\gamma^{\frac{3}{2}}}\right) e^{\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda}), \tag{2.4.15}$$

$$\widetilde{g}_2(\lambda) = \left(\sqrt{\gamma}\lambda^2 - \frac{1}{2\gamma^2} - \frac{1}{2\gamma^{\frac{3}{2}}}\right) e^{-\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda}), \tag{2.4.16}$$

$$\widetilde{g}_3(\lambda) = \left(-\frac{\lambda}{\sqrt{\gamma}} + \frac{1}{2\gamma^2}\right) e^{\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda}) \tag{2.4.17}$$

and

$$\widetilde{g}_4(\lambda) = \left(\frac{\lambda}{\sqrt{\gamma}} + \frac{1}{2\gamma^2}\right) e^{-\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda}). \tag{2.4.18}$$

Similarly, we get

$$\widetilde{h}_1(\lambda) = \frac{1}{\sqrt{\gamma}}\left(1+\frac{1}{\sqrt{\gamma}}+\frac{\alpha}{\lambda}\right)e^{\frac{1}{\sqrt{\gamma}}}+O(\frac{1}{\lambda^2}), \tag{2.4.19}$$

$$\widetilde{h}_2(\lambda) = \frac{1}{\sqrt{\gamma}}\left(-1+\frac{1}{\sqrt{\gamma}}-\frac{\alpha}{\lambda}\right)e^{-\frac{1}{\sqrt{\gamma}}}+O(\frac{1}{\lambda^2}), \tag{2.4.20}$$

$$\widetilde{h}_3(\lambda) = \left(\gamma\lambda^2+(-\frac{1}{2\sqrt{\gamma}}+\sqrt{\gamma})\lambda+\frac{1-12\gamma+8\gamma^2\sqrt{\gamma}\alpha}{8\gamma^2}\right)e^{\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda}) \tag{2.4.21}$$

and

$$\widetilde{h}_4(\lambda) = \left(\gamma\lambda^2+(\frac{1}{2\sqrt{\gamma}}-\sqrt{\gamma})\lambda+\frac{1-12\gamma-8\gamma^2\sqrt{\gamma}\alpha}{8\gamma^2}\right)e^{-\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda}). \tag{2.4.22}$$

Combining (2.4.13)-(2.4.22) and (2.4.4), we can write the system (2.4.4) as follows:

$$N(\lambda)\widetilde{C}(\lambda)=0,$$

where $N(\lambda)$ is given by

$$N(\lambda)=\begin{pmatrix} 1 & 1 & 1 & 1 \\ \widetilde{P}_1+O(\frac{1}{\lambda^2}) & \widetilde{P}_2+O(\frac{1}{\lambda^2}) & \widetilde{P}_3+O(\frac{1}{\lambda^4}) & \widetilde{P}_4+O(\frac{1}{\lambda^4}) \\ e^{\frac{1}{\sqrt{\gamma}}}\widetilde{P}_5+O(\frac{1}{\lambda}) & e^{-\frac{1}{\sqrt{\gamma}}}\widetilde{P}_6+O(\frac{1}{\lambda}) & e^{\sqrt{\gamma}\lambda}\widetilde{P}_7+O(\frac{1}{\lambda^2}) & e^{-\sqrt{\gamma}\lambda}\widetilde{P}_8+O(\frac{1}{\lambda^2}) \\ e^{\frac{1}{\sqrt{\gamma}}}\widetilde{P}_9+O(\frac{1}{\lambda}) & e^{-\frac{1}{\sqrt{\gamma}}}\widetilde{P}_{10}+O(\frac{1}{\lambda}) & e^{\sqrt{\gamma}\lambda}\widetilde{P}_{11}+O(\frac{1}{\lambda^2}) & e^{-\sqrt{\gamma}\lambda}\widetilde{P}_{12}+O(\frac{1}{\lambda^2}) \end{pmatrix},$$

with

$$\widetilde{P}_1=\frac{1}{\sqrt{\gamma}}+\frac{1}{2\gamma^{\frac{5}{2}}\lambda^2},\quad \widetilde{P}_2=-\frac{1}{\sqrt{\gamma}}-\frac{1}{2\gamma^{\frac{5}{2}}\lambda^2},\quad \widetilde{P}_3=\lambda\sqrt{\gamma}-\frac{1}{2\lambda\gamma^{\frac{3}{2}}},$$

$$\widetilde{P}_4=-\lambda\sqrt{\gamma}+\frac{1}{2\lambda\gamma^{\frac{3}{2}}},\quad \widetilde{P}_5=-\sqrt{\gamma}\lambda^2-\frac{1}{2\gamma^2}+\frac{1}{2\gamma^{\frac{3}{2}}}\quad \widetilde{P}_6=\sqrt{\gamma}\lambda^2-\frac{1}{2\gamma^2}-\frac{1}{2\gamma^{\frac{3}{2}}},$$

$$\widetilde{P}_7=-\frac{\lambda}{\sqrt{\gamma}}+\frac{1}{2\gamma^2},\quad \widetilde{P}_8=\frac{\lambda}{\sqrt{\gamma}}+\frac{1}{2\gamma^2},\quad \widetilde{P}_9=\frac{1}{\sqrt{\gamma}}\left(1+\frac{1}{\sqrt{\gamma}}+\frac{\alpha}{\lambda}\right),$$

$$\widetilde{P}_{10} = \frac{1}{\sqrt{\gamma}}\left(-1 + \frac{1}{\sqrt{\gamma}} - \frac{\alpha}{\lambda}\right),$$

$$\widetilde{P}_{11} = \gamma\lambda^2 + \left(-\frac{1}{2\sqrt{\gamma}} + \sqrt{\gamma}\right)\lambda + \frac{1}{8\gamma^2} - \frac{3}{2\gamma} + \sqrt{\gamma}\alpha$$

and

$$\widetilde{P}_{12} = \gamma\lambda^2 + \left(\frac{1}{2\sqrt{\gamma}} - \sqrt{\gamma}\right)\lambda + \frac{1}{8\gamma^2} - \frac{3}{2\gamma} - \sqrt{\gamma}\alpha.$$

Then, after some computations, we find the following asymptotic development of $\widetilde{f}(\lambda) = \det N(\lambda)$

$$\widetilde{f}(\lambda) = \lambda^5 \widetilde{f}_0(\lambda) + \lambda^4 \widetilde{f}_1(\lambda) + \lambda^3 \widetilde{f}_2(\lambda) + O(\lambda^2), \tag{2.4.23}$$

where

$$\widetilde{f}_0(\lambda) = -4\gamma^2 \cosh(\frac{1}{\sqrt{\gamma}}) \cosh(\sqrt{\gamma}\lambda), \tag{2.4.24}$$

$$\widetilde{f}_1(\lambda) = l_1(\gamma) \sinh(\sqrt{\gamma}\lambda), \tag{2.4.25}$$

with

$$l_1(\gamma) = 2\sqrt{\gamma}\left((1-2\gamma)\cosh(\frac{1}{\sqrt{\gamma}}) + 2\sqrt{\gamma}\sinh(\frac{1}{\sqrt{\gamma}})\right) \tag{2.4.26}$$

and where

$$\widetilde{f}_2(\lambda) = 8 - 4\alpha\gamma^{\frac{3}{2}} \cosh(\frac{1}{\sqrt{\gamma}}) \cosh(\sqrt{\gamma}\lambda) + l_2(\gamma) \sinh(\sqrt{\gamma}\lambda), \tag{2.4.27}$$

with

$$l_2(\gamma) = (10 - \frac{1}{2\gamma}) \cosh(\frac{1}{\sqrt{\gamma}}) + (4\sqrt{\gamma} - \frac{4}{\sqrt{\gamma}}) \sinh(\frac{1}{\sqrt{\gamma}}). \tag{2.4.28}$$

As the real part of λ is bounded, then the functions $\widetilde{f}_i$ are bounded for $i \in \{0, 1, 2\}$. For convenience, we set

$$\widetilde{S}(\lambda) = \frac{\widetilde{f}(\lambda)}{\lambda^5} = \widetilde{f}_0(\lambda) + \frac{\widetilde{f}_1(\lambda)}{\lambda} + \frac{\widetilde{f}_2(\lambda)}{\lambda^2} + O(\frac{1}{\lambda^3}). \tag{2.4.29}$$

Step 2. We look at the roots of $\widetilde{S}$. It is easy to see that the roots of $\widetilde{f}_0$ are simple and given by

$$z_k = i\alpha_k, \quad k \in \mathbb{Z}, \tag{2.4.30}$$

where α_k is defined in (2.3.11). Then, with the help of Rouché's theorem there exists k_1 large enough such that for all $|k| \geq k_1$, the large eigenvalues of $\sigma(\mathcal{A}_\alpha)$ (denoted by λ_k) are simple and close to z_k. More precisely, there exists $k_1 \in \mathbb{N}^*$ large enough, such that the splitting of $\sigma(\mathcal{A}_\alpha)$ given by (2.4.8)-(2.4.9) holds and we have

$$\lambda_k = i\alpha_k + o(1), \quad |k| \to \infty. \tag{2.4.31}$$

Equivalently, we can write

$$\lambda_k = i\alpha_k + \epsilon_k, \quad \lim_{|k|\to\infty} \epsilon_k = 0. \tag{2.4.32}$$

Step 3. Asymptotic behavior of ϵ_k. First, using (2.4.29) and the identities (2.4.24)-(2.4.27) we have

$$\begin{aligned}
0 = \widetilde{S}(\lambda_k) =& \widetilde{f}_0(\lambda_k) + \frac{\widetilde{f}_1(\lambda_k)}{\lambda_k} + \frac{\widetilde{f}_2(\lambda_k)}{\lambda_k^2} + O(\frac{1}{\lambda_k^3}) \\
=& -4\gamma^2 \cosh(\frac{1}{\sqrt{\gamma}}) \cosh(\sqrt{\gamma}\lambda_k) + \frac{l_1(\gamma)\sinh(\sqrt{\gamma}\lambda_k)}{\lambda_k} \\
&+ \frac{8}{\lambda_k^2} - \frac{4\alpha\gamma^{\frac{3}{2}} \cosh(\frac{1}{\sqrt{\gamma}}) \cosh(\sqrt{\gamma}\lambda_k)}{\lambda_k^2} \\
&+ \frac{l_2(\gamma)\sinh(\sqrt{\gamma}\lambda_k)}{\lambda_k^2} + O(\frac{1}{\lambda_k^3}).
\end{aligned} \tag{2.4.33}$$

On the other hand, using (2.4.32) we find

$$\cosh(\sqrt{\gamma}\lambda_k) = i(-1)^k \sqrt{\gamma}\epsilon_k + O(\epsilon_k^3) \tag{2.4.34}$$

and

$$\sinh(\sqrt{\gamma}\lambda_k) = i(-1)^k + O(\epsilon_k^2). \tag{2.4.35}$$

Then, substituting (2.4.34) into (2.4.24) and (2.4.35) into (2.4.25) with $\lambda = \lambda_k$ yields

$$\widetilde{f}_0(\lambda_k) = -4i\gamma^{\frac{5}{2}}(-1)^k \cosh(\frac{1}{\sqrt{\gamma}})\epsilon_k + O(\epsilon_k^3) \tag{2.4.36}$$

and

$$\widetilde{f}_1(\lambda_k) = i(-1)^k l_1(\gamma) + O(\epsilon_k^2). \tag{2.4.37}$$

Similarly, we get

$$\widetilde{f}_2(\lambda_k) = 8 - 4i\alpha\gamma^{\frac{3}{2}}\cosh(\frac{1}{\sqrt{\gamma}}) + \sqrt{\gamma}i(-1)^k l_2(\gamma)\epsilon_k + O(\epsilon_k^3). \qquad (2.4.38)$$

Now, using (2.4.32), (2.4.37) and (2.4.38) we get

$$\frac{\widetilde{f}_1(\lambda_k)}{\lambda_k} = \frac{(-1)^k l_1(\gamma)}{\alpha_k} + O(\frac{\epsilon_k}{k}) \qquad (2.4.39)$$

and

$$\frac{\widetilde{f}_2(\lambda_k)}{\lambda_k^2} = -\frac{8}{\alpha_k^2} + \frac{4\alpha i\gamma^{\frac{3}{2}}\cosh(\frac{1}{\sqrt{\gamma}})(-1)^k}{\alpha_k^2} + O(\frac{\epsilon_k}{k}). \qquad (2.4.40)$$

Next, substituting (2.4.36), (2.4.39) and (2.4.40) into (2.4.33) yields

$$\begin{aligned} 0 &= -4i\gamma^{\frac{5}{2}}(-1)^k\cosh(\frac{1}{\sqrt{\gamma}})\epsilon_k + \frac{(-1)^k l_1(\gamma)}{\alpha_k} - \frac{8}{\alpha_k^2} \qquad (2.4.41)\\ &\quad + \frac{4\alpha i\gamma^{\frac{3}{2}}\cosh(\frac{1}{\sqrt{\gamma}})(-1)^k}{\alpha_k^2} + O(\frac{\epsilon_k}{k}). \end{aligned}$$

Therefore

$$\epsilon_k = -\frac{\frac{8}{\alpha_k^2} - \frac{(-1)^k l_1(\gamma)}{\alpha_k}}{4i\gamma^{\frac{5}{2}}(-1)^k\cosh(\frac{1}{\sqrt{\gamma}})} + \frac{\alpha}{\gamma\alpha_k^2} + O(\frac{\epsilon_k}{k}). \qquad (2.4.42)$$

Moreover, substituting (2.3.11) and (2.4.26) into (2.4.42) then a long computation gives

$$\epsilon_k = i\left(\frac{D}{k} + \frac{E}{k^2}\right) + \frac{\alpha}{\pi^2 k^2} + o(\frac{1}{k^2}) \qquad (2.4.43)$$

where D and E are given by (2.4.11)-(2.4.12). Finally, substituting (2.4.43) into (2.4.32), we directly get (2.4.10). ■

Numerical validation. The asymptotic behavior of λ_k in (2.4.10) can be numerically validated. For instance, with $\alpha = 1$ and $\gamma = 2$ then from (2.4.10) we have

$$\lim_{k\to+\infty} k^2\Re(\lambda_k) = \frac{1}{\pi^2}(\approx 0.101321).$$

The table below confirms this behavior.

k	100	150	200	250	300	350	400	450	500
$k^2\Re(\lambda_k)$	0.100312	0.100647	0.100816	0.100917	0.100984	0.101032	0.101068	0.101096	0.101119

In addition, figure 2.1 represents some eigenvalues in this case. Note

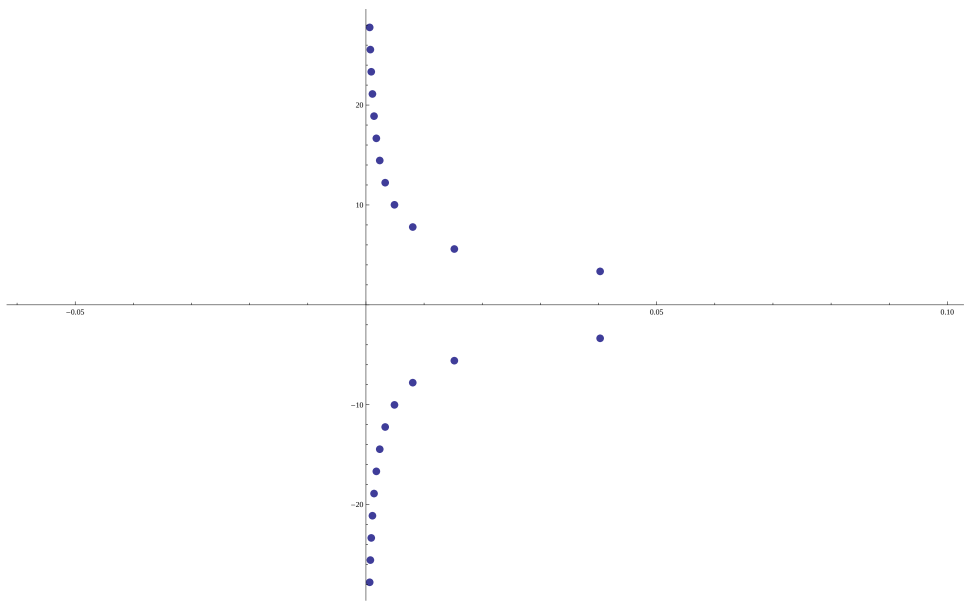

Figure 2.1: Eigenvalues of $\mathcal{A}_1$ with $\gamma = 2$

that for a scale reason three eigenvalues (with a small imaginary part) do not appear in the previous figure. Their approximated value are

$$0.13825 \pm i1.30223, \qquad \text{and} \quad 0.54640.$$

Proof of Theorem 2.4.1. Let $\epsilon > 0$ and set $l = \dfrac{\epsilon}{1+\epsilon}$. First, for $|k| \geq k_1$, let λ_k be an eigenvalue of the operator $\mathcal{A}_\alpha$ and $U_k \in D(\mathcal{A}_0)$ the associated normalized eigenfunction. Moreover, we introduce the following sequence

$$\beta_k = -\Im(\lambda_k), \quad |k| \geq k_1.$$

Next, using (2.4.10), we have

$$(iI\beta_k + \mathcal{A}_\alpha)U_k = (iI\beta_k + \lambda_k)U_k = \left(\frac{\alpha}{\pi^2 k^2} + o(\frac{1}{k^2})\right) U_k, \quad \forall |k| \geq k_1.$$

Therefore

$$\beta_k^{2-2l}\|(i\beta_k I+\mathcal{A}_\alpha)U_k\|_{\mathcal{H}} \sim \frac{\alpha}{\pi^2}\times\frac{1}{k^{\frac{2\epsilon}{1+\epsilon}}},\quad \forall |k|\geq k_1.$$

Thus, we deduce

$$\lim_{k\to+\infty}\beta_k^{2-2l}\|(i\beta_k I+\mathcal{A}_\alpha)U_k\|_{\mathcal{H}}=0.$$

Finally, thanks to Theorem 2.4 in [20], we deduce that the trajectory $e^{-t\mathcal{A}_\alpha}U_0$ decays slower that $\dfrac{1}{t^{\frac{1}{2-2l}}}$ on the time $t\to+\infty$. Then we cannot expect the energy decay rate $\dfrac{1}{t^{1+\epsilon}}$. ■

2.5 Open problems

The extension of the results of this chapter to space dimensions greater than or equal to 2 are widely open problems. We maybe apply a multiplier method, but we can not show the optimality using the spectrum study, since it is difficult to analyze it in the multidimensional problem. The exact controllability of Rayleigh beam equation and the stabilization of the numerical approximation schemes with static or dynamic boundary control moment, are also and open problems.
Moreover, a Rayleigh beam equation with only one dynamical boundary control force is also an open problem. We deal this case in the next chapter.

3

Rayleigh beam equation with only one dynamical boundary control force

Abstract: In [92], Wehbe considered a Rayleigh beam equation with two dynamical boundary controls and established the optimal polynomial energy decay rate of type $\frac{1}{t}$. The proof exploits in an explicit way the presence of two boundary controls, hence the case of the Rayleigh beam damped by only one dynamical boundary control remained open. In this chapter, we fill this gap by considering a clamped Rayleigh beam equation subject to only one dynamical boundary control force. We use a Riesz basis approach. First, we start by giving the asymptotic expansion of the eigenvalues and the eigenfunctions of the damped and undamped systems. Next, we show that the system of eigenvectors of the damped problem form a Riesz basis. Finally, we deduce the optimal energy decay rate of polynomial type in $\frac{1}{\sqrt{t}}$.

3.1 Introduction

In [92], Wehbe considered a Rayleigh beam clamped at one end and subjected to two dynamical boundary controls at the other end, namely

$$\begin{aligned}
y_{tt} - \gamma y_{xxtt} + y_{xxxx} &= 0, & 0 < x < 1, \quad t > 0, && (3.1.1)\\
y(0,t) = y_x(0,t) &= 0, & t > 0, && (3.1.2)\\
y_{xx}(1,t) + a\eta(t) &= 0, & t > 0, && (3.1.3)\\
y_{xxx}(1,t) - \gamma y_{xtt}(1,t) - b\xi(t) &= 0, & t > 0, && (3.1.4)
\end{aligned}$$

where $\gamma > 0$ is the coefficient of moment of inertia, $a > 0$ and $b > 0$ are constants, η and ξ denote respectively the dynamical boundary control moment and force. The damping of the system is made via the indirect damping mechanism at the right extremity of the beam that involves the following two first order differential equations:

$$\begin{aligned}
\eta_t(t) - y_{xt}(1,t) + \alpha\eta(t) &= 0, & t > 0, && (3.1.5)\\
\xi_t(t) - y_t(1,t) + \beta\xi(t) &= 0, & t > 0, && (3.1.6)
\end{aligned}$$

where $\alpha > 0$ and $\beta > 0$. The notion of indirect damping mechanisms has been introduced by Russell in [85] and since that time, it retains the attention of many authors. The lack of uniform stability was proved by a compact perturbation argument of Gibson [35] and a polynomial energy decay rate of type $\frac{1}{t}$ is obtained by a multiplier method usually used for nonlinear problems. Finally, using a spectral method, he proved that the obtained energy decay is optimal in the sense that for any $\varepsilon > 0$, we cannot expect a decay rate of type $\frac{1}{t^{1+\varepsilon}}$. But in [92] the effect of each control separately on the stability of the Rayleigh beam equation is not investigated. Indeed, the multiplier method exploits in an explicit way the presence of the two boundary controls. Furthermore, the lack of one of this two controls yield this method ineffective. Then, the important and interesting case when the Rayleigh beam equation is damped by only one dynamical boundary control ($a = 0$ and $b > 0$ or $a > 0$ and $b = 0$) remained open. In chapter 2, we have considered a Rayleigh beam equation damped at one end and subjected to one dynamic boundary control moment at the other end, *i.e.* when $a > 0$, $b = 0$ and η solution of (3.1.5). First, we applied a methodology introduced in [12] to establish an energy decay rate of polynomial type $\frac{1}{t}$. Next, using the analysis of the spectrum of our dissipative operator and from a frequency domain approach given in [20], we have proved that the obtained energy decay rate is optimal. In this chapter, we consider the second case, a Rayleigh beam equation subject to only one dynamical boundary control force, *i.e.* when $a = 0$ and $b > 0$. Rao in [79] studied the stabilization of system (3.1.1)-(3.1.4) with $a = 0$, $b > 0$ and $\xi(t) = y_1(1,t)$. He first proved the lack of exponential stability of the system (2.1.1)-(2.1.4). Next, he proved that the Rayleigh beam equation can be strongly stabilized by only one control force if and only if the inertia coefficient γ is large enough, but he did not studied the decay rate of the energy of the system. In [17], Bassam *and al.* studied the decay rate of energy of system (3.1.1)-(3.1.4) with $a = 0$, $b > 0$ and $\xi(t) = y_t(1,t)$. First, using an explicit approximation, they gave the asymptotic expansion of eigenvalues and eigenfunctions of the undamped system corresponding to (3.1.1)-(3.1.4), then they established the optimal polynomial energy decay rate of type $\frac{1}{t}$ via an observability inequality of solution of the undamped system and the boundedness of the transfer function associated with the undamped problem.

In this chapter, we consider the Rayleigh beam equation (3.1.1)-(3.1.4) with only one dynamical boundary control force, *i.e.* when $a = 0$, $b = 1$ and ξ solution of (3.1.6). Here, we prefer to use a Riesz basis approach. First, we give the asymptotic expansion of the eigenvalues and the eigenfunctions of the damped and undamped systems. Next, we show that the system of eigenvectors of high frequencies of the damped problem is quadratically closed to the system of eigenvectors of high frequencies of the undamped one. This yields, from Theorem 1.2.10 given in [2] (see also [43, Theorem 6.3]) that the system of generalized eigenvectors of the damped problem forms a Riesz basis of the energy space. Finally, by applying Theorem 2.1 given in [63], we establish the optimal energy decay rate of polynomial type $\frac{1}{\sqrt{t}}$.

The plan of this chapter is as follows:
In section 3.2 we transform our system into an evolution equation, we deduce the well-posedness property of the problem by the semigroup approach and we recall the condition to reach the strong stability of our

system (see [79]). In section 3.3, we propose an explicit approximation of the characteristic equation determining the eigenvalues of the damped and undamped system. Then, we give an asymptotic expansion of eigenvalues and eigenfunctions of the corresponding operators. In section 3.4, we show that the system of eigenvectors of the damped problem forms a Riesz basis and we establish the optimal polynomial energy decay rate of type $\dfrac{1}{\sqrt{t}}$. In the last section, we give some open problems.

3.2 Well-posedness and strong stability

In this section, we study the existence, uniqueness and the asymptotic behavior of the solution of Rayleigh beam equation with only one dynamical boundary control force:

$$\begin{cases} y_{tt} - \gamma y_{xxtt} + y_{xxxx} & = 0, \quad 0 < x < 1, \quad t > 0, \\ y(0,t) = y_x(0,t) & = 0, \qquad\qquad\qquad t > 0, \\ y_{xx}(1,t) & = 0, \qquad\qquad\qquad t > 0, \\ y_{xxx}(1,t) - \gamma y_{xtt}(1,t) - \xi(t) & = 0, \qquad\qquad\qquad t > 0, \\ \xi_t(t) - y_t(1,t) + \beta\xi(t) & = 0, \qquad\qquad\qquad t > 0. \end{cases} \tag{3.2.1}$$

First, let y and ξ be smooth solutions of system (3.2.1). We define its associated energy by

$$E(t) = \frac{1}{2}\left(\int_0^1 (|y_t|^2 + \gamma|y_{xt}|^2 + |y_{xx}|^2)dx + |\xi(t)|^2\right), \quad t \geq 0. \tag{3.2.2}$$

A direct computation gives

$$\frac{d}{dt}E(t) = -\beta|\xi(t)|^2 \leq 0, \quad t \geq 0.$$

Then the system (3.2.1) is dissipative in the sense that its energy $E(t)$ is a nonincreasing function of the time variable t. We next introduce the following spaces:

$$V = \left\{y \in H^1(0,1);\ y(0) = 0\right\}, \quad \|y\|_V^2 = \int_0^1 (|y|^2 + \gamma|y_x|^2)dx, \tag{3.2.3}$$

$$W = \left\{y \in H^2(0,1);\ y(0) = y_x(0) = 0\right\}, \quad \|y\|_W^2 = \int_0^1 |y_{xx}|^2 dx \tag{3.2.4}$$

and the energy space

$$\mathcal{H} = W \times V \times \mathbb{C}, \tag{3.2.5}$$

endowed with the usual inner product

$$((y_1,\, z_1,\, \eta_1), (y_2,\, z_2,\, \eta_2))_{\mathcal{H}} = (y_1, y_2)_W + (z_1, z_2)_V + \eta_1 \overline{\eta_2},$$

$$\forall (y_1,\, z_1,\, \eta_1), (y_2,\, z_2,\, \eta_2) \in \mathcal{H}.$$

Identify $L^2(0,1)$ with its dual so that we have the following continuous embedding:

$$W \subset V \subset L^2(0,1) \subset V' \subset W'. \tag{3.2.6}$$

Multiplying the first equation of (3.2.1) by $\overline{\Phi} \in W$ and integrating by parts, we transform (3.2.1) into a variational equation:

$$\int_0^1 (y_{tt}\overline{\Phi} + \gamma y_{xtt}\overline{\Phi_x})dx + \int_0^1 y_{xx}\overline{\Phi_{xx}}dx + \xi\overline{\Phi(1)} = 0. \tag{3.2.7}$$

According, we define the following linear operators $A \in \mathcal{L}(W, W')$, $\widetilde{B} \in \mathcal{L}(\mathbb{C}, V')$ and $C \in \mathcal{L}(V, V')$ by:

$$< Ay, \Phi >_{W' \times W} = (y, \Phi)_W, \quad \forall y, \Phi \in W, \tag{3.2.8}$$

$$< \widetilde{B}\xi, \Phi >_{V' \times V} = \xi\overline{\Phi(1)}, \quad \forall \xi \in \mathbb{C}, \forall \Phi \in V \tag{3.2.9}$$

and

$$< Cy, \Phi >_{V' \times V} = (y, \Phi)_V, \quad \forall y, \Phi \in V. \tag{3.2.10}$$

Assume that $Ay \in V'$, then we can formulate the variational equation (3.2.7) as

$$y_{tt} + C^{-1}Ay + C^{-1}\widetilde{B}\xi = 0. \tag{3.2.11}$$

Later, we introduce the linear unbounded operator $\widetilde{\mathcal{A}}_0$ and the linear bounded operator $\widetilde{\mathcal{B}}$ as follows:

$$D(\widetilde{\mathcal{A}}_0) = \{(y, z, \xi) \in \mathcal{H};\, z \in W \text{ and } Ay \in V'\}, \tag{3.2.12}$$

$$\widetilde{\mathcal{A}}_0 U = \begin{pmatrix} -z \\ C^{-1}Ay + C^{-1}\widetilde{B}\xi \\ -z(1) \end{pmatrix}, \qquad U = (y, z, \xi) \in D(\widetilde{\mathcal{A}}_0) \tag{3.2.13}$$

and

$$\widetilde{\mathcal{B}}U = \begin{pmatrix} 0 \\ 0 \\ \xi \end{pmatrix}, \qquad U = (y, z, \xi) \in \mathcal{H}. \tag{3.2.14}$$

Then, denoting by $U = (y,\, y_t,\, \xi)$ the state of system (3.2.1) and define $\widetilde{\mathcal{A}}_\beta = \widetilde{\mathcal{A}}_0 + \beta\widetilde{\mathcal{B}}$ with $D(\widetilde{\mathcal{A}}_\beta) = D(\widetilde{\mathcal{A}}_0)$, we can formulate system (3.2.1) into an evolution equation

$$\begin{cases} U_t(t) + \widetilde{\mathcal{A}}_\beta U(t) &= \quad 0, \qquad t > 0, \\ \qquad U(0) &= U_0 \in \mathcal{H}. \end{cases} \tag{3.2.15}$$

It is easy to prove that $-\widetilde{\mathcal{A}}_\beta$ is a maximal dissipative operator in the energy space $\mathcal{H}$, therefore it generates a C_0-semigroup $(e^{-t\widetilde{\mathcal{A}}_\beta})_{t\geq 0}$ of contractions in the energy space $\mathcal{H}$ following Lumer-Phillips' theorem (see Pazy [75]). Thus, we have the following results concerning the existence and uniqueness of the solution of the problem (3.2.15):

Theorem 3.2.1. *For any initial data $U_0 \in \mathcal{H}$, the problem (3.2.15) has a unique weak solution $U(t) = e^{-t\widetilde{\mathcal{A}}_\beta}U_0$ such that $U \in C^0\left([0,\infty[, \mathcal{H}\right)$. Moreover, if $U_0 \in D(\widetilde{\mathcal{A}}_0)$, then the problem (3.2.15) has a strong solution $U(t) = e^{-t\widetilde{\mathcal{A}}_\beta}U_0$ such that $U \in C^1\left([0,\infty[, \mathcal{H}\right) \cap C^0\left([0,\infty[, D(\widetilde{\mathcal{A}}_0)\right)$.* ■

In addition, it is easy to show that an element $U = (y, z, \xi) \in D(\widetilde{\mathcal{A}}_0)$ if and only if $y \in H^3(0,1) \cap W$, $z \in W$ and $y_{xx}(1) = 0$. In particular, the resolvent $(I + \widetilde{\mathcal{A}}_0)^{-1}$ of $-\widetilde{\mathcal{A}}_0$ is compact in the energy space $\mathcal{H}$ (compare with Proposition 2.2.2). This implies with the compactness of $\widetilde{\mathcal{B}}$ that the resolvent $(I + \widetilde{\mathcal{A}}_\beta)^{-1}$ of $-\mathcal{A}_\beta$ is also compact in the Hilbert space $\mathcal{H}$. Consequently, the spectrum of $\widetilde{\mathcal{A}}_\beta$ (respectively $\widetilde{\mathcal{A}}_0$) consists entirely of isolated eigenvalues with finite multiplicities (see [50]). Moreover, since the coefficients of $\widetilde{\mathcal{A}}_\beta$ (respectively $\widetilde{\mathcal{A}}_0$) are real, their eigenvalues appear by conjugate pairs.

Now, we investigate the strong stability of the problem (3.2.15). Theorem 4.2 of [79] shows that the semigroup of contractions $(e^{-t\widetilde{\mathcal{A}}_\beta})_{t\geq 0}$ is strongly asymptotically stable in the energy space $\mathcal{H}$, *i.e.* for any $U_0 \in \mathcal{H}$, we have $\lim_{t\to+\infty} \|e^{-t\widetilde{\mathcal{A}}_\beta}U_0\|^2_{\mathcal{H}} = 0$ if $\gamma \geq \gamma_0$ where $\sqrt{\gamma_0}\sinh^{-1}(\sqrt{\gamma_0}\pi) = 0$. Using a numerical program we find

$$\gamma_0 \simeq 0.45001246517627713.$$

Moreover, from Theorem 4.3 of [79] there exists an infinite numbers of $0 < \gamma < \gamma_0$ such that the operator $\widetilde{\mathcal{A}}_\beta$ has eigenvalues on the imaginary axis and therefore for which problem (3.2.15) is not stable. Further, we know that the Rayleigh beam is not uniformly exponentially stable neither with one boundary direct control force (see [79]) nor with two dynamical boundary control (see [92]). Then, we look for the optimal polynomial energy decay rate for smooth initial data.

3.3 Spectral analysis of the operator $\widetilde{\mathcal{A}}_\beta$ for $\beta \geq 0$

In this section, we study the eigenvalues and the eigenvectors of the operator $\widetilde{\mathcal{A}}_\beta$ for $\beta \geq 0$. First, let $\lambda \neq \beta$ be an eigenvalue of the operator $\widetilde{\mathcal{A}}_\beta$ and $U = (y,\ z,\ \xi)$ be an associated eigenfunction, then we have $\widetilde{\mathcal{A}}_\beta U = \lambda U$. Equivalently, λ and y verify the following system:

$$\begin{cases} y_{xxxx} - \gamma\lambda^2 y_{xx} + \lambda^2 y & = & 0, \\ y_{xxx}(1) - \gamma\lambda^2 y_x(1) - \dfrac{\lambda}{\lambda-\beta} y(1) & = & 0, \\ y(0) = y_x(0) = y_{xx}(1) & = & 0. \end{cases} \tag{3.3.1}$$

The general solution of the system (3.3.1) is

$$y = \sum_{i=1}^{4} c_i(\lambda) e^{R_i(\lambda)x}, \tag{3.3.2}$$

where $R_i(\lambda)$, $i = 1, .., 4$ are given by

$$R_1(\lambda) = \sqrt{\frac{\gamma\lambda^2 - \lambda\sqrt{\gamma^2\lambda^2 - 4}}{2}},\ R_2(\lambda) = -R_1(\lambda), \tag{3.3.3}$$

$$R_3(\lambda) = \sqrt{\frac{\gamma\lambda^2 + \lambda\sqrt{\gamma^2\lambda^2 - 4}}{2}},\ R_4(\lambda) = -R_3(\lambda).$$

Next, using the boundary conditions, we may write the system (3.3.1) as follows:

$$M_\beta(\lambda) \cdot C(\lambda) = 0, \tag{3.3.4}$$

where

$$M_\beta(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ R_1(\lambda) & R_2(\lambda) & R_3(\lambda) & R_4(\lambda) \\ R_1^2(\lambda)e^{R_1(\lambda)} & R_2^2(\lambda)e^{R_2(\lambda)} & R_3^2(\lambda)e^{R_3(\lambda)} & R_4^2(\lambda)e^{R_4(\lambda)} \\ T_{1,\beta}(\lambda) & T_{2,\beta}(\lambda) & T_{3,\beta}(\lambda) & T_{4,\beta}(\lambda) \end{pmatrix}, \tag{3.3.5}$$

$$C(\lambda) = \begin{pmatrix} c_1(\lambda) \\ c_2(\lambda) \\ c_3(\lambda) \\ c_4(\lambda) \end{pmatrix},$$

where

$$T_{i,\beta}(\lambda) = \left(R_i(\lambda)^3 - \gamma\lambda^2 R_i(\lambda) - \frac{\lambda}{\lambda - \beta} \right) e^{R_i(\lambda)},$$

for $i = 1, 2, 3, 4$.

Remark 3.3.1. *First, like we did in Proposition 2.4.2, we find that the real part of any eigenvalue λ of $\widetilde{\mathcal{A}}_\beta$ is bounded,* i.e.

$$\exists c > 0, \quad \forall \lambda \in \sigma(\widetilde{\mathcal{A}}_\beta), \quad 0 < \Re(\lambda) \leq c.$$

Next, let λ^0 be an eigenvalue of $\widetilde{\mathcal{A}}_0$ and $U^0 = (y^0, z^0, \xi^0) \in D(\widetilde{\mathcal{A}}_0)$ an associated eigenvector. Then, as we did in Proposition 2.3.2, we can easily prove that λ^0 is simple and $\xi^0 \neq 0$. ■

Next, we study the asymptotic behavior of the eigenvalues of the operators $\widetilde{\mathcal{A}}_\beta$ for $\beta \geq 0$ in the following proposition:

Proposition 3.3.2. *(Spectrum of $\widetilde{\mathcal{A}}_\beta$)*
Let $\beta \geq 0$. Then there exists $k_\beta \in \mathbb{N}^*$ sufficiently large such that the spectrum $\sigma(\mathcal{A}_\beta)$ of $\mathcal{A}_\beta$ is given by

$$\sigma(\mathcal{A}_\beta) = \sigma_{\beta,0} \cup \sigma_{\beta,1}, \tag{3.3.6}$$

where

$$\sigma_{\beta,0} = \{\kappa_{\beta,j}\}_{j \in J_\beta}, \quad \sigma_{\beta,1} = \{\lambda_{\beta,k}\}_{\substack{k \in \mathbb{Z} \\ |k| \geq k_\beta}}, \quad \sigma_{\beta,0} \cap \sigma_{\beta,1} = \emptyset, \tag{3.3.7}$$

where J_β is a finite set and $\lambda_{\beta,k}$ is simple and satisfies the following asymptotic behavior:

$$\begin{aligned}\lambda_{\beta,k} &= i\left(\alpha_k - \frac{(\frac{1}{2\sqrt{\gamma}} + \tanh(\frac{1}{\sqrt{\gamma}}))}{\gamma^{\frac{3}{2}}\alpha_k} + \frac{2(-1)^k}{\gamma^{\frac{5}{2}}\cosh(\frac{1}{\sqrt{\gamma}})\alpha_k^2}\right. \\ &\quad \left. + \frac{E}{\alpha_k^3} + \frac{F}{\alpha_k^4}\right) + \frac{\beta}{\pi^4\cosh(\frac{1}{\sqrt{\gamma}})} \times \frac{1}{k^4} + o(\frac{1}{k^4}),\end{aligned} \tag{3.3.8}$$

with

$$\alpha_k = \frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}, \tag{3.3.9}$$

$$\begin{aligned}E &= \frac{1}{3\gamma^{\frac{7}{2}}}\tanh(\frac{1}{\sqrt{\gamma}})^3 + \frac{1}{\gamma^{\frac{7}{2}}}\tanh(\frac{1}{\sqrt{\gamma}}) \\ &\quad -\frac{1}{\gamma^2}\left(1 + \frac{1}{\gamma} + \frac{1}{2\gamma^2}\right)\tanh(\frac{1}{\sqrt{\gamma}})^2 + \frac{1}{\gamma^2} + \frac{1}{\gamma^4},\end{aligned} \tag{3.3.10}$$

$$\begin{aligned}F &= \frac{(-1)^k}{\gamma^3\cosh(\frac{1}{\sqrt{\gamma}})}\left[(2 + \frac{6}{\gamma} + \frac{1}{\gamma^2})\tanh(\frac{1}{\sqrt{\gamma}}) - \right. \\ &\quad \left. \frac{1}{\gamma^{\frac{3}{2}}}(1 + \tanh(\frac{1}{\sqrt{\gamma}})^2)\right].\end{aligned} \tag{3.3.11}$$

Proof: The proof uses the same strategy than the one from Proposition 2.4.3. In Step 1 we furnishe an asymptotic development of the characteristic equation for large λ. Step 2 uses Rouché's theorem to localize high frequency eigenvalues. In step 3, we perform a limited development stopped when a non zero real part appear. For the sake of completeness, we give the details. For simplicity, we denote $R_i(\lambda)$ by R_i.
Step 1. First, we start by the expansion of R_1 and R_3 when $|\lambda| \to \infty$

$$R_1 = \frac{1}{\sqrt{\gamma}} + \frac{1}{2\gamma^{\frac{5}{2}}\lambda^2} + O(\frac{1}{\lambda^4}) \tag{3.3.12}$$

and

$$R_3 = \lambda\sqrt{\gamma} - \frac{1}{2\lambda\gamma^{\frac{3}{2}}} + O(\frac{1}{\lambda^3}). \tag{3.3.13}$$

Using the expansions (3.3.12) and (3.3.13), we find the following asymptotic behavior:

$$R_1^2 e^{R_1} = \left(\frac{1}{\gamma} + (\frac{1}{2\gamma^{\frac{7}{2}}} + \frac{1}{\gamma^3})\frac{1}{\lambda^2}\right) e^{\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda^4}), \tag{3.3.14}$$

$$R_1^2 e^{R_1} = \left(\frac{1}{\gamma} + (\frac{1}{2\gamma^{\frac{7}{2}}} + \frac{1}{\gamma^3})\frac{1}{\lambda^2}\right) e^{\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda^4}), \tag{3.3.15}$$

$$R_3^2 e^{R_3} = \left(\gamma\lambda^2 - \frac{\lambda}{2\sqrt{\gamma}} + \frac{1}{8\gamma^2} - \frac{1}{\gamma} - (\frac{1}{8\gamma^{\frac{5}{2}}} + \frac{1}{48\gamma^{\frac{7}{2}}})\frac{1}{\lambda}\right) e^{\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda^2}) \tag{3.3.16}$$

and

$$R_4^2 e^{R_4} = \left(\gamma\lambda^2 + \frac{\lambda}{2\sqrt{\gamma}} + \frac{1}{8\gamma^2} - \frac{1}{\gamma} + (\frac{1}{8\gamma^{\frac{5}{2}}} + \frac{1}{48\gamma^{\frac{7}{2}}})\frac{1}{\lambda}\right) e^{-\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda^2}). \tag{3.3.17}$$

Similarly, we get

$$T_{\beta,1}(\lambda) = \left(-\sqrt{\gamma}\lambda^2 - \frac{1}{2\gamma^2} + \frac{1}{2\gamma^{\frac{3}{2}}} - 1 - \frac{\beta}{\lambda}\right) e^{\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda^2}), \tag{3.3.18}$$

$$T_{\beta,2}(\lambda) = \left(\sqrt{\gamma}\lambda^2 - \frac{1}{2\gamma^2} - \frac{1}{2\gamma^{\frac{3}{2}}} - 1 - \frac{\beta}{\lambda}\right) e^{-\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda^2}), \tag{3.3.19}$$

$$\begin{aligned} T_{\beta,3}(\lambda) = \Bigg[&-\frac{\lambda}{\sqrt{\gamma}} + \frac{1}{2\gamma^2} - 1 + \left(\frac{1}{2\gamma^{\frac{3}{2}}} - \frac{1}{2\gamma^{\frac{5}{2}}} - \frac{1}{8\gamma^{\frac{7}{2}}} - \beta\right)\frac{1}{\lambda} \\ &+ \left(-\beta^2 + \frac{\beta}{2\gamma^{\frac{3}{2}}} - \frac{1}{8\gamma^3} + \frac{7}{8\gamma^4} + \frac{1}{48\gamma^5}\right)\frac{1}{\lambda^2}\Bigg] e^{\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda^3}) \end{aligned} \tag{3.3.20}$$

and

$$\begin{aligned} T_{\beta,4}(\lambda) = \Bigg[&\frac{\lambda}{\sqrt{\gamma}} + \frac{1}{2\gamma^2} - 1 + \left(-\frac{1}{2\gamma^{\frac{3}{2}}} + \frac{1}{2\gamma^{\frac{5}{2}}} + \frac{1}{8\gamma^{\frac{7}{2}}} - \beta\right)\frac{1}{\lambda} \\ &+ \left(-\beta^2 - \frac{\beta}{2\gamma^{\frac{3}{2}}} - \frac{1}{8\gamma^3} + \frac{7}{8\gamma^4} + \frac{1}{48\gamma^5}\right)\frac{1}{\lambda^2}\Bigg] e^{-\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda^3}). \end{aligned} \tag{3.3.21}$$

Combining (3.3.12)-(3.3.21) and (3.3.5), we can write

$$M_\beta(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ q_1 + O(\frac{1}{\lambda^4}) & q_2 + O(\frac{1}{\lambda^4}) & q_3 + O(\frac{1}{\lambda^2}) & q_4 + O(\frac{1}{\lambda^2}) \\ q_5 e^{\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda^4}) & q_6 e^{-\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda^4}) & q_7 e^{\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda^2}) & q_8 e^{-\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda^2}) \\ q_9 e^{\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda^2}) & q_{10} e^{-\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\lambda^2}) & q_{11} e^{\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda^3}) & q_{12} e^{-\sqrt{\gamma}\lambda} + O(\frac{1}{\lambda^3}) \end{pmatrix},$$

where

$$q_1 = \frac{1}{\sqrt{\gamma}} + \frac{1}{2\gamma^{\frac{5}{2}}\lambda}, \quad q_2 = -\frac{1}{\sqrt{\gamma}} - \frac{1}{2\gamma^{\frac{5}{2}}\lambda^2}, \quad q_3 = \lambda\sqrt{\gamma} - \frac{1}{2\lambda\gamma^{\frac{3}{2}}}, \quad q_4 = -\lambda\sqrt{\gamma} + \frac{1}{2\lambda\gamma^{\frac{3}{2}}},$$

$$q_5 = \frac{1}{\gamma} + (\frac{1}{2\gamma^{\frac{7}{2}}} + \frac{1}{\gamma^3})\frac{1}{\lambda^2}, \; q_6 = \frac{1}{\gamma} + (-\frac{1}{2\gamma^{\frac{7}{2}}} + \frac{1}{\gamma^3})\frac{1}{\lambda^2},$$

$$q_7 = \gamma\lambda^2 - \frac{\lambda}{2\sqrt{\gamma}} + \frac{1}{8\gamma^2} - \frac{1}{\gamma} - (\frac{1}{8\gamma^{\frac{5}{2}}} + \frac{1}{48\gamma^{\frac{7}{2}}})\frac{1}{\lambda},$$

$$q_8 = \gamma\lambda^2 + \frac{\lambda}{2\sqrt{\gamma}} + \frac{1}{8\gamma^2} - \frac{1}{\gamma} + (\frac{1}{8\gamma^{\frac{5}{2}}} + \frac{1}{48\gamma^{\frac{7}{2}}})\frac{1}{\lambda},$$

$$q_9 = -\sqrt{\gamma}\lambda^2 - \frac{1}{2\gamma^2} + \frac{1}{2\gamma^{\frac{3}{2}}} - 1 - \frac{\beta}{\lambda}, \quad q_{10} = \sqrt{\gamma}\lambda^2 - \frac{1}{2\gamma^2} - \frac{1}{2\gamma^{\frac{3}{2}}} - 1 - \frac{\beta}{\lambda},$$

$$\begin{aligned} q_{11} = &-\frac{\lambda}{\sqrt{\gamma}} + \frac{1}{2\gamma^2} - 1 + \left(\frac{1}{2\gamma^{\frac{3}{2}}} - \frac{1}{2\gamma^{\frac{5}{2}}} - \frac{1}{8\gamma^{\frac{7}{2}}} - \beta\right)\frac{1}{\lambda} \\ &+ \left(-\beta^2 + \frac{\beta}{2\gamma^{\frac{3}{2}}} - \frac{1}{8\gamma^3} + \frac{7}{8\gamma^4} + \frac{1}{48\gamma^5}\right)\frac{1}{\lambda^2} \end{aligned}$$

and

$$\begin{aligned} q_{12} = &\frac{\lambda}{\sqrt{\gamma}} + \frac{1}{2\gamma^2} - 1 + \left(-\frac{1}{2\gamma^{\frac{3}{2}}} + \frac{1}{2\gamma^{\frac{5}{2}}} + \frac{1}{8\gamma^{\frac{7}{2}}} - \beta\right)\frac{1}{\lambda} \\ &+ \left(-\beta^2 + \frac{\beta}{2\gamma^{\frac{3}{2}}} - \frac{1}{8\gamma^3} + \frac{7}{8\gamma^4} + \frac{1}{48\gamma^5}\right)\frac{1}{\lambda^2}. \end{aligned}$$

Then, after long computations, we find the following asymptotic development of $f_\beta(\lambda) = \det(M_\beta(\lambda))$:

$$f_\beta(\lambda) = \lambda^5 f_0(\lambda)+\lambda^4 f_1(\lambda)+\lambda^3 f_2(\lambda)+\lambda^2 f_{\beta,3}(\lambda)+\lambda f_{\beta,4}(\lambda)+O(1), \quad (3.3.22)$$

where

$$f_0(\lambda) = L_0(\gamma)\cosh(\sqrt{\gamma}\lambda),\ L_0(\gamma) = 4\gamma^2\cosh(\frac{1}{\sqrt{\gamma}}), \quad (3.3.23)$$

$$f_1(\lambda) = L_1(\gamma)\sinh(\sqrt{\gamma}\lambda), \quad (3.3.24)$$

$$L_1(\gamma) = -2\sqrt{\gamma}\left(\cosh(\frac{1}{\sqrt{\gamma}}) + 2\sqrt{\gamma}\sinh(\frac{1}{\sqrt{\gamma}})\right),$$

$$f_2(\gamma) = -8 + L_2(\gamma)\cosh(\sqrt{\gamma}\lambda), \quad (3.3.25)$$

$$L_2(\gamma) = \left(\frac{1}{2\gamma} - 8\right)\cosh(\frac{1}{\sqrt{\gamma}}) + \left(\frac{4}{\sqrt{\gamma}} + 4\gamma\sqrt{\gamma}\right)\sinh(\frac{1}{\sqrt{\gamma}}),$$

$$f_{\beta,3}(\lambda) = L_{\beta,3c}(\gamma)\cosh(\sqrt{\gamma}\lambda) + L_{3s}(\gamma)\sinh(\sqrt{\gamma}\lambda), \quad (3.3.26)$$

$$L_{\beta,3c}(\gamma) = 4\beta\gamma^{\frac{3}{2}}\sinh(\frac{1}{\sqrt{\gamma}}),$$

$$\begin{aligned} L_{3s}(\gamma) = &-\left(2 + \frac{3}{2\gamma^2}\right)\sinh(\frac{1}{\sqrt{\gamma}}) \\ &- \left(\frac{1}{12\gamma^{\frac{5}{2}}} + \frac{1}{2\gamma\sqrt{\gamma}} + 4\sqrt{\gamma}\right)\cosh(\frac{1}{\sqrt{\gamma}}) \end{aligned} \quad (3.3.27)$$

and where

$$f_{\beta,4}(\lambda) = L_{4c}(\gamma)\cosh(\sqrt{\gamma}\lambda) + L_{\beta,4s}(\gamma)\sinh(\sqrt{\gamma}\lambda), \quad (3.3.28)$$

$$\begin{aligned} L_{4c}(\gamma) \quad = \quad &\left(\frac{1}{3\gamma^{\frac{7}{2}}} - \frac{1}{2\gamma^{\frac{5}{2}}} + \frac{1}{2\gamma^{\frac{3}{2}}} - \frac{10}{\sqrt{\gamma}}\right)\sinh(\frac{1}{\sqrt{\gamma}}) \\ &+ \left(\frac{2}{\gamma} + \frac{13}{2\gamma^2} + \frac{1}{2\gamma^3}\right)\cosh(\frac{1}{\sqrt{\gamma}}), \end{aligned} \quad (3.3.29)$$

$$L_{\beta,4s}(\gamma) = -4\sqrt{\gamma}\beta\left(\cosh(\frac{1}{\sqrt{\gamma}}) + \frac{1}{\sqrt{\gamma}}\sinh(\frac{1}{\sqrt{\gamma}})\right). \quad (3.3.30)$$

Since the real part of λ is bounded, the functions f_i, $i \in \{0,1,2,3,4\}$ are also bounded. For convenience we set

$$S_\beta(\lambda) = \frac{f_\beta(\lambda)}{\lambda^5} = f_0(\lambda) + \frac{f_1(\lambda)}{\lambda} + \frac{f_2(\lambda)}{\lambda^2} + \frac{f_{\beta,3}(\lambda)}{\lambda^3} + \frac{f_{\beta,4}(\lambda)}{\lambda^4} + O(\frac{1}{\lambda^5}). \tag{3.3.31}$$

Step 2. Large eigenvalues of $\mathcal{A}_\beta$. We look at the roots of S_β. It is easy to see that the roots of f_0 are simple and given by

$$z_k = i\alpha_k = i\left(\frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}\right).$$

Then, with the help of Rouché's theorem, there exists $k_\beta \in \mathbb{N}^*$ large enough, such that $\forall |k| \geq k_\beta$ the large eigenvalues of $\mathcal{A}_\beta$ (denoted by $\lambda_{\beta,k}$) are simple and close to z_k, *i.e.*

$$\lambda_{\beta,k} = i\alpha_k + o_\beta(1), \quad |k| \to \infty. \tag{3.3.32}$$

Equivalently we can write

$$\lambda_{\beta,k} = i\alpha_k + \zeta_{\beta,k}, \quad \lim_{|k|\to\infty} \zeta_{\beta,k} = 0. \tag{3.3.33}$$

Step 3. Asymptotic behavior of $\zeta_{\beta,k}$. First, using (3.3.31) and the identities (3.3.23)-(3.3.30) we have

$$\begin{aligned} 0 = S_\beta(\lambda_{\beta,k}) =& L_0(\gamma)\cosh(\sqrt{\gamma}\lambda_{\beta,k}) + \frac{L_1(\gamma)\sinh(\sqrt{\gamma}\lambda_{\beta,k})}{\lambda_{\beta,k}} \\ &+ \frac{-8 + L_2(\gamma)\cosh(\sqrt{\gamma}\lambda_{\beta,k})}{\lambda_{\beta,k}^2} + \frac{L_{\beta,3c}(\gamma)\cosh(\sqrt{\gamma}\lambda_{\beta,k})}{\lambda_{\beta,k}^3} \\ &+ \frac{L_{3s}(\gamma)\sinh(\sqrt{\gamma}\lambda_{\beta,k})}{\lambda_{\beta,k}^3} + \frac{L_{4c}(\gamma)\cosh(\sqrt{\gamma}\lambda_{\beta,k})}{\lambda_{\beta,k}^4} \\ &+ \frac{L_{\beta,4s}(\gamma)\sinh(\sqrt{\gamma}\lambda_{\beta,k})}{\lambda_{\beta,k}^4} + O(\frac{1}{\lambda_{\beta,k}^5}). \end{aligned} \tag{3.3.34}$$

On the other hand, using (3.3.33) we obtain

$$\cosh(\sqrt{\gamma}\lambda_{\beta,k}) = i(-1)^k \sinh(\sqrt{\gamma}\zeta_{\beta,k}) \tag{3.3.35}$$
$$= i(-1)^k \left(\sqrt{\gamma}\zeta_{\beta,k} + \frac{\gamma\sqrt{\gamma\zeta_{\beta,k}^3}}{9} + o(\zeta_{\beta,k}^4)\right),$$
$$\sinh(\sqrt{\gamma}\zeta_{\beta,k}) = i(-1)^k \cosh(\sqrt{\gamma}\zeta_{\beta,k}) \tag{3.3.36}$$
$$= i(-1)^k \left(1 + \frac{\gamma\zeta_{\beta,k}^2}{2} + \frac{\gamma^2\zeta_{\beta,k}^4}{6} + o(\zeta_{\beta,k}^4)\right)$$

and

$$\frac{1}{\lambda_{\beta,k}} = \frac{1}{i\alpha_k}\left(1 - \frac{\zeta_{\beta,k}}{i\alpha_k} + o(\frac{\zeta_{\beta,k}^2}{\alpha_k^2})\right)$$
$$= -\frac{i}{\alpha_k} + \frac{\zeta_{\beta,k}}{\alpha_k^2} + o(\frac{\zeta_{\beta,k}^2}{\alpha_k^2}). \tag{3.3.37}$$

Similarly, we get

$$\frac{1}{\lambda_{\beta,k}^2} = -\frac{1}{\alpha_k^2} - 2i\frac{\zeta_{\beta,k}^2}{\alpha_k^3} + o(\frac{\zeta_{\beta,k}^2}{\alpha_k^2}), \tag{3.3.38}$$
$$\frac{1}{\lambda_{\beta,k}^3} = \frac{i}{\alpha_k^3} - 3\frac{\zeta_{\beta,k}}{\alpha_k^4} + o(\frac{\zeta_{\beta,k}^2}{\alpha_k^2}) \tag{3.3.39}$$

and

$$\frac{1}{\lambda_{\beta,k}^4} = \frac{1}{\alpha_k^4} + 4i\frac{\zeta_{\beta,k}}{\alpha_k^5} + o(\frac{\zeta_{\beta,k}^2}{\alpha_k^2}). \tag{3.3.40}$$

Then, substituting (3.3.35)-(3.3.40) into (3.3.34) and after some computation yields

$$\begin{aligned} 0 =& iL_0(\gamma)\sqrt{\gamma}\zeta_{\beta,k} + \frac{i\gamma\sqrt{\gamma}L_0(\gamma)}{6}\zeta_{\beta,k}^3 + \frac{L_1(\gamma)}{\alpha_k} + \frac{iL_1(\gamma)}{\alpha_k^2}\zeta_{\beta,k} \\ &+ \frac{\gamma L_1(\gamma)}{2\alpha_k}\zeta_{\beta,k}^2 + \frac{8(-1)^k}{\alpha_k^2} + \frac{16i(-1)^k}{\alpha_k^3}\zeta_{\beta,k} - \frac{iL_2(\gamma)}{\alpha_k^2}\zeta_{\beta,k} \\ &- \frac{\sqrt{\gamma}L_{\beta,3c}(\gamma)}{\alpha_k^3}\zeta_{\beta,k} - \frac{L_{3s}(\gamma)}{\alpha_k^3} + \frac{iL_{\beta,4s}(\gamma)}{\alpha_k^4} \\ &+ o(\zeta_{\beta,k}^4) + o(\frac{\zeta_{\beta,k}^2}{\alpha_k^2}) + o(\frac{1}{\alpha_k^4}). \end{aligned} \tag{3.3.41}$$

Next, using (3.3.41) we find the first development of $\zeta_{k,\beta}$ given by

$$\zeta_{\beta,k} = \frac{iL_1(\gamma)}{\sqrt{\gamma}L_0(\gamma)\alpha_k} + e_{\beta,1}, \tag{3.3.42}$$

where $e_{\beta,1} = O_\beta(\frac{1}{\alpha_k^2})$. Then, inserting (3.3.42) in (3.3.41) we obtain

$$e_{\beta,1} = \frac{8i(-1)^k}{\sqrt{\gamma}L_0(\gamma)\alpha_k^2} + e_{\beta,2}, \tag{3.3.43}$$

where $e_{\beta,2} = O_\beta(\frac{1}{\alpha_k^3})$. Substituting (3.3.43) into (3.3.42) yields

$$\zeta_{\beta,k} = \frac{iL_1(\gamma)}{\sqrt{\gamma}L_0(\gamma)\alpha_k} + \frac{8i(-1)^k}{\sqrt{\gamma}L_0(\gamma)\alpha_k^2} + e_{\beta,2}. \tag{3.3.44}$$

Next, inserting (3.3.44) in (3.3.41) we obtain

$$e_{\beta,2} = \frac{iQ_1}{\alpha_k^3} + e_{\beta,3}, \tag{3.3.45}$$

where

$$\begin{aligned} Q_1 \quad = \quad & \frac{1}{3\gamma L_0^3(\gamma)}\Big[-\sqrt{\gamma}L_1^3(\gamma) - 3L_0(\gamma)L_1^2(\gamma) \\ & +3\sqrt{\gamma}L_0(\gamma)L_1(\gamma)L_2(\gamma) - 3\sqrt{\gamma}L_0^2(\gamma)L_{3s}(\gamma)\Big] \end{aligned} \tag{3.3.46}$$

and where $e_{\beta,3} = O_\beta(\frac{1}{\alpha_k^4})$. Then, substituting (3.3.45) into (3.3.44) yields

$$\zeta_{\beta,k} = \frac{iL_1(\gamma)}{\sqrt{\gamma}L_0(\gamma)\alpha_k} + \frac{8i(-1)^k}{\sqrt{\gamma}L_0(\gamma)\alpha_k^2} + \frac{iQ_1}{\alpha_k^3} + e_{3,\beta}. \tag{3.3.47}$$

Later, inserting (3.3.47) in (3.3.41) we obtain

$$e_{\beta,3} = \frac{iQ_2}{\alpha_k^4} + \frac{Q_{\beta,3}}{\alpha_k^4} + o(\frac{1}{\alpha_k^4}), \tag{3.3.48}$$

where

$$Q_2 = \frac{4(-1)^k}{\gamma L_0^3(\gamma)}\left[2\sqrt{\gamma}L_0(\gamma)L_2(\gamma) - \sqrt{\gamma}L_1^2(\gamma) - 6L_0(\gamma)L_1(\gamma)\right] \tag{3.3.49}$$

and where

$$Q_{\beta,3} = \frac{L_{\beta,3c}(\gamma)L_1(\gamma) - L_0(\gamma)L_{\beta,4s}(\gamma)}{\sqrt{\gamma}L_0^2}. \tag{3.3.50}$$

Then, substituting (3.3.48) into (3.3.47) yields

$$\zeta_{\beta,k} = \frac{iL_1(\gamma)}{\sqrt{\gamma}L_0(\gamma)\alpha_k} + \frac{8i(-1)^k}{\sqrt{\gamma}L_0(\gamma)\alpha_k^2} + \frac{iQ_1}{\alpha_k^3} + i\frac{Q_2}{\alpha_k^4} + \frac{Q_{\beta,3}}{\alpha_k^4} + o(\frac{1}{k^4}). \tag{3.3.51}$$

Moreover, using (3.3.23)-(3.3.27) and (3.3.30), then from (3.3.51) and after long computations we obtain

$$\begin{aligned}\zeta_{\beta,k} &= i\left(-\frac{(\frac{1}{2\sqrt{\gamma}} + \tanh(\frac{1}{\sqrt{\gamma}}))}{\gamma^{\frac{3}{2}}\alpha_k} + \frac{2(-1)^k}{\gamma^{\frac{5}{2}}\cosh(\frac{1}{\sqrt{\gamma}})\alpha_k^2} + \frac{E}{\alpha_k^3} + \frac{F}{\alpha_k^4}\right)\\ &\quad + \frac{\beta}{\pi^4\cosh(\frac{1}{\sqrt{\gamma}})} \times \frac{1}{k^4} + o(\frac{1}{k^4}),\end{aligned}$$

where E and F are given by (3.3.10) and (3.3.11) respectively. Finally inserting the previous identity in (3.3.33) we directly get (3.3.8). ■

Graphical Interpretation. Figure 3.1 represents the eigenvalues of $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_0$ for $\gamma = 10$.

Note that for a scale reason seven eigenvalues do no appear in the previous figure. Their approximates values are

$0.0152039 \pm 5.58917i$, $0.0402791 \pm 3.3494i$, $0.138254 \pm 1.30223i$ and 0.546406.

From Proposition 3.3.2 we denote that

$$\Phi_{\beta,k} = (y_{\beta,k}, -\lambda_{\beta,k}y_{\beta,k}, y_{\beta,k}(1)) \tag{3.3.52}$$

is the eigenvector associated with the eigenvalue $\lambda_{\beta,k}$ of high frequency and by $\{\Phi_{\beta,j,l}\}_{l=1}^{m_{\beta,j}}$ the Jordan chain of root vectors associated with the eigenvalue $\lambda_{\beta,j}$ of low frequency ($\Phi_{0,j,l}$ are in fact eigenvectors of $\mathcal{A}_0$) . Thus we obtain a system of root vectors of $\mathcal{A}_\beta$

$$\{\Phi_{\beta,k},\ |k| \geq k_\beta\} \cup \{\Phi_{\beta,j,l},\ 1 \leq l \leq m_{\beta,j},\ j \in J_\beta\}. \tag{3.3.53}$$

Now, we solve the problem (3.3.1) for $\lambda = \lambda_{\beta,k}$ (for $\beta \geq 0$) and we give a solution up to factor by the following proposition:

Proposition 3.3.3. *For $\beta \geq 0$ and $|k| \geq k_\beta$, a solution $y_{\beta,k}$ of the*

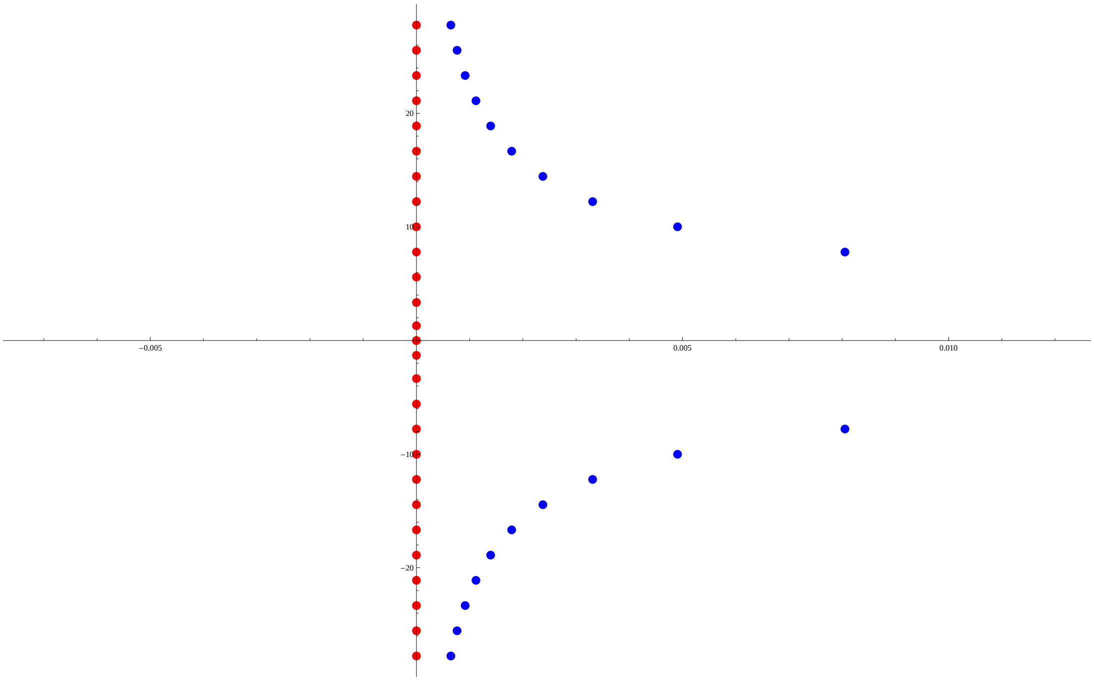

Figure 3.1: Eigenvalues of $\widetilde{\mathcal{A}}_1$ (in blue) and $\widetilde{\mathcal{A}}_0$ (in red) with $\beta = 1$ and $\gamma = 10$

problem (3.3.1) *with* $\lambda = \lambda_{\beta,k}$ *satisfies the following estimations:*

$$\begin{cases} y_{\beta,k}(1) = -\dfrac{2}{\cosh(\frac{1}{\sqrt{\gamma}})} + o(1) \neq 0, \\ \|y_{\beta,k}\|_W \sim |k|^2, \ \|y_{\beta,k}\|_V \sim |k|, \ |k| \to \infty. \end{cases} \tag{3.3.54}$$

and we deduce that

$$\|\Phi_{\beta,k}\|_{\mathcal{H}} \sim |k|^2, \quad |k| \to \infty. \tag{3.3.55}$$

Proof: For simplicity, in this proof we denote $\lambda_{\beta,k}$ by λ_k and $y_{\beta,k}$ by y_k. For $\beta \geq 0$, $\lambda = \lambda_k$ and $|k| \geq k_\beta$, solving (3.3.1) amounts to find a solution $C(\lambda_k) \neq 0$ of system (3.3.4) of rank three. For clarity, we divide the proof to several steps.

Step 1. Determination of y_k. Since we search $C(\lambda_k)$ up to factor we choose $c_4(\lambda_k) = 1$, the possibility of this choice will be justify later.

Therefore (3.3.4) becomes

$$\begin{cases} c_1(\lambda_k) + c_2(\lambda_k) + c_3(\lambda_k) & = & -1, \\ R_1(\lambda_k)c_1(\lambda_k) + R_2(\lambda_k)c_2(\lambda_k) + R_3(\lambda_k)c_3(\lambda_k) & = & -R_4(\lambda_k), \\ R_1^2(\lambda_k)e^{R_1(\lambda_k)}c_1(\lambda_k) + R_2^2(\lambda_k)e^{R_2(\lambda_k)}c_2(\lambda_k) + R_3^2(\lambda_k)e^{R_3(\lambda_k)} & = & -R_4^2(\lambda_k)e^{R_4(\lambda_k)}. \end{cases}$$

Next, using Cramer's rule, we obtain

$$c_1(\lambda_k) = \frac{b_1}{b_4}, \quad c_2(\lambda_k) = \frac{b_2}{b_4}, \quad c_3(\lambda_k) = \frac{b_3}{b_4}, \tag{3.3.56}$$

where

$$\begin{aligned} b_1 =& 2R_1(\lambda_k)R_3(\lambda_k)^2\sinh(R_3(\lambda_k)) - 2R_3(\lambda_k)^3\cosh(R_3(\lambda_k)) \\ &+ 2R_1(\lambda_k)^2R_3(\lambda_k)e^{-R_1(\lambda_k)}, \end{aligned} \tag{3.3.57}$$

$$\begin{aligned} b_2 =& 2R_1(\lambda_k)R_3(\lambda_k)^2\sinh(R_3(\lambda_k)) + 2R_3(\lambda_k)^3)\cosh(R_3(\lambda_k)) \\ &- 2R_1(\lambda_k)^2R_3(\lambda_k)e^{R_1(\lambda_k)}, \end{aligned} \tag{3.3.58}$$

$$\begin{aligned} b_3 =& 2R_1(\lambda_k)^2R_3(\lambda_k)\sinh(R_1(\lambda_k)) - 2R_1(\lambda_k)^3\cosh(R_1(\lambda_k)) \\ &+ R_1(\lambda_k)R_3(\lambda_k)^2e^{-R_3(\lambda_k)} \end{aligned} \tag{3.3.59}$$

and where

$$\begin{aligned} b_4 =& 2R_1(\lambda_k)^2R_3(\lambda_k)\sinh(R_1(\lambda_k)) + 2R_1(\lambda_k)^3\cosh(R_1(\lambda_k)) \\ &- R_1(\lambda_k)R_3(\lambda_k)^2e^{R_3(\lambda_k)}. \end{aligned} \tag{3.3.60}$$

First, we study the behavior of b_1. Inserting (3.3.12) and (3.3.13) (with $\lambda = \lambda_k$) in (3.3.57) we find after some computations

$$b_1 = -2\gamma^{\frac{3}{2}}\lambda_k^3\cosh(\sqrt{\gamma}\lambda_k) + (1 + 2\sqrt{\gamma})\lambda_k^2\sinh(\sqrt{\gamma}\lambda_k). \tag{3.3.61}$$

Now, using the asymptotic behavior (3.3.8) we find

$$\begin{cases} \cosh(\sqrt{\gamma}\lambda_k) & = & \dfrac{i(-1)^k(1 + 2\sqrt{\gamma}\tanh(\frac{1}{\sqrt{\gamma}}))}{2\gamma^{\frac{3}{2}}\lambda_k} + O(\dfrac{1}{\lambda_k^2}), \\ \sinh(\sqrt{\gamma}\lambda_k) & = & i(-1)^k + O(\dfrac{1}{\lambda_k^2}). \end{cases} \tag{3.3.62}$$

Then, inserting (3.3.62) in (3.3.61) we find again after some computations

$$b_1 = 2\sqrt{\gamma} i(-1)^k \left(1 - \tanh(\frac{1}{\sqrt{\gamma}})\right) \lambda_k^2 + O(\lambda_k). \tag{3.3.63}$$

Similarly, long computations left to the reader yield

$$b_2 = 2i(-1)^k \sqrt{\gamma} \left(1 + \tanh(\frac{1}{\sqrt{\gamma}})\right) \lambda_k^2 + O(\lambda_k), \tag{3.3.64}$$

$$b_3 = -2\sqrt{\gamma} i(-1)^k \lambda_k^2 + O(\lambda_k) \tag{3.3.65}$$

and

$$b_4 = -2\sqrt{\gamma} i(-1)^k \lambda_k^2 + O(\lambda_k). \tag{3.3.66}$$

Remark that $b_4 \neq 0$ provided we have chosen k_β large enough, for this reason our choice $c_4(\lambda_k) = 1$ is valid. Substituting (3.3.63)-(3.3.66) into (3.3.56), we deduce

$$\begin{cases} c_1(\lambda_k) &= & -1 + \tanh(\frac{1}{\sqrt{\gamma}}) + O(\dfrac{1}{|\lambda_k|}), \\ c_2(\lambda_k) &= & -1 - \tanh(\frac{1}{\sqrt{\gamma}}) + O(\dfrac{1}{|\lambda_k|}), \\ c_3(\lambda_k) &= & 1 + O(\dfrac{1}{|\lambda_k|}), \\ c_4(\lambda_k) &= & 1. \end{cases} \tag{3.3.67}$$

Finally, we have found that a solution of (3.3.4) has the form

$$C(\lambda_k) = C_0 + O(\frac{1}{|\lambda_k|}), \tag{3.3.68}$$

where

$$C_0 = (-1 + \tanh(\frac{1}{\sqrt{\gamma}}),\ -1 - \tanh(\frac{1}{\sqrt{\gamma}}),\ 1,\ 1). \tag{3.3.69}$$

Note that the corresponding solution y_k of (3.3.1) is given by

$$y_k = \sum_{i=1}^{4} c_i(\lambda_k) e^{R_i(\lambda_k)}. \tag{3.3.70}$$

Step 2. Estimate of $y_k(1)$. From equation (3.3.70), we have

$$y_k(1) = c_1(\lambda_k)e^{R_1(\lambda_k)} + c_2(\lambda_k)e^{R_2(\lambda_k)} + c_3(\lambda_k)e^{R_3(\lambda_k)} + c_4(\lambda_k)e^{R_4(\lambda_k)},$$

where we recall that for $i \in \{1,2,3,4\}$ $R_i(\lambda_k)$ are given by (3.3.3) and c_i satisfy (3.3.67). Therefore using the series expansions (3.3.8) and (3.3.12)-(3.3.13) for $\lambda = \lambda_k$ we easily find

$$y_k(1) = -\frac{2}{\cosh(\frac{1}{\sqrt{\gamma}})} + o(1) \neq 0. \tag{3.3.71}$$

Step 3. Estimates of $\|y_k\|_W$ and $\|y_k\|_V$. We start with

$$\begin{aligned}
\|y_k\|_W^2 &= \int_0^1 |y_{k,xx}|^2 dx \\
&= \sum_{i=1}^4 \sum_{j=1}^4 c_i(\lambda_k) R_i(\lambda_k)^2 \left(\int_0^1 e^{R_i(\lambda_k)x} \overline{e^{R_j(\lambda_k)x}} dx \right) \overline{c_j(\lambda_k) R_j(\lambda_k)^2} \\
&= C_k G_k \overline{C_k}^T,
\end{aligned} \tag{3.3.72}$$

where

$$G_k = \left(\int_0^1 e^{(R_i(\lambda_k) + \overline{R_j(\lambda_k)})x} dx \right)_{1\le i,j \le 4} \quad \text{and where} \quad C_k = (c_i(\lambda_k) R_i(\lambda_k)^2)_{i=1,..,4}.$$

First, using (3.3.8), then from (3.3.12) and (3.3.13) we can write $R_1(\lambda_k)$ and $R_3(\lambda_k)$ as follows

$$R_1(\lambda_k) = q_1 + i r_1, \tag{3.3.73}$$

where

$$q_1 = \frac{1}{\sqrt{\gamma}} + O(\frac{1}{\lambda_k^2}), \quad r_1 = -\frac{\gamma^{\frac{7}{2}}\beta}{\cosh(\frac{1}{\sqrt{\gamma}})\lambda_k^7} + O(\frac{1}{\lambda_k^8})$$

and

$$R_3(\lambda_k) = q_3 + i r_3, \tag{3.3.74}$$

where

$$q_3 = \frac{\gamma^{\frac{5}{2}}\beta}{\lambda_k^4} + O(\frac{1}{\lambda_k^6}), \quad r_3 = \sqrt{\gamma}\lambda_k + O(1).$$

Then, the fact that $R_2(\lambda_k) = -R_1(\lambda_k)$ and $R_4(\lambda_k) = -R_3(\lambda_k)$ and using the asymptotic behavior (3.3.73)-(3.3.74) we directly find

$$\begin{aligned}\int_0^1 e^{(R_1(\lambda_k)+\overline{R_2(\lambda_k)})x}dx &= \int_0^1 e^{(R_2(\lambda_k)+\overline{R_1(\lambda_k)})x}dx \\ &= \int_0^1 e^{(R_3(\lambda_k)+\overline{R_3(\lambda_k)})x}dx \\ &= \int_0^1 e^{(R_4(\lambda_k)+\overline{R_4(\lambda_k)})x}dx = 1 + O(\frac{1}{\lambda_k^4}).\end{aligned}$$

Moreover, using the asymptotic behavior (3.3.73)- (3.3.74) we find that G_k is given as follows:

$$G_k = G_0 + O(\frac{1}{\lambda_k}), \tag{3.3.75}$$

where

$$G_0 = \begin{pmatrix} \frac{\sqrt{\gamma}}{2}(e^{\frac{2}{\sqrt{\gamma}}}-1) & 1 & 0 & 0 \\ 1 & \frac{\sqrt{\gamma}}{2}(1-e^{-\frac{2}{\sqrt{\gamma}}}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.3.76}$$

and where $O(\frac{1}{\lambda_k})$ is a matrix where all entries are if order $\frac{1}{\lambda_k}$. Next, using (3.3.67) and (3.3.73)-(3.3.74) we obtain

$$C_k = (0, 0, \gamma\lambda_k^2, \gamma\lambda_k^2) + O(1). \tag{3.3.77}$$

Finally, using (3.3.8), (3.3.75) and (3.3.77) then from (3.3.72) we deduce

$$\|y_k\|_W^2 = \gamma^2|\lambda_k|^4 + O(|\lambda_k|^3) \sim |k|^4, \quad |k| \to \infty. \tag{3.3.78}$$

Similarly, we easily prove that

$$\|y_k\|_{L^2(0,1)}^2 \sim 1, \quad \|y_{k,x}\|_{L^2(0,1)}^2 \sim |k|^2, \quad |k| \to \infty.$$

Therefore, we deduce that

$$\|y_k\|_V \sim |k|, \quad |k| \to \infty. \tag{3.3.79}$$

Consequently, using the estimations (3.3.71), (3.3.78) and (3.3.79) then from (3.3.52) we deduce (3.3.55). This completes the proof. ■

3.4 Riesz basis and optimal energy decay rate

Our main result is the following optimal polynomial-type decay estimation:

Theorem 3.4.1. *(Optimal energy decay rate)*
Assume that $\beta > 0$ and that $\gamma \geq \gamma_0$. Then, for all initial data $U_0 \in D(\widetilde{\mathcal{A}}_0)$, there exists a constant $c > 0$ independent of U_0, such that the energy of the problem (3.2.15) satisfies the following estimation:

$$E(t) \leq \frac{c}{\sqrt{t}} \|U_0\|^2_{D(\widetilde{\mathcal{A}}_0)}. \tag{3.4.1}$$

Moreover, the energy decay rate (3.4.1) is optimal.

■

First, we prove that the set of the generalized eigenvectors associated with $\widetilde{\mathcal{A}}_\beta$ forms a Riesz basis of $\mathcal{H}$ by the following theorem:

Theorem 3.4.2. *The set of generalized eigenvectors associated with $\sigma(\widetilde{\mathcal{A}}_\beta)$ forms a Riesz basis of $\mathcal{H}$.*

Proof: First, since $\widetilde{\mathcal{A}}_0$ is a skew-adjoint operator, its set of normalized eigenvectors form an orthonormal basis in $\mathcal{H}$. Next, we prove the following property:

$$\sum_{k=\max\{k_0,k_\beta\}}^{+\infty} \|\widetilde{\Phi}_{\beta,k} - \widetilde{\Phi}_{0,k}\|_{\mathcal{H}} < +\infty \tag{3.4.2}$$

where

$$\widetilde{\Phi}_{\beta,k} = (\widetilde{y}_{\beta,k}, \widetilde{z}_{\beta,k}, \widetilde{\xi}_{\beta,k}) = \frac{1}{\|\Phi_{0,k}\|_{\mathcal{H}}} \Phi_{\beta,k}, \quad \forall |k| \geq k_\beta \tag{3.4.3}$$

and where

$$\widetilde{\Phi}_{0,k} = (\widetilde{y}_{0,k}, \widetilde{z}_{0,k}, \widetilde{\xi}_{0,k}) = \frac{1}{\|\Phi_{0,k}\|_{\mathcal{H}}} \Phi_{0,k}, \quad \forall |k| \geq k_0. \tag{3.4.4}$$

We first estimate

$$\|\widetilde{\Phi}_{\beta,k} - \widetilde{\Phi}_{0,k}\|^2_{\mathcal{H}} = \|\widetilde{y}_{\beta,k} - \widetilde{y}_{0,k}\|^2_W + \|\widetilde{z}_{\beta,k} - \widetilde{z}_{0,k}\|^2_V + |\widetilde{\xi}_{\beta,k} - \widetilde{\xi}_{0,k}|^2. \tag{3.4.5}$$

For clarity, we divide the proof into several steps.
Step 1. Estimate of $\|\widetilde{y}_{\beta,k} - \widetilde{y}_{0,k}\|^2_W$. First, since $\|\Phi_{0,k}\|_{\mathcal{H}} \sim |k|^2$ then

from (3.4.3) and (3.4.4) we obtain

$$\|\widetilde{y}_{\beta,k}-\widetilde{y}_{0,k}\|_W \sim \frac{1}{|k|^2}\|y_{\beta,k}-y_{0,k}\|_W. \tag{3.4.6}$$

Next, using (3.3.70) we obtain

$$\begin{aligned}
\widetilde{y}_{\beta,k,xx}-\widetilde{y}^0_{0,k,xx} \sim & \frac{1}{|k|^2}\left(R_1^2(\lambda_{\beta,k})c_1(\lambda_{\beta,k})e^{R_1(\lambda_{\beta,k})x}-R_1^2(\lambda_{0,k})c_1(\lambda_{0,k})e^{R_1(\lambda_{0,k})x}\right)\\
&+\frac{1}{|k|^2}\left(R_1^2(\lambda_{\beta,k})c_2(\lambda_{\beta,k})e^{-R_1(\lambda_{\beta,k})x}-R_1^2(\lambda_{0,k})c_2(\lambda_{0,k})e^{-R_1(\lambda_{0,k})x}\right)\\
&+\frac{1}{|k|^2}\left(R_3^2(\lambda_{\beta,k})c_3(\lambda_{\beta,k})e^{R_3(\lambda_{\beta,k})x}-R_3^2(\lambda_{0,k})c_3(\lambda_{0,k})e^{R_3(\lambda_{0,k})x}\right)\\
&+\frac{1}{|k|^2}\left(R_3^2(\lambda_{\beta,k})c_4(\lambda_{\beta,k})e^{-R_3(\lambda_{\beta,k})x}-R_3^2(\lambda_{0,k})c_4(\lambda_{0,k})e^{-R_3(\lambda_{0,k})x}\right).
\end{aligned}$$

For simplicity we denote $c_i(\lambda_{\beta,k})$ by $c_i^{\beta,k}$ and $c_i(\lambda_{0,k})$ by $c_i^{0,k}$ for $i\in\{1,2,3,4\}$. Then, a direct computation gives

$$\|\widetilde{y}_{\beta,k}-\widetilde{y}_{0,k}\|_W^2 \lesssim J_1+J_2+J_3+J_4 \tag{3.4.7}$$

where

$$\begin{aligned}
J_1 =&\frac{1}{|k|^4}\int_0^1 |R_1^2(\lambda_{\beta,k})-R_1^2(\lambda_{0,k})|^2|c_1^{\beta,k}|^2|e^{R_1(\lambda_{\beta,k})x}|^2dx \\
&+\frac{1}{|k|^4}\int_0^1 |R_1^2(\lambda_{0,k})|^2|c_1^{\beta,k}-c_1^{0,k}|^2|e^{R_1(\lambda_{\beta,k})x}|^2dx\\
&+\frac{1}{|k|^4}\int_0^1 |R_1^2(\lambda_{0,k})|^2|c_1^{0,k}|^2|e^{R_1(\lambda_{\beta,k})x}-e^{R_1(\lambda_{0,k})x}|^2dx,
\end{aligned} \tag{3.4.8}$$

$$
\begin{aligned}
J_2 =& \frac{1}{|k|^4}\int_0^1 |R_1^2(\lambda_{\beta,k}) - R_1^2(\lambda_{0,k})|^2 |c_2^{\beta,k}|^2 |e^{-R_1(\lambda_{\beta,k})x}|^2 dx \qquad (3.4.9)\\
&+ \frac{1}{|k|^4}\int_0^1 |R_1^2(\lambda_{0,k})|^2 |c_2^{\beta,k} - c_2^{0,k}|^2 |e^{-R_1(\lambda_{\beta,k})x}|^2 dx\\
&+ \frac{1}{|k|^4}\int_0^1 |R_1^2(\lambda_{0,k})|^2 |c_2^{0,k}|^2 |e^{-R_1(\lambda_{\beta,k})x} - e^{-R_1(\lambda_{0,k})x}|^2 dx,\\
J_3 =& \frac{1}{|k|^4}\int_0^1 |R_3^2(\lambda_{\beta,k}) - R_3^2(\lambda_{0,k})|^2 |c_3^{\beta,k}|^2 |e^{R_3(\lambda_{\beta,k})x}|^2 dx \qquad (3.4.10)\\
&+ \frac{1}{|k|^4}\int_0^1 |R_3^2(\lambda_{0,k})|^2 |c_3^{\beta,k} - c_3^{0,k}|^2 |e^{R_3(\lambda_{\beta,k})x}|^2 dx\\
&+ \frac{1}{|k|^4}\int_0^1 |R_3^2(\lambda_{0,k})|^2 |c_3^{0,k}|^2 |e^{R_3(\lambda_{\beta,k})x} - e^{R_3(\lambda_{0,k})x})|^2 dx
\end{aligned}
$$

and

$$
\begin{aligned}
J_4 =& \frac{1}{|k|^4}\int_0^1 |R_3^2(\lambda_{\beta,k}) - R_3^2(\lambda_{0,k})|^2 |e^{-R_3(\lambda_{\beta,k})x}|^2 dx \qquad (3.4.11)\\
&+ \frac{1}{|k|^4}\int_0^1 |R_3(\lambda_{0,k})|^2 |c_4^{\beta,k} - c_4^{0,k}|^2 |e^{-R_3(\lambda_{\beta,k})x}|^2 dx\\
&+ \frac{1}{|k|^4}\int_0^1 |R_3^2(\lambda_{0,k})|^2 |e^{-R_3(\lambda_{\beta,k})x} - e^{-R_3(\lambda_{0,k})x}|^2 dx.
\end{aligned}
$$

Now, using (3.3.8) ,(3.3.67) and the asymptotic behaviors (3.3.12)-(3.3.13) for $\lambda = \lambda_{\beta,k}$ and for $\lambda = \lambda_{0,k}$ we find

$$
\begin{cases}
R_1(\lambda_{\beta,k})^2 - R_1(\lambda_{0,k})^2 & \sim \dfrac{1}{|k|^4},\\
e^{R_1(\lambda_{\beta,k})} - e^{R_1(\lambda_{0,k})} & \sim \dfrac{1}{|k|^2},\\
c_1^{\beta,k} - c_1^{0,k} & \sim \dfrac{1}{|k|}
\end{cases} \qquad (3.4.12)
$$

and

$$
\begin{cases}
R_3(\lambda_{\beta,k})^2 - R_3(\lambda_{0,k})^2 & \sim \dfrac{1}{|k|^3},\\
e^{R_3(\lambda_{\beta,k})x} - e^{R_3(\lambda_{0,k})x} & \sim \dfrac{1}{|k|},\\
c_3^{\beta,k} - c_3^{0,k} & \sim \dfrac{1}{|k|}.
\end{cases} \qquad (3.4.13)
$$

Since $c_1^{\beta,k} \sim c_1^{0,k} \sim e^{R_1(\lambda_{\beta,k})x} \sim R_1^2(\lambda_{0,k}) \sim 1$ and using (3.4.12), then from (3.4.8) we obtain

$$J_1 \sim \frac{1}{|k|^4}\int_0^1 \frac{1}{|k|^8}dx + \frac{1}{|k|^4}\int_0^1 \frac{1}{|k|^2}dx + \frac{1}{|k|^4}\int_0^1 \frac{1}{|k|^4}dx \sim \frac{1}{|k|^6}. \tag{3.4.14}$$

Similarly, we get

$$J_2 \sim \frac{1}{|k|^8}. \tag{3.4.15}$$

In the same way, since $c_3^{\beta,k} \sim c_3^{0,k} \sim e^{R_3(\lambda_{\beta,k})} \sim 1$, $R_1(\lambda_{\beta,k})^2 \sim |k|^2$ and using (3.4.13) then from (3.4.10) we obtain

$$J_3 \sim \frac{1}{|k|^4}\int_0^1 \frac{1}{|k|^6}dx + \frac{1}{|k|^4}\int_0^1 |k|^2 dx + \frac{1}{|k|^4}\int_0^1 |k|^2 dx \sim \frac{1}{|k|^2}. \tag{3.4.16}$$

Similarly, we get

$$J_4 \sim \frac{1}{|k|^2}. \tag{3.4.17}$$

Finally, using (3.4.14)-(3.4.17), from (3.4.7) we deduce

$$\|\widetilde{y}_{\beta,k} - \widetilde{y}_{0,k}\|_W^2 \lesssim \frac{1}{|k|^2}. \tag{3.4.18}$$

Step 2. Estimates $\|\widetilde{z}_{\beta,k} - \widetilde{z}_{0,k}\|_V^2$ and $|\widetilde{\xi}_{\beta,k} - \widetilde{\xi}_{0,k}|^2$. First, since $\|\Phi_{0,k}\|_{\mathcal{H}} \sim |k|^2$ and using (3.4.3)-(3.4.4) we obtain

$$\|\widetilde{z}_k - \widetilde{z}_k^0\|_V^2 \sim \frac{1}{|k|^4}\|z_k - z_k^0\|_V^2. \tag{3.4.19}$$

Then, using (3.3.52) we obtain

$$\begin{aligned} \|\widetilde{z}_{\beta,k} - \widetilde{z}_{0,k}\|_V^2 \ &\sim \ \frac{1}{|k|^4}\|\lambda_{\beta,k}y_{\beta,k} - \lambda_{0,k}y_{0,k}\|_V^2 \\ &\leq \ \frac{1}{|k|^4}|\lambda_{\beta,k} - \lambda_{0,k}|^2\|y_{\beta,k}\|_V^2 + \frac{|\lambda_{0,k}|^2}{|k|^4}\|y_{\beta,k} - y_{0,k}\|_V^2. \end{aligned} \tag{3.4.20}$$

Now, since $|\lambda_{\beta,k} - \lambda_{0,k}| \sim \dfrac{1}{|k|^4}$ and $\|y_{\beta,k}\|_V \sim |k|^2$ we get

$$\frac{1}{|k|^4}|\lambda_{\beta,k} - \lambda_{0,k}|^2\|y_{\beta,k}\|_V^2 \sim \frac{1}{|k|^8}. \tag{3.4.21}$$

Next, using the same strategy as in Step 1, we find after long computations that

$$\|y_{\beta,k} - y_{0,k}\|_V^2 \sim 1. \tag{3.4.22}$$

Then inserting (3.4.21)-(3.4.22) in (3.4.20) and the fact that $|\lambda_{0,k}|^2 \sim |k|^2$ we deduce

$$\|\widetilde{z}_{\beta,k} - \widetilde{z}_{0,k}\|_V^2 \lesssim \frac{1}{|k|^2}. \tag{3.4.23}$$

Similarly, we can easily find that

$$|\widetilde{\xi}_{\beta,k} - \widetilde{\xi}_{0,k}|^2 \lesssim \frac{1}{|k|^{10}}. \tag{3.4.24}$$

Step 3. Inserting the estimations (3.4.18), (3.4.23) and (3.4.24) into (3.4.5) we obtain

$$\|\widetilde{\Phi}_{\beta,k} - \widetilde{\Phi}_{0,k}\|_{\mathcal{H}}^2 \lesssim \frac{1}{|k|^2}$$

and consequently

$$\sum_{k=\max\{k_0,k_\beta\}}^{\infty} \|\widetilde{\Phi}_{\beta,k} - \widetilde{\Phi}_{0,k}\|_{\mathcal{H}}^2 < +\infty.$$

Therefore, using a clarified form of Guo's theorem given by Theorem 1.2.10 in [2] (see also [43, Theorem 6.3]), we deduce that the set of generalized eigenvectors associated with $\sigma(\widetilde{\mathcal{A}}_\beta)$ forms a Riesz basis in $\mathcal{H}$. ■

Proof of Theorem 3.4.1: First, using (3.3.8) we have $\Re(\lambda_k) \sim \frac{1}{k^4}$. Next, from Theorem 3.4.2 we know that the set of generalized eigenvectors associated with $\sigma(\widetilde{\mathcal{A}}_\beta)$ form a Riesz basis of $\mathcal{H}$. Then, applying Theorem 2.1 given in [63] (see also [60], [63] and [91]), we deduce the optimal polynomial energy decay rate (3.4.1) for smooth initial data. ■

3.5 Open problems

The extensions of our results to multidimensional space are open problems. We cannot use the same strategy since it is based on the spectrum study and it is difficult to analyze it in these type of space. We maybe can apply a multiplier method like these given in [12, 20]. The exact controllability of Rayleigh beam equation and the stabilization of the numerical approximation schemes with static or dynamic boundary control force are also an open problems.

4

Indirect Stability of the wave equation with a dynamic boundary control

Abstract: In this chapter, we consider a damped wave equation with a dynamic boundary control. First, using Arendt and Batty's theorem (see [90]) we show the strong stability of our system. Next, we show that our system is not uniformly stable in general, since it is the case of the unit disk. Hence, we look for a polynomial decay rate for smooth initial data for our system by applying a frequency domain approach. In a first step, by giving some sufficient conditions on the boundary of our domain and by using the exponential decay of the wave equation with a standard damping, we prove a polynomial decay in $\frac{1}{t^{\frac{1}{4}}}$ of the energy. In a second step, under appropriated condition on the boundary of our system named by multiplier control conditions, we establish a polynomial decay in $\frac{1}{t}$ of the energy. Later, we show that such a polynomial decay seems to be also available even if the previous conditions is not satisfied. For this aim, we consider our system on the unit square of the plane. Using a spectral analysis, we show that the decay rate to zero of the energy is not exponential. Then, using a method based on a Fourier analysis, a specific analysis of the obtained 1-d problem combining Ingham's inequality and an interpolation method, we establish a polynomial decay in $\frac{1}{t}$ of the energy for sufficiently smooth initial data.

4.1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^d$, $d \geq 2$, with a Lipschitz boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, with Γ_0 and Γ_1 open subsets of Γ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$ and Γ_1 is non empty. In [30, 31, 41], N. Fourrier, I. Lasiecka and P. Graber studied the following problem (under the assumption that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$):

$$\begin{cases} u_{tt} - \Delta u - k_\Omega \Delta u_t + c_\Omega u_t & = 0, \quad \text{in} \quad \Omega \times \mathbb{R}_+^*, \\ u & = 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+^*, \\ u - w & = 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+^*, \\ w_{tt} - k_\Gamma \Delta_T(\alpha w_t + w) + \partial_\nu(u + k_\Omega u_t) + c_\Gamma w_t & = 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+^*, \\ w & = 0, \quad \text{on} \quad \partial\Gamma_1 \times \mathbb{R}_+^*, \\ u(\cdot,\cdot,0) = u_0, \quad u_t(\cdot,\cdot,0) = u_1, & \qquad \text{in} \quad \Omega, \\ w(\cdot,0) = w_0, \qquad w_t(\cdot,0) = w_1, & \qquad \text{on} \quad \Gamma_1, \end{cases} \tag{4.1.1}$$

where ∂_ν means the normal derivative on Γ_1, ν is the unit outward normal vector along the boundary and Δ_T denotes the Laplace-Beltrami operator on Γ. In system (4.1.1), two types of dissipation appear: internal (if $c_\Omega > 0$) and boundary (if $k_\Gamma > 0$) frictional ones and internal (if $k_\Omega > 0$) and boundary (if $k_\Gamma \alpha > 0$) viscoelastic ones. A physical description of this model is first described in [66]. In [30, 31], it is shown that system (4.1.1) is exponentially stable if one of the following three conditions is satisfied: $k_\Omega > 0$ (interior viscoelastic damping), or $c_\Omega > 0$ and $c_\Gamma > 0$ (internal and boundary frictional damping) or $c_\Omega > 0$ and $k_\Gamma \alpha > 0$ (internal frictional damping and boundary viscoelastic damping). The first case corresponds to a direct damping, while the other cases correspond to a phenomenon of overdamping. This phenomenon was the motivation of these authors to study the balance between the competiting dampings. On the contrary, in this chapter, we are interested in the important case where only a boundary frictional damping occurs, *i.e.* $k_\Omega = c_\Omega = \alpha = 0$ and $k_\Gamma = c_\Gamma = 1$. More precisely, we consider the following problem:

$$\begin{cases} u_{tt} - \Delta u & = 0, \quad \text{in} \quad \Omega \times \mathbb{R}_+^*, \\ u & = 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+^*, \\ u - w & = 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+^*, \\ w_{tt} - \Delta_T w + \partial_\nu u + w_t & = 0, \quad \text{in} \quad \Gamma_1 \times \mathbb{R}_+^*, \\ w & = 0, \quad \text{on} \quad \partial\Gamma_1 \times \mathbb{R}_+^*, \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, & \qquad \text{in} \quad \Omega, \\ w(\cdot,0) = w_0, \qquad w_t(\cdot,0) = w_1, & \qquad \text{on} \quad \Gamma_1. \end{cases} \tag{4.1.2}$$

In this case, the damping term is the term w_t in the fourth equation of (4.1.2) and therefore the system in Ω is only damped indirectly. The notion of the indirect damping mechanisms has been introduced by Russell in [83, 85], and since that time it retains the attention of many authors, because several models from acoustic theory enter in this framework.

There are many results concerning the wave equation with different models of damping:
In [24], M. Cavalcanti *and al.* considered the following wave equation with Wentzell boundary conditions:

$$\begin{cases} \Delta u + a(x)g(u_t) &= 0, & \text{in } \Omega\times]0,T[, \\ \beta u_{tt} + b(x)g_0(u_t) + \partial_\nu u + u &= \gamma\Delta_T u, & \text{on } \Gamma\times]0,T[, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & & \text{in } \Omega, \end{cases} \tag{4.1.3}$$

where Ω denotes a bounded class-C^2 domain in $\mathbb{R}^3$ with boundary Γ; the feedback maps g, g_0 are continuous monotone increasing, both vanishing at 0; $a(x) \in L^\infty(\Omega)$ and $b(x) \in C^1(\Gamma)$ are non-negative functions localizing the effect of the feedbacks to some subset of the domain and its boundary; the constants β, γ are non-negative. The system (4.1.3) can be seen as a hybrid system described by two potentially independent (but coupled) evolutions: one in Ω and another one on the boundary Γ, or else as single wave equation with higher-order tangential boundary conditions. M. Cavalcanti *and al.* have shown that system (4.1.3) for $\beta > 0$ (Wentzell dynamic boundary conditions) or for $\beta = 0$ (Wentzell static boundary conditions), can be exponentially stable under appropriated conditions.
In [4], N. Aissa and D. Hamroun considered the following system of coupled wave equations:

$$\begin{cases} u_{tt} - \Delta u &= d\cdot\nabla u_t, & \text{in } \Omega\times\mathbb{R}_+^*, \\ u &= 0, & \text{on } \Gamma_0\times\mathbb{R}_+^*, \\ \partial_\nu u &= -(-\partial_{xx})^{\frac{1}{2}} v_t, & \text{on } \Gamma_1\times\mathbb{R}_+^*, \\ v_{tt} - v_{xx} - (-\partial_{xx})^{\frac{1}{2}}(u_t) &= 0, & \text{on } \Gamma_1\times\mathbb{R}_+^*, \\ v(0,t) = v(1,t) &= 0, & \text{in } \mathbb{R}_+^*, \\ u(0) = u_0, \quad u_t(0) = u_1, & & \text{in } \Omega, \\ v(0) = v_0, \quad v_t(0) = v_1, & & \text{on } \Gamma_1, \end{cases} \tag{4.1.4}$$

where Ω is a square in $\mathbb{R}^2$, $\Gamma_1 =]0,1[\times\{0\}$, $\Gamma_0 = \partial\Omega\backslash\Gamma_1$ and $d \in C^1(\mathbb{R}^2, \mathbb{R})$. This system can be seen as a hybrid system of equation arising in the control of noise where the dissipation is located both on Ω and Γ_1. Using the multiplier method and under some hypothesis on d, N. Aissa and D. Hamroun proved that the energy of system (4.1.4) decays exponentially to 0.

The most popular model is the wave equation with acoustic boundary conditions, that takes the following form:

$$\begin{cases} u_{tt} - \Delta u & = & 0, & \text{in} & \Omega \times \mathbb{R}_+^*, \\ u & = & 0, & \text{on} & \Gamma_0 \times \mathbb{R}_+^*, \\ \partial_\nu u & = & w_t, & \text{on} & \Gamma_1 \times \mathbb{R}_+^*, \\ m w_{tt} + d w_t + k w + \rho u_t & = & 0, & \text{on} & \Gamma_1 \times \mathbb{R}_+^*, \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, & & & \text{in} & \Omega, \\ w(\cdot,0) = w_0, & & & \text{on} & \Gamma_1. \end{cases} \tag{4.1.5}$$

In [18], Beale showed that this problem is governed by a C_0-semigroup of contraction, while in [82], the authors obtained, under some geometrical conditions, a polynomial stability.

In [65], S. Micu and E. ZuaZua considered the following simple model arising in the control of noise consisting of two coupled hyperbolic equations of dimensions two and one respectively:

$$\begin{cases} u_{tt} - \Delta u & = & 0, & \text{in} & \Omega \times \mathbb{R}_+^*, \\ \partial_\nu u & = & 0, & \text{on} & \Gamma_0 \times \mathbb{R}_+^*, \\ \dfrac{\partial u}{\partial y} & = & -w_t, & \text{on} & \Gamma_1 \times \mathbb{R}_+^*, \\ w_{tt} - w_{xx} + w_t + u_t & = & 0, & \text{on} & \Gamma_1 \times \mathbb{R}_+^*, \\ w_x(0,t) = w_x(1,t) & = & 0, & \text{for} & t > 0, \\ u(0) = u^0, \quad u_t(0) = u_1, & & & \text{in} & \Omega, \\ w(0) = w_0, \quad w_t(0) = w_1, & & & \text{on} & \Gamma_1, \end{cases} \tag{4.1.6}$$

where $\Gamma_1 = \{(x,0);\ x \in (0,1)\}$ and $\Gamma_0 = \Gamma\backslash\Gamma_1$. This system is nothing else than system (4.1.5) where the Dirichlet boundary conditions on Γ_0 have been replaced by the Neumann ones. Using separation of variables method, they studied the asymptotic behavior of the eigenvalues and eigenfunctions of system (4.1.6). Since there exists a sequence of eigenvalues which approach the imaginary axis, E. ZuaZua and S. Micu proved that the decay rate of the energy of (4.1.6) is not exponential in the energy space. Later, they proved that system (4.1.6) can be exponentially stable in a subspace of the energy space. This subspace is generated by the eigenfunctions corresponding to a sequence of eigenvalues with uniformly bounded negative real parts. For a generalization of system (4.1.5) and polynomial decay rates, we refer to [1], while an abstract framework is extensively studied in [67]. For other related problems we refer to [4, 24, 32, 34, 60].

In a first step, using Arendt and Batty's Theorem (see [90]) and with help of Holmgren's theorem, we show the strong stability of system (4.1.2), but for the simple example like the case when Ω is the unit disc of $\mathbb{R}^2$ and $\Gamma_0 = \emptyset$, we show that our system is not uniformly stable, since the corresponding spatial operator has a sequence of eigenvalues that approach the imaginary axis. Hence, we are interested in proving a weaker decay of the energy, for that purpose, we will apply a frequency domain approach (see [20]) based on the growth of the resolvent on the imaginary axis. More precisely, we will give sufficient conditions that guarantee the polynomial decay of the energy of our system (for sufficiently smooth initial data). We actually obtain two different decay rates. In the first case, we will use the exponential decay of the wave equation with the standard damping

$$\frac{\partial y}{\partial \nu} = -y_t, \ \text{ on } \Gamma_1 \times \mathbb{R}_+^*,$$

and establish a polynomial energy decay rate of type $\dfrac{1}{t^{\frac{1}{4}}}$. In the second case, under a stronger geometrical conditions on Γ_0 and Γ_1, we establish a polynomial energy decay rate of type $\dfrac{1}{t}$.

In a second step, we want to show that such a polynomial decay seems to be also available even if the previous geometrical assumption is not satisfied. Therefore, we consider the case of the unit square of the plane where Γ_1 is only one edge of the boundary. In this case, using the separation of variables method, we study the spectrum of system (4.1.2) and we show that the decay of the energy to zero is not uniform. Then, using a method based on a Fourier analysis (compare with [71] where a similar method was used for the wave equation with Ventcel 's boundary conditions), a specific analysis of the obtained 1-d problem combining

Ingham's inequality and an interpolation method from [12], we establish a polynomial energy decay rate of type $\frac{1}{t}$ for sufficiently smooth initial data.

This chapter is organized as follows:

Section 4.2 deals with the well-posedness of the problem obtained by using semigroup theory. We further characterize the domain of the associated operator in some particular cases and obtain the strong stability. In section 4.3, we show that our system is not uniformly stable in the unit disc. Section 4.4 is devoted to the proof of the polynomial decay in the general setting by using the frequency domain approach. In section 4.5, we show that the energy of our system is not uniform stable in the unit square. In section 4.6, we obtain the polynomial stability result for a 1-d model with a parameter associated with (4.1.2). This result is then used in section 4.7 to show for the unit square a polynomial decay in $1/t$ of the energy for sufficiently smooth initial data.

Let us finish this section with some notations used in the remainder of the chapter. For a bounded domain D, the usual norm and semi-norm of $H^s(D)$ $(s \geq 0)$ are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively. For $s = 0$, we will drop the index s.

4.2 Well-posedness and strong stability

In this section, we study the existence, uniqueness and the asymptotic behavior of the solution of system (4.1.2).
If Γ_0 is non empty, we introduce the space $H^1_{\Gamma_0}(\Omega)$ as follows:

$$H^1_{\Gamma_0}(\Omega) = \left\{ u \in H^1(\Omega);\, u = 0 \quad \text{on} \quad \Gamma_0 \right\}, \tag{4.2.1}$$

which is a Hilbert space with the norm

$$\|u\|_{1,\Omega} = \|\nabla u\|_{\Omega}. \tag{4.2.2}$$

Next, we introduce the Hilbert space

$$\mathcal{H} = \left\{ \begin{array}{c} (u, v, w, z) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times H^1_0(\Gamma_1) \times L^2(\Gamma_1); \\ \gamma u = w \text{ on } \Gamma_1, \end{array} \right\} \tag{4.2.3}$$

endowed with the product

$$\Big((u^1,\, v^1,\, w^1,\, z^1), (u^2,\, v^2,\, w^2,\, z^2)\Big)_{\mathcal{H}} = (\nabla u^1, \nabla u^2)_\Omega + (v^1,\, v^2)_\Omega \\ + (\nabla_T w^1,\, \nabla_T w^2)_{\Gamma_1} + (z^1,\, z^2)_{\Gamma_1}, \tag{4.2.4}$$

$\forall (u^1,\, v^1,\, w^1,\, z^1),\, (u^2,\, v^2,\, w^2,\, z^2) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times H^1_0(\Gamma_1) \times L^2(\Gamma_1)$,

and the associated norm $\|\cdot\|_{\mathcal{H}} = (\cdot,\cdot)^{\frac{1}{2}}_{\mathcal{H}}$, γ being the usual trace operator from $H^1(\Omega)$ into $H^{\frac{1}{2}}(\Gamma)$. For simplicity, we will denote γu by u.

If Γ_0 is empty, we define $\mathcal{H}$ is the same manner, in this case we equip it with its natural norm: $\|(u, v, w, z)\|^2 := \|(u, v, w, z)\|^2_{\mathcal{H}} + \|u\|^2_\Omega + \|w\|^2_\Gamma$.

The energy of the solution of (4.1.2) is defined by

$$E(t) = \frac{1}{2}\|(u, u_t, w, w_t)\|^2_{\mathcal{H}}. \tag{4.2.5}$$

For smooth solution, a direct computation gives

$$\frac{d}{dt}E(t) = -\|w_t\|^2_{\Gamma_1}. \tag{4.2.6}$$

Then, system (4.1.2) is dissipative in the sense that its energy is a non-increasing function of the time variable t. We can now introduce the unbounded operator $\mathcal{A}$ on $\mathcal{H}$ with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (u,\, v,\, w,\, z) \in \mathcal{H}; \\ \Delta_T w - \partial_\nu u \in L^2(\Gamma_1), \\ \quad v \in H^1_{\Gamma_0}(\Omega), \quad \Delta u \in L^2(\Omega), \\ \quad z \in H^1_0(\Gamma_1), \quad v = z \text{ on } \Gamma_1 \end{array} \right\}, \tag{4.2.7}$$

defined by

$$\mathcal{A}U = \begin{pmatrix} v \\ \Delta u \\ z \\ \Delta_T w - \partial_\nu u - z \end{pmatrix}, \forall U = \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} \in D(\mathcal{A}). \tag{4.2.8}$$

Then, denoting (u, u_t, w, w_t) the state of system (4.1.2), we can rewrite system (4.1.2) into a first-order evolution equation

$$\begin{cases} U_t(t) &= \mathcal{A}U(t), \quad t > 0, \\ U(0) &= U_0 \in \mathcal{H}, \end{cases} \tag{4.2.9}$$

where $U_0 = (u_0, v_0, w_0, z_0) \in \mathcal{H}$. It is easy to show that $\mathcal{A}$ is a maximal dissipative operator, therefore its generates a C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ of contractions on the energy space $\mathcal{H}$ following Lumer-Phillips' theorem (see [75]). Hence, semigroup theory allows to show the next existence and uniqueness results:

Theorem 4.2.1. *For any initial data $U_0 \in \mathcal{H}$, the problem (4.2.9) has a unique weak solution $U(t) = e^{t\mathcal{A}}U_0$ such that $U \in C^0([0, +\infty[, \mathcal{H})$. Moreover, if $U_0 \in D(\mathcal{A})$, then the problem (4.2.9) has a strong solution $U(t) = e^{t\mathcal{A}}U_0$ such that $U \in C^1([0, +\infty[, \mathcal{H}) \cap C^0([0, +\infty[, D(\mathcal{A}))$.* ■

Now, we characterize the domain $D(\mathcal{A})$ of $\mathcal{A}$ in two different cases: either Γ is smooth enough and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ or Ω is the unit square. We start with the first situation:

Proposition 4.2.2. *If the boundary Γ of Ω is $C^{1,1}$ and if $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, then*

$$D(\mathcal{A}) = \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right) \times H^1_{\Gamma_0}(\Omega) \times \left(H^2(\Gamma_1) \cap H^1_0(\Gamma_1)\right) \times H^1_0(\Gamma_1),$$

with

$$\|(u, v, w, z)\|_{D(\mathcal{A})} \sim \|u\|_{2,\Omega} + \|v\|_{1,\Omega} + \|w\|_{2,\Gamma_1} + \|z\|_{1,\Gamma_1}, \forall (u, v, w, z) \in D(\mathcal{A}).$$

In particular, the resolvent $(I - \mathcal{A})^{-1}$ of $\mathcal{A}$ is compact on the energy space $\mathcal{H}$.

Proof: The proof is based on a bootstrap argument. Let us fix $U = (u, v, w, z) \in D(\mathcal{A})$, and set $h = \Delta_T w - \partial_\nu u - z$, that belongs to $L^2(\Gamma_1)$. Then by definition, $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$. Hence, by a result of Lions and Magenes (see the end of subsection 1.5 of [42]), we will have $\partial_\nu u \in H^{-\frac{1}{2}}(\Gamma_1)$ (as $\overline{\Gamma_0}$ and $\overline{\Gamma_1}$ are disjoint) with

$$\|\partial_\nu u\|_{-\frac{1}{2},\Gamma_1} \lesssim \|u\|_{1,\Omega} + \|\Delta u\|_\Omega. \tag{4.2.10}$$

Therefore $w \in H^1(\Gamma_1)$ satisfies

$$\Delta_T w = h + \partial_\nu u + z \in H^{-\frac{1}{2}}(\Gamma_1). \tag{4.2.11}$$

Hence by a standard shift theorem, we deduce that $w \in H^{\frac{3}{2}}(\Gamma_1)$ with

$$\begin{aligned}\|w\|_{\frac{3}{2},\Gamma_1} &\lesssim \|w\|_{1,\Gamma_1} + \|h + \partial_\nu u + z\|_{-\frac{1}{2},\Gamma_1} \\ &\lesssim \|w\|_{1,\Gamma_1} + \|h\|_{\Gamma_1} + \|\partial_\nu u\|_{-\frac{1}{2},\Gamma_1} + \|z\|_{\Gamma_1}.\end{aligned}$$

Thus by (4.2.10), we get

$$\|w\|_{\frac{3}{2},\Gamma_1} \lesssim \|w\|_{1,\Gamma_1} + \|h\|_{\Gamma_1} + \|u\|_{1,\Omega} + \|\Delta u\|_\Omega + \|z\|_{\Gamma_1}. \tag{4.2.12}$$

Now this improved regularity on w allows to look at $u \in H^1(\Omega)$ as the solution of the next boundary value problem:

$$\begin{cases} \Delta u & \in L^2(\Omega), \\ u = 0, & \text{on} \quad \Gamma_0, \\ u = w \in H^{\frac{3}{2}}(\Gamma_1), & \text{on} \quad \Gamma_1. \end{cases} \tag{4.2.13}$$

Hence again a standard shift theorem yields $u \in H^2(\Omega)$ with

$$\|u\|_{2,\Omega} \lesssim \|\Delta u\|_\Omega + \|w\|_{\frac{3}{2},\Gamma_1},$$

and hence by (4.2.12), we get

$$\|u\|_{2,\Omega} \lesssim \|w\|_{1,\Gamma_1} + \|h\|_{\Gamma_1} + \|u\|_{1,\Omega} + \|\Delta u\|_\Omega + \|z\|_{\Gamma_1}. \tag{4.2.14}$$

By a trace theorem, we deduce that $\partial_\nu u \in H^{\frac{1}{2}}(\Gamma_1)$ and coming back to (4.2.11), we deduce that

$$\Delta_T w = h + \partial_\nu u + z \in L^2(\Gamma_1).$$

Again a shift theorem yields $w \in H^2(\Gamma_1)$ with

$$\|w\|_{2,\Gamma_1} \lesssim \|w\|_{1,\Gamma_1} + \|\Delta u\|_\Omega + \|h + \partial_\nu u + z\|_{\Gamma_1}.$$

And by (4.2.14), we deduce that

$$\|w\|_{2,\Gamma_1} \lesssim \|w\|_{1,\Gamma_1} + \|h\|_{\Gamma_1} + \|u\|_{1,\Omega} + \|\Delta u\|_\Omega + \|z\|_{\Gamma_1}. \tag{4.2.15}$$

We have shown that

$$D(\mathcal{A}) \subset \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right) \times H^1_{\Gamma_0}(\Omega) \times \left(H^2(\Gamma_1) \cap H^1_0(\Gamma_1)\right) \times H^1_0(\Gamma_1).$$

On the other hand the estimates (4.2.14)-(4.2.15) yield

$$\|u\|_{2,\Omega}+\|v\|_{1,\Omega}+\|w\|_{2,\Gamma_1}+\|z\|_{1,\Gamma_1} \lesssim \|(u,v,w,z)\|_{D(\mathcal{A})}, \forall (u,v,w,z) \in D(\mathcal{A}),$$

reminding that $\|U\|_{D(\mathcal{A})} = \|U\|_{\mathcal{H}} + \|\mathcal{A}U\|_{\mathcal{H}}$. The converse inclusion and estimate being trivial, the proof is complete. ■

Corollary 4.2.3. *If the boundary Γ of Ω is $C^{2,1}$ and if $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, then*

$$\begin{aligned} D(\mathcal{A}^2) =& \Big(H^3(\Omega) \cap H^1_{\Gamma_0}(\Omega)\Big) \times \Big(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\Big) \\ & \times \Big(H^3(\Gamma_1) \cap H^1_0(\Gamma_1)\Big) \times \Big(H^2(\Gamma_1) \cap H^1_0(\Gamma_1)\Big), \end{aligned}$$

with

$$\|(u,v,w,z)\|_{D(\mathcal{A}^2)} \sim \|u\|_{3,\Omega}+\|v\|_{2,\Omega}+\|w\|_{3,\Gamma_1}+\|z\|_{2,\Gamma_1}, \forall (u,v,w,z) \in D(\mathcal{A}^2).$$

Proof: First, $U = (u,v,w,z)$ belongs to $D(\mathcal{A}^2)$ if and only if $U \in D(\mathcal{A})$ and $\mathcal{A}U \in D(\mathcal{A})$. Hence by the previous result we will have

$$\Delta u \in H^1(\Omega),$$

and $h = \Delta_T - \partial_\nu u - z \in H^1_0(\Gamma_1)$. Next, as the previous characterization yields $u \in H^2(\Omega)$, we know that $\partial_\nu u$ belongs to $H^{\frac{1}{2}}(\Gamma_1)$ and coming back to (4.2.11), we deduce that

$$\Delta_T w = h + \partial_\nu u + z \in H^{\frac{1}{2}}(\Gamma_1).$$

A shift theorem will lead to $w \in H^{\frac{5}{2}}(\Gamma_1)$. Then coming back to (4.2.13), the improved regularity on Δu and w, combined with a shift theorem give $u \in H^3(\Omega)$. Again coming back to (4.2.11), we deduce that $\Delta_T w = h + \partial_\nu u + z \in H^1(\Gamma_1)$, and therefore $w \in H^3(\Gamma_1)$.
This proves the result (for shortness we have skipped the estimates). ■

Proposition 4.2.4. *If Ω is the unit square with $\Gamma_1 = \{(0,y),\ y \in (0,1)\}$, and $\Gamma_0 = \Gamma \backslash \overline{\Gamma_1}$, then the statements of Proposition 4.2.2 and Corollary 4.2.3 are valid.*

Proof: The difficulty stays on the fact that Ω has a non smooth boundary and that $\overline{\Gamma_0} \cap \overline{\Gamma_1}$ is not empty. But we take advantage of the particular geometry.

Let us start with the characterization of $D(\mathcal{A})$. Let $U = (u,v,w,z)$ be in $D(\mathcal{A})$. Then by a localization argument and Proposition 4.2.2, we directly see that u (resp. w) belongs to $H^2(\Omega \setminus W)$ (resp. $H^2(\Gamma_1 \setminus W)$), where W is any neighborhood of the corners. Hence it remains to improve the regularity of u and w near the corners. But in a small neigborhood V of the corner $(1,0)$ (or $(1,1)$), as u is solution of a homogeneous Dirichlet problem with $\Delta u \in L^2$, it is wellknown (see Theorem 3.2.1.2 of [42] for instance) that $u \in H^2(V)$. Hence, the main difficulty is to show the regularity of u and w in a neighborhood V of the corner $(0,0)$ (or $(0,1)$). By symmetry, it suffices to look at the case of the corner $(0,0)$. Now fix a cut-off function $\eta \in \mathcal{D}(\mathbb{R}^2)$ such that $\eta = 1$ in the disc of center $(0,0)$ and radius $1/4$ and equal to 0 outside the disc of center $(0,0)$ and radius $1/2$. Then we easily check that ηU belongs to $D(\mathcal{A}_0)$, the operator $\mathcal{A}_0$ being our operator $\mathcal{A}$ but defined in the quater plane $Q = \{(x,y) \in \mathbb{R}^2; x, y > 0\}$, with $\Gamma_1 = \{(0,y) \in \mathbb{R}^2; y > 0\}$ and $\Gamma_0 = \{(x,0) \in \mathbb{R}^2; x > 0\}$.

Now the first statement holds if we show that

$$D(\mathcal{A}_0) \subset H^2(Q) \times H^1(Q) \times H^2(\Gamma_1) \times H^1_0(\Gamma_1). \tag{4.2.16}$$

For that purpose, we use a reflexion technique. Let us fix $(u, v, w, z) \in D(\mathcal{A})$ and introduce the function

$$\tilde{u}(x,y) = \begin{cases} u(x,y) & \text{if } y > 0, \\ -u(x,-y) & \text{if } y < 0, \end{cases}$$

defined in the half-plane $\mathbb{R}^2_+ := \{(x,y) \in \mathbb{R}^2 : x > 0\}$, and similarly

$$\tilde{w}(y) = \begin{cases} w(y) & \text{if } y > 0, \\ -w(-y) & \text{if } y < 0, \end{cases}$$

defined in the line $\{(0,y) \in \mathbb{R}^2; y \in \mathbb{R}\}$.

Now we denote by $\tilde{\mathcal{A}}_0$ our operator $\mathcal{A}$ but defined in the half-plane $\mathbb{R}^2_+$, with $\Gamma_1 = \{(0,y) \in \mathbb{R}^2; y \in \mathbb{R}\}$. Then by Proposition 4.2.2, it is clear that

$$D(\tilde{\mathcal{A}}_0) = H^2(\mathbb{R}^2_+) \times H^1(\mathbb{R}^2_+) \times H^2(\Gamma_1) \times H^1(\Gamma_1).$$

Hence (4.2.16) holds if we can show that $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})$ belongs to $D(\tilde{\mathcal{A}}_0)$. The only non trivial properties are to check that $\Delta\tilde{u}$ belongs to $L^2(\mathbb{R}^2_+)$ and that $\tilde{w}_{yy} - \partial_\nu \tilde{u}$ belongs to $L^2(\Gamma_1)$. For the first assertion, we show that

$$\Delta\tilde{u}(x,y) = \begin{cases} \Delta u(x,y) & \text{if } y > 0, \\ -\Delta u(x,-y) & \text{if } y < 0. \end{cases} \tag{4.2.17}$$

Indeed we take $\varphi \in \mathcal{D}(\mathbb{R}^2_+)$, we clearly have

$$\langle \Delta\tilde{u}, \varphi\rangle = \int_Q u\Delta d_\varphi,$$

where $d_\varphi \in H^1_{\Gamma_0}(Q)$ is defined by

$$d_\varphi(x,y) = \varphi(x,y) - \varphi(x,-y), \forall (x,y) \in Q.$$

Since d_φ is zero in a neighborhood of $(0,0)$, we can apply Theorem 1.5.3.6 of [42] and deduce that

$$\langle \Delta\tilde{u}, \varphi\rangle = \int_Q \Delta u d_\varphi,$$

and (4.2.17) follows.

Similarly we show that

$$\tilde{w}_{yy}(y) = \begin{cases} w_{yy}(y) & \text{if } y > 0, \\ -w_{yy}(-y) & \text{if } y < 0. \end{cases} \tag{4.2.18}$$

Finally for any $v \in H^1(\mathbb{R}^2_+)$, we have

$$\langle \partial_\nu \tilde{u}, v\rangle = \int_{\mathbb{R}^2_+} (\Delta\tilde{u}v + \nabla\tilde{u}\cdot\nabla v).$$

Hence by the previous argument, we have

$$\langle \partial_\nu \tilde{u}, v\rangle = \int_Q (\Delta u d_v + \nabla\ u\cdot\nabla d_v).$$

where $d_v \in H^1_{\Gamma_0}(Q)$. Hence by the definition of $\partial_\nu u$, we deduce that

$$\langle \partial_\nu \tilde{u}, v\rangle = \langle \partial_\nu u, d_v\rangle,$$

which means that

$$\partial_\nu\tilde{u}(y) = \begin{cases} \partial_\nu u(y) & \text{if } y > 0, \\ -\partial_\nu u(-y) & \text{if } y < 0. \end{cases} \tag{4.2.19}$$

For the second assertion $\tilde{w}_{yy} - \partial_\nu\tilde{u} \in L^2(\Gamma_1)$, if we dente by $h =$

$w_{yy} - \partial_\nu u$ that by assumption belongs to L^2, then (4.2.18) and (4.2.19) imply that

$$(\tilde{w}_{yy} - \partial_\nu \tilde{u})(y) = \begin{cases} h(y) & \text{if } y > 0, \\ -h(-y) & \text{if } y < 0, \end{cases} \tag{4.2.20}$$

and consequently it belongs to L^2 as well.

For the characterization of $D(\mathcal{A}^2)$, it suffices to notice that for $(u, v, w, z) \in D(\mathcal{A}_0^2)$, then

$$\Delta u \in H^1_{\Gamma_0}(Q).$$

In a neighborhood of the corner $(0, 0)$, we first notice that $\Delta\tilde{u}$ given by (4.2.17) belongs to $H^1(\mathbb{R}^2_+)$. Similarly $h = w_{yy} - \partial_\nu u$ belongs to H^1_0, and hence $\tilde{w}_{yy} - \partial_\nu \tilde{u}$ given by (4.2.20) belongs to H^1. This means that $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})$ belongs to $D(\tilde{\mathcal{A}}_0^2)$ and we conclude by Corollary 4.2.3. In a neighborhood of the corners $(1, 0)$ or $(1, 1)$, we simply use the same reflexion technique as before (see Lemma 2.4 of [45]) to get the H^3 regularity of u. ■

Now, we investigate the strong stability of system (4.2.9). But before going on, if Γ_0 is empty, we need to introduce the closed subspace

$$\mathcal{H}_0 = \{(u, v, w, z) \in \mathcal{H} : \int_\Omega v dx + \int_{\Gamma_1} z d\Gamma + \int_{\Gamma_1} w d\Gamma = 0\}$$

of $\mathcal{H}$ and the restriction $\mathcal{B}$ of $\mathcal{A}$ to $\mathcal{H}_0$, defined by $D(\mathcal{B}) = D(\mathcal{A}) \cap \mathcal{H}_0$, and

$$\mathcal{B}U = \mathcal{A}U, \quad \forall U \in D(\mathcal{B}).$$

Note that this definition is meaningful because for all $U \in D(\mathcal{A})$, $\mathcal{A}U$ belongs to $\mathcal{H}_0$. Hence $\mathcal{B}$ also generates a C_0-semigroup of contractions that is simply the restriction of $(e^{t\mathcal{A}})_{t\geq 0}$ to $\mathcal{H}_0$.

Theorem 4.2.5. *If Γ_0 is non empty, then the semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable on the energy space $\mathcal{H}$, i.e. for any $U_0 \in \mathcal{H}$, we have*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0. \tag{4.2.21}$$

If $\Gamma_0 = \emptyset$, then the semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable on the space $\mathcal{H}_0$. Further, for any $U_0 = (u_0, v_0, w_0, z_0) \in \mathcal{H}$, if $\alpha = \frac{1}{|\Gamma_1|}(\int_\Omega v_0 dx + \int_{\Gamma_1} z_0 d\Gamma + \int_{\Gamma_1} w_0 d\Gamma)$ (where $|\Gamma_1|$ means the measure of Γ_1), then

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}} U_0 - \alpha(1, 0, 1, 0)\|_{\mathcal{H}} = 0. \tag{4.2.22}$$

■

To prove the above theorem, we apply the strategy used in [72]. It is based on the theorem of Arendt and Batty in [90]. We need to proof the following two lemmas:

Lemma 4.2.6. *For all $\lambda \in \mathbb{R}^*$, we have*

$$\ker(i\lambda I - \mathcal{A}) = \{0\},$$

while

$$\ker(\mathcal{A}) = \{0\},$$

if Γ_0 is non empty, and

$$\ker(\mathcal{A}) = \textit{ Span } \{(1,0,1,0)\},$$

if Γ_0 is empty, but

$$\ker(i\lambda I - \mathcal{B}) = \{0\}, \forall \lambda \in \mathbb{R}. \tag{4.2.23}$$

Proof: Let $U = (u,\, v,\, w,\, z) \in D(\mathcal{A})$ and let $\lambda \in \mathbb{R}$, such that

$$\mathcal{A}U = i\lambda U. \tag{4.2.24}$$

First, by detailing (4.2.24) we get

$$\begin{cases} v & = & i\lambda u, \\ \Delta u & = & i\lambda v, \\ z & = & i\lambda w, \\ \Delta_T w - \frac{\partial u}{\partial \nu} - z & = & i\lambda z. \end{cases} \tag{4.2.25}$$

Next, a straightforward computation gives

$$\Re(\mathcal{A}U, U)_{\mathcal{H}} = -\int_{\Gamma_1} |z|^2 dx. \tag{4.2.26}$$

Then, using (4.2.24) and (4.2.26) we deduce that

$$z = 0, \quad \text{on} \quad \Gamma_1. \tag{4.2.27}$$

Now, we distinguish two cases:
Case 1: $\lambda \neq 0$**.** Using (4.2.27) and the third equation of system (4.2.25), we deduce that $u = w = 0$ on Γ_1. Thus, by eliminating v, the system

(4.2.25) implies that

$$\begin{cases} \Delta u+\lambda^2 u &= 0, \quad \text{in} \quad \Omega, \\ u &= 0, \quad \text{on} \quad \Gamma, \\ \partial_\nu u &= 0, \quad \text{on} \quad \Gamma_1. \end{cases} \tag{4.2.28}$$

Therefore, using Holmgren's theorem, we deduce that $u=0$ and consequently, $U=0$.

Case 2: $\lambda = 0$**.** The system (4.2.25) becomes

$$\begin{cases} v &= 0, \quad \text{in} \quad \Omega, \\ \Delta u &= 0, \quad \text{in} \quad \Omega, \\ z &= 0, \quad \text{in} \quad \Gamma_1, \\ \Delta_T w-\partial_\nu u &= 0, \quad \text{on} \quad \Gamma_1. \end{cases} \tag{4.2.29}$$

By integrating by parts and using the boundary conditions $u=0$ on Γ_0 and $w=0$ on $\partial\Gamma_1$, we have

$$\begin{aligned} 0=\int_\Omega \Delta u\bar{u} &= -\int_\Omega |\nabla u|^2+\int_{\Gamma_1}\partial_\nu u\bar{u} \\ &= -\int_\Omega |\nabla u|^2-\int_{\Gamma_1}|\nabla_T u|^2. \end{aligned}$$

Hence u is constant in the whole domain Ω. Therefore if Γ_0 is non empty we deduce that $u=w=0$ and directly conclude that $\ker(i\lambda I-\mathcal{A})=\{0\}$. On the other hand, if Γ_0 is empty, then $u=w$ constant is allowed and we find that $(1,0,1,0)$ is the sole eigenvector of $\mathcal{A}$ of eigenvalue 0. But since $(1,0,1,0)$ does not belongs to $\mathcal{H}_0$, 0 is not an eigenvalue of $\mathcal{B}$ and consequently we deduce that (4.2.23) holds. ■

Lemma 4.2.7. *If $\Gamma_0 \neq \emptyset$, for all $\lambda\in\mathbb{R}$, we have*

$$\mathrm{R}(i\lambda I-\mathcal{A})=\mathcal{H},$$

while if $\Gamma_0=\emptyset$, for all $\lambda\in\mathbb{R}$, we have

$$\mathrm{R}(i\lambda I-\mathcal{B})=\mathcal{H}_0.$$

Proof: We give the proof in the case $\Gamma_0\neq\emptyset$, the proof of the second statement is fully similar by using (4.2.23). Let $F=(f,g,h,k)\in\mathcal{H}$, then we look for $U=(u,v,w,z)\in D(\mathcal{A})$ such that

$$i\lambda U-\mathcal{A}U=F, \tag{4.2.30}$$

or equivalently

$$\begin{cases} i\lambda u - v & = & f, \\ i\lambda v - \Delta u & = & g, \\ i\lambda w - z & = & h, \\ i\lambda z - \Delta_T w + \partial_\nu u + z & = & k. \end{cases} \tag{4.2.31}$$

From the first and the third identities of (4.2.31) and the fact that $w = u$ on Γ_1, we get

$$\begin{cases} -\Delta u - \lambda^2 u & = & g + i\lambda f, & \text{in} & \Omega, \\ -\lambda^2 u - \Delta_T u + \partial_\nu u + i\lambda u & = & k + (i\lambda - 1)h, & \text{on} & \Gamma_1. \end{cases} \tag{4.2.32}$$

Next, we define the space V by

$$V = \left\{ u \in H^1_{\Gamma_0}(\Omega) : u \in H^1_0(\Gamma_1) \right\},$$

endowed with the norm

$$\|u\|_V^2 = \|\nabla u\|_\Omega^2 + \|\nabla_T u\|_{\Gamma_1}^2.$$

Multiplying the first equation of (4.2.32) by $\tilde{u} \in V$, integrating in Ω and using the second equation of the same problem, and formal integration by parts, we get formally the following identity:

$$a_\lambda(u, \tilde{u}) = L_\lambda(\tilde{u}), \tag{4.2.33}$$

where a_λ is a bilinear form from $V \times V$ into $\mathbb{C} \times \mathbb{C}$ given by

$$\begin{aligned} a_\lambda(u, \tilde{u}) = & \int_\Omega (\nabla u \cdot \nabla \tilde{u} - \lambda^2 u \tilde{u}) dx \\ & + \int_{\Gamma_1} (\nabla_T u \cdot \nabla_T \tilde{u} + (i\lambda - \lambda^2) u \tilde{u}) d\Gamma, \end{aligned} \tag{4.2.34}$$

and L_λ is a linear form from V into $\mathbb{C}$ defined by

$$L_\lambda(\tilde{u}) = \int_\Omega (g + i\lambda f)\tilde{u} dx + \int_{\Gamma_1} (k + (i\lambda - 1)h)\tilde{u} d\Gamma. \tag{4.2.35}$$

Now, we introduce the operator $\mathcal{A}_\lambda : V \to V'$ by

$$< \mathcal{A}_\lambda u, \tilde{u} >_{V', V} = a_\lambda(u, \tilde{u}), \quad \forall \tilde{u} \in V.$$

For λ, $\lambda' \in \mathbb{R}$, we have

$$
\begin{aligned}
|< (\mathcal{A}_\lambda - \mathcal{A}_{\lambda'})u, \tilde{u} >_{V',V}| &= |a_\lambda(u,\tilde{u}) - a_{\lambda'}(u,\tilde{u})| \\
&\leq \left| \int_\Omega (\lambda'^2 - \lambda^2) u\tilde{u} dx \right| \\
&\quad + \left| \int_{\Gamma_1} (i(\lambda - \lambda') + (\lambda'^2 - \lambda^2)) u\tilde{u} d\Gamma \right| \\
&\leq C_{\lambda,\lambda',\Omega} \|u\|_V (\|\tilde{u}\|_{L^2(\Omega)} + \|\tilde{u}\|_{L^2(\Gamma_1)}) \\
&\leq C_{\lambda,\lambda',\Omega} \|u\|_V \|\tilde{u}\|_{H^{1/2+\varepsilon}_{\Gamma_0}(\Omega)}.
\end{aligned}
$$

This implies that

$$
\mathcal{A}_\lambda - \mathcal{A}_{\lambda'} \in \mathcal{L}\left(V; H^{1/2+\varepsilon}_{\Gamma_0}(\Omega)' \right)
$$

and thus $\mathcal{A}_\lambda - \mathcal{A}_{\lambda'}$ is a compact operator from V into V'. On the other hand, since $\Gamma_0 \neq \emptyset$, then, it is easy to see that the operator $\mathcal{A}_0$ is an isomorphism and consequently, it is a Fredholm operator of index zero. It follows, from the compactness of $\mathcal{A}_\lambda - \mathcal{A}_{\lambda'}$, that $\mathcal{A}_\lambda$ is also a Fredholm operator of index zero for all λ. Therefore, $\mathcal{A}_\lambda$ is surjective if and only if it is injective. Using Lemma 4.2.6 we deduce the injectivity of the operator $\mathcal{A}_\lambda$ (compare with Proposition 3.3 in [72]). This means that $\mathcal{A}_\lambda$ is an isomorphism for all $\lambda \in \mathbb{R}$ and therefore problem (4.2.33) has a unique solution $u \in V$. By choosing appropriated test functions in (4.2.33), we see that u satisfies (4.2.32). By defining $w = u$, $z = i\lambda w - h$ on Γ_1 and $v = i\lambda u - f$ in Ω, we deduce that $U = (u, v, w, z)$ belongs to $D(\mathcal{A})$ and is solution of (4.2.30). This completes the proof. ■

Proof of Theorem 4.2.5: We distinguish two cases:

Case 1. $\Gamma_0 \neq \emptyset$**.** Using Lemmas 4.2.6 and 4.2.7, we directly deduce that the imaginary axis is included in the resolvent set of $\mathcal{A}$. We then conclude (4.2.21) with the help of Arendt-Batty's theorem [90].

Case 2. $\Gamma_0 = \emptyset$**.** As before using Lemmas 4.2.6 and 4.2.7 and Arendt-Batty's theorem, we conclude that the semigroup generated by $\mathcal{B}$ is stable, in other words

$$
\lim_{t \to +\infty} \|e^{t\mathcal{B}} \widetilde{U}_0\|_{\mathcal{H}} = 0, \quad \forall \widetilde{U}_0 \in \mathcal{H}_0.
$$

But, for $U_0 \in \mathcal{H}$ and α given as in the second statement of Theorem

4.2.5, we notice that

$$\widetilde{U}_0 := U_0 - \alpha(1,0,1,0)$$

belongs to $\mathcal{H}_0$. The conclusion then follows by noticing that $e^{t\mathcal{A}}(1,0,1,0) = (1,0,1,0)$. The proof is thus completed (compare with Theorem 4.3.2 of [30]). ∎

4.3 Non-uniform stability result

In this section we show that the uniform stability (*i.e.* exponential stability) of problem (4.2.9) does not hold in general, since it is already the case for the unit disk D of $\mathbb{R}^2$ and $\Gamma_0 = \emptyset$ as shown below. This result is due to the fact that a subsequence of eigenvalues of $\mathcal{A}$ which is close to the imaginary axis. First, let $U = (u, v, w, z) \in D(\mathcal{A})$ such that $\mathcal{A}U = \lambda U$. Equivalently we have

$$\begin{cases} v & = & \lambda u, & \text{in} & D, \\ \Delta u & = & \lambda v, & \text{in} & D, \\ z & = & \lambda w, & \text{on} & \partial D, \\ \Delta_T w - \partial_\nu u - z & = & \lambda z, & \text{on} & \partial D. \end{cases}$$

Next, by eliminating v and z from the above system and using the fact that $u = w$ on Γ_1 we get the following system:

$$\begin{cases} \Delta u - \lambda^2 u & = & 0, & \text{in} & D, \\ \Delta_T u - \partial_\nu u - \lambda(\lambda+1)u & = & 0, & \text{on} & \partial D. \end{cases} \tag{4.3.1}$$

A radial solution $u(r,\theta) = f(r)$ of (4.3.1) is a solution of

$$\begin{cases} f''(r) + \dfrac{1}{r} f'(r) - \lambda^2 f(r) & = & 0, & r \in (0,1), \\ f'(1) + (\lambda^2 + \lambda) f(1) & = & 0. \end{cases} \tag{4.3.2}$$

If $\lambda \neq 0$, the general solution of the first equation of (4.3.2) is given by

$$f(r) = c_J J_0(i\lambda r) + c_Y Y_0(i\lambda r), \quad c_J,\ c_Y \in \mathbb{C},$$

where J_0 (resp. Y_0) is the Bessel of first (resp. second) kind. Since u is regular in D, necessarily we have $c_Y = 0$ and $c_J \neq 0$. Therefore, using the second equation of (4.3.2), we find that if $\lambda \in \mathbb{C}$ and $\lambda \neq 0$ satisfies

$$- i\lambda J_1(i\lambda) + (\lambda^2 + \lambda)J_0(i\lambda) = 0, \tag{4.3.3}$$

then λ is an eigenvalue of $\mathcal{A}$ where J_1 is the Bessel function which satisfies $J_1 = -J_0'$. Our goal is to find large eigenvalues which are closed to the imaginary axis and to give their expansion. For that reason, we fix $c > 0$ large enough and we consider the solution of (4.3.3) which are in the strip

$$S = \{\lambda \in \mathbb{C}; -c \leq \Re|\lambda| \leq c\}.$$

For convenience, we set $\phi(\lambda) = \dfrac{1}{\lambda^2}\sqrt{\dfrac{i\pi\lambda}{2}}(-i\lambda J_1(i\lambda) + (\lambda^2 + \lambda)J_0(i\lambda))$, thus (4.3.3) is equivalent to the following characteristic equation:

$$\phi(\lambda) = 0. \tag{4.3.4}$$

By the following proposition we give the asymptotic behavior of the eigenvalue of high frequency associated to the radial solution $f(r)$ of problem (4.3.2) in S:

Proposition 4.3.1. *There exists $k_0 \in \mathbb{N}^*$ and a sequence $(\lambda_k)_{k\geq k_0}$ of simple roots of ϕ (that are also simple eigenvalues of $\mathcal{A}$) and satisfying the following asymptotic behavior:*

$$\lambda_k = i\left(k\pi - \frac{\pi}{4} + \frac{9}{8k\pi} + \frac{9}{32\pi k^2}\right) - \frac{1}{\pi^2 k^2} + o(\frac{1}{k^2}). \tag{4.3.5}$$

Proof: For clarity, the proof is divided into two steps.
Step 1. First, using the asymptotic expansions of Bessel's functions (see equation 10.7.3 [74] for instance) for $\lambda \in S$ we have

$$\begin{aligned}\sqrt{\frac{i\pi\lambda}{2}}J_0(i\lambda) =& i\cos(\frac{\pi}{4} + i\lambda) + \frac{1}{8\lambda}\cos(\frac{\pi}{4} - i\lambda)\\ &+ \frac{9i}{128\lambda^2}\cos(\frac{\pi}{4} + i\lambda) + O(\frac{1}{|\lambda|^3}), \qquad |\lambda| \to \infty\end{aligned} \tag{4.3.6}$$

and

$$\begin{aligned}\sqrt{\frac{i\pi\lambda}{2}}(-i\lambda J_1(i\lambda)) =& \lambda\cos(\frac{\pi}{4} - i\lambda) - \frac{3}{8}i\cos(\frac{\pi}{4} + i\lambda)\\ &+ O(\frac{1}{|\lambda|}), \qquad |\lambda| \to \infty.\end{aligned} \tag{4.3.7}$$

Next, from (4.3.6) and (4.3.7) it follows that for $\lambda \in S$ we have

$$\begin{aligned}\phi(\lambda) =& i\cos(\frac{\pi}{4}+i\lambda) \\ &+\left[\frac{9}{8}\cos(\frac{\pi}{4}-i\lambda)+i\cos(\frac{\pi}{4}+i\lambda)\right]\frac{1}{\lambda} \\ &+\left[-\frac{39}{128}i\cos(\frac{\pi}{4}+i\lambda)+\frac{1}{8}\cos(\frac{\pi}{4}-i\lambda)\right]\frac{1}{\lambda^2} \\ &+O(\frac{1}{|\lambda|^3}).\end{aligned} \tag{4.3.8}$$

Then, since the roots of the analytic function $\lambda \mapsto \cos(\frac{\pi}{4}+i\lambda)$ are $\lambda_k^0 = ik\pi - i\frac{\pi}{4}, k \in \mathbb{Z}$, using Rouché's theorem, we deduce from (4.3.8) that ϕ admits an infinity of simple roots in S denoted by λ_k, with $|k| \geq k_0$, k_0 large enough, such that

$$\lambda_k = \lambda_k^0 + o(1) = ik\pi - i\frac{\pi}{4} + o(1),\ |k| \to \infty.$$

Equivalently, we have

$$\lambda_k = ik\pi - i\frac{\pi}{4} + \epsilon_k \quad \text{and} \quad \lim_{|k|\to\infty} \epsilon_k = 0. \tag{4.3.9}$$

Step 2. Asymptotic behavior of ϵ_k: First, using (4.3.9) we obtain

$$\cos(\frac{\pi}{4}+i\lambda_k) = -i(-1)^k\epsilon_k + o(\epsilon_k), \tag{4.3.10}$$

$$\cos(\frac{\pi}{4}-i\lambda_k) = (-1)^k + o(\epsilon_k) \tag{4.3.11}$$

and

$$\frac{1}{\lambda_k} = -\frac{i}{k\pi} + o(\frac{1}{k}). \tag{4.3.12}$$

Next, by inserting (4.3.10)-(4.3.12) in the identity $\phi(\lambda_k) = 0$ and keeping only the terms of order $\frac{1}{k}$, we find after a simplification

$$(-1)^k\epsilon_k + o(\epsilon_k) - \frac{9i(-1)^k}{8k\pi} + o(\frac{1}{k}) = 0,$$

thus

$$\epsilon_k = \frac{9i}{8k\pi} + o(\frac{1}{k}).$$

Later, from the above equality we can write $\lambda_k = ik\pi - i\frac{\pi}{4} + \frac{9i}{8k\pi} + \frac{\epsilon_k}{k}$, $\lim_{|k|\to\infty} \epsilon_k = 0$. That implies

$$\cos(\frac{\pi}{4} + i\lambda_k) = \frac{9(-1)^k}{8k\pi} - \frac{i(-1)^k\epsilon_k}{k} + o(\frac{1}{k^2}), \tag{4.3.13}$$

$$\cos(\frac{\pi}{4} - i\lambda_k) = (-1)^k - \frac{81(-1)^k}{128\pi^2 k^2} + o(\frac{1}{k^2}), \tag{4.3.14}$$

$$\frac{1}{\lambda_k} = -\frac{i}{k\pi} - \frac{i}{4k^2\pi} + o(\frac{1}{k^2}) \tag{4.3.15}$$

and

$$\frac{1}{\lambda_k^2} = -\frac{1}{k^2\pi^2} + o(\frac{1}{k^2}). \tag{4.3.16}$$

Inserting (4.3.13)-(4.3.16) in the equation $\phi(\lambda_k) = 0$ and keeping only the terms of order $\frac{1}{k^2}$ with find after simplifications

$$\frac{(-1)^k\epsilon_k}{k} + \frac{(-1)^k}{k^2\pi^2} - \frac{9i(-1)^k}{32k^2\pi} + o(\frac{1}{k^2}) = 0,$$

thus

$$\epsilon_k = \frac{9i}{32k\pi} - \frac{1}{k\pi^2} + o(\frac{1}{k}).$$

Finally, we find

$$\lambda_k = i(k\pi - \frac{\pi}{4} + \frac{9}{8k\pi} + \frac{9}{32k^2\pi}) - \frac{1}{k^2\pi^2} + o(\frac{1}{k^2}).$$

■

As (4.3.5) shows that the eigenvalues λ_k of $\mathcal{A}$ approach the imaginary axis as k goes to infinity, system (4.2.9) in the unit disc is clearly not uniformly stable.

The asymptotic behavior of λ_k in (4.3.5) can be numerically validated. Namely, from (4.3.5) we have

$$-\lim_{k\to+\infty} k^2\pi^2\Re(\lambda_k) = 1.$$

The table below confirms this behavior.

k	100	150	200	250	300	350	400	450	500
$-\pi^2k^2\Re(\lambda_k)$	1.00495	1.00331	1.00249	1.00199	1.00166	1.00142	1.00125	1.00111	1.001

4.4 Polynomial energy decay rate

In this section we study the polynomial decay rate of the energy of problem (4.2.9) under appropriated conditions. First, we consider the following auxiliary problem:

$$\begin{cases} \varphi_{tt}(x,t) - \Delta\varphi(x,t) & = & 0, & x \in \Omega, & t > 0, \\ \varphi(x,t) & = & 0, & x \in \Gamma_0, & t > 0, \\ \partial_\nu \varphi(x,t) & = & -\varphi_t(x,t), & x \in \Gamma_1, & t > 0. \end{cases} \tag{4.4.1}$$

Next, we denote by (H_1) the following condition:

$$(H_1): \text{ the problem (4.4.1) is uniformly stable in } H^1_{\Gamma_0}(\Omega) \times L^2(\Omega),$$

or equivalently there exist two positive constants C and w such that for any $(\varphi_0, \varphi_1) \in H^1_{\Gamma_0} \times L^2(\Omega)$, the solution φ of (4.4.1) with initial conditions

$$\varphi(\cdot, 0) = \varphi_0, \quad \varphi_t(\cdot, 0) = \varphi_1,$$

satisfies

$$|\varphi(\cdot,t)|^2_{1,\Omega} + \|\varphi_t(\cdot,t)\|^2_\Omega \le Ce^{-wt}(|\varphi_0|^2_{1,\Omega} + \|\varphi_1\|^2_\Omega), \quad \forall t \ge 0.$$

Alternatively, we recall the multiplier control condition **MCC** by the following definition:

Definition 4.4.1. *We say that the boundary Γ of Ω satisfies the multiplier control condition* **MCC**, *if there exists $x_0 \in \mathbb{R}^d$ and a positive constant $m_0 > 0$ such that*

$$m \cdot \nu \le 0 \quad on \quad \Gamma_0 \quad and \quad m \cdot \nu \ge m_0 \quad on \quad \Gamma_1,$$

with $m(x) = x - x_0 \in \mathbb{R}^d$. ■

Remark 4.4.2. *In [16], Bardos* and al., *proved that (H_1) holds if Γ is smooth (of class C^∞), $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and under a geometric control condition named by* **GCC**. *We say that Γ satisfies the geometric control condition* **GCC**, *if every ray of geometrical optics, starting at any point $x \in \Omega$ at time $t = 0$, hits Γ_1 in finite time T. For less regular domains, namely of class C^2, (H_1) holds if the vector field assumptions described in [54] (see (i), (ii), (iii) of Theorem* 1 *in [54]) hold. Moreover, in Theorem* 1.2 *of [56] the authors prove that (H_1) holds for smooth domains under* *weaker geometric conditions than in [54] (without (ii) of Theorem 1). It is easy to see that the multiplier control condition* **MCC** *implies that the vector field assumptions described in [54] are satisfied and therefore the condition* (H_1) *holds if* **MCC** *holds.* ■

Next, we present our main result of this section by the following theorem:

Theorem 4.4.3. *Assume that* $\Gamma_0 \neq \emptyset$ *and* $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$.

1. *Assume that the boundary* Γ *of* Ω *is Lipschitz and that the condition* (H_1) *holds. Then for all initial data* $U_0 \in D(\mathcal{A})$*, there exists a constant* $c > 0$ *independent of* U_0*, such that the solution of the problem* (4.2.9) *satisfies the following estimation:*

$$E(t) \leq \frac{c}{t^{\frac{1}{4}}} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{4.4.2}$$

2. *Assume that the boundary* Γ *of* Ω *is* $C^{1,1}$ *and that the multiplier control condition* **MCC** *on* Γ_1 *holds. Then for all initial data* $U_0 \in D(\mathcal{A})$*, there exists a constant* $c > 0$ *independent of* U_0*, such that the solution of problem* (4.2.9) *satisfies the following estimation:*

$$E(t) \leq \frac{c}{t} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{4.4.3}$$

■

In order to prove our results, we will use Theorem 2.4 of [20]. A C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ in a Hilbert space $\mathcal{H}$ satisfies (4.4.2) (respectively (4.4.3)) if

$$(H_2) : i\mathbb{R} \subset \rho(\mathcal{A}),$$
$$(H_3) : \sup_{|\beta|\geq 1} \frac{1}{|\beta|^l} \left\|(i\beta I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < +\infty$$

hold with $l = 8$ (respectively with $l = 2$). As condition (H_2) was already checked in Theorem 4.2.5, we now prove that condition (H_3) holds, using an argument of contradiction. For this aim, we suppose that there exists a sequence $\beta_n \in \mathbb{R}$ such that $\beta_n \xrightarrow[n\to+\infty]{} +\infty$, and a sequence $U_n = (u_n, v_n, w_n, z_n) \in D(\mathcal{A})$ such that

$$\|U_n\|_{\mathcal{H}} = 1 \tag{4.4.4}$$

and

$$\beta_n^l \|(i\beta_n I - \mathcal{A})U_n\|_{\mathcal{H}} \xrightarrow[n\to+\infty]{} 0. \tag{4.4.5}$$

For simplification, we denote β_n by β, $U_n = (u_n, v_n, w_n, z_n)$ by $U = (u, v, w, z)$ and $F_n = \beta_n^l(i\beta_n I - \mathcal{A})U_n = (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n})$ by $F = (f_1, f_2, f_3, f_4)$. Next, by detailing (4.4.5) we obtain

$$\begin{cases} \beta^l\,(i\beta u - v) & = & f_1 \longrightarrow 0 & \text{in} & H^1_{\Gamma_0}(\Omega), \\ \beta^l\,(i\beta v - \Delta u) & = & f_2 \longrightarrow 0 & \text{in} & L^2(\Omega), \\ \beta^l\,(i\beta w - z) & = & f_3 \longrightarrow 0 & \text{in} & H^1_0(\Gamma_1), \\ \beta^l\,(i\beta z - \Delta_T w + \partial_\nu u + z) & = & f_4 \longrightarrow 0 & \text{in} & L^2(\Gamma_1). \end{cases} \tag{4.4.6}$$

Later, by eliminating v and z from system (4.4.6) and since $u = w$ on Γ_1 we obtain

$$\begin{cases} \beta^2 u + \Delta u & = & -\dfrac{f_2 + i\beta f_1}{\beta^l}, \\ \beta^2 u + \Delta_T u - \partial_\nu u - i\beta u & = & -\dfrac{f_4 + (1 + i\beta) f_3}{\beta^l}. \end{cases} \tag{4.4.7}$$

Lemma 4.4.4. *The solution $(u, v, w, z) \in D(\mathcal{A})$ of system (4.4.6) satisfies the following estimation:*

$$\int_{\Gamma_1} |u|^2\, d\Gamma = \frac{o(1)}{\beta^{l+2}}. \tag{4.4.8}$$

Proof: First, multiplying equation (4.4.5) by U in $\mathcal{H}$, we get

$$\int_{\Gamma_1} |z|^2 d\Gamma = \Re\,(i\beta U - \mathcal{A}U, U)_{\mathcal{H}} = \frac{o(1)}{\beta^l}. \tag{4.4.9}$$

Next, using the third equation of system (4.4.6) and using (4.4.9), we get

$$\int_{\Gamma_1} |w|^2 d\Gamma = \frac{o(1)}{\beta^{l+2}}. \tag{4.4.10}$$

Finally, since $u = w$ on Γ_1, from (4.4.10) we deduce directly (4.4.8). ■

Before going on, we give a relation between u and $\partial_\nu u$ by the following lemma:

Lemma 4.4.5. *Let Ω be a bounded domain of $\mathbb{R}^d$, $d \geq 1$, with Lipschitz* *boundary. Let $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$. Then*

$$u \in H^1(\Gamma) \Longleftrightarrow \partial_\nu u \in L^2(\Gamma) \tag{4.4.11}$$

and in this case we have

$$\|\Delta u\|_\Omega + \|u\|_{1,\Gamma} \sim \|\partial_\nu u\|_\Gamma + \|\Delta u\|_\Omega. \tag{4.4.12}$$

Proof: First, we denote by $h = \Delta u$ and we set

$$\widetilde{h} = \begin{cases} h & \text{in} \quad \Omega, \\ 0 & \text{in} \quad \mathbb{R}^d \backslash \Omega. \end{cases}$$

Moreover, we consider O a smooth domain such that $\overline{\Omega} \subset O$. Next, let $\tilde{w} \in H_0^1(O)$ be a solution of

$$\Delta \tilde{w} = \widetilde{h} \quad \text{in} \quad O.$$

Then $\tilde{w} \in H^2(O)$ and we have

$$\|\tilde{w}\|_{2,O} \lesssim \|\widetilde{h}\|_O \lesssim \|h\|_\Omega. \tag{4.4.13}$$

Consequently $v = u - \tilde{w} \in H^1(\Gamma)$ and satisfies $\Delta v = 0$ in Ω. On the other hand, using Lemma 1 of [26], we deduce that

$$v \in H^1(\Gamma) \Longleftrightarrow \partial_\nu v \in L^2(\Gamma) \tag{4.4.14}$$

and

$$\|v\|_{1,\Gamma} \sim \|\partial_\nu v\|_\Gamma. \tag{4.4.15}$$

As $u = v + \tilde{w}$ and $\partial_\nu u = \partial_\nu v + \partial_\nu \tilde{w}$ and since by (4.4.13) $\tilde{w} \in H^1(\Gamma)$ and $\partial_\nu \tilde{w} \in L^2(\Gamma)$, using (4.4.14)-(4.4.15) we deduce that

$$u \in H^1(\Gamma) \Longleftrightarrow \partial_\nu u \in L^2(\Gamma).$$

Now, to prove estimate (4.4.12), we notice that

$$\begin{aligned} \|u\|_{1,\Gamma} &= \|v + \tilde{w}\|_{1.\Gamma} \\ &\leq \|v\|_{1,\Gamma} + \|\tilde{w}\|_{1,\Gamma}. \end{aligned}$$

Hence, using (4.4.15) we get

$$\begin{aligned}\|u\|_{1,\Gamma} &\lesssim \|\partial_\nu v\|_\Gamma + \|\tilde{w}\|_{1,\Gamma}\\ &\lesssim \|\partial_\nu u\|_\Gamma + \|\partial_\nu \tilde{w}\|_\Gamma + \|\tilde{w}\|_{1,\Gamma}.\end{aligned}$$

Finally, by using trace result theorem and (4.4.13) we obtain

$$\begin{aligned}\|u\|_{1,\Gamma} &\lesssim \|\partial_\nu u\|_\Gamma + \|\tilde{w}\|_{2,\Omega}\\ &\lesssim \|\partial_\nu u\|_\Gamma + \|h\|_\Omega.\end{aligned}$$

The converse inequality is proved similarly. ■

Lemma 4.4.6. *Assume that the boundary Γ of Ω is Lipschitz, $\Gamma_0 \neq \emptyset$ and $l \geq 1$. Then, the solution $(u, v, w, z) \in D(\mathcal{A})$ of system (4.4.6) satisfies the following estimation:*

$$\int_{\Gamma_1} |\partial_\nu u|^2 d\Gamma = O(\beta^2). \tag{4.4.16}$$

Proof: First, since $u \in H_0^1(\Gamma_1)$ and since $u = 0$ on Γ_0, we have $u \in H^1(\Gamma)$. Next, as $\Delta u \in L^2(\Omega)$ and the boundary Γ of Ω is Lipschitz, then using (4.4.12) and Poincaré's inequality we obtain

$$\begin{aligned}\|\partial_\nu u\|_\Gamma &\lesssim \|\Delta u\|_\Omega + \|u\|_{1,\Gamma}\\ &\lesssim \|\Delta u\|_\Omega + \|\nabla_T u\|_{\Gamma_1}.\end{aligned} \tag{4.4.17}$$

Next, using the first equation of system (4.4.7) we have

$$\|\Delta u\|_\Omega \lesssim \beta^2 \|u\|_\Omega + \frac{o(1)}{\beta^{l-1}}. \tag{4.4.18}$$

Moreover, since by the first equation of (4.4.6) and by (4.4.4), we have βu and $\nabla_T u$ are uniformly bounded in $L^2(\Omega)$ and in $L^2(\Gamma_1)$ respectively. Finally, combining (4.4.17)-(4.4.18) with $l \geq 1$, we deduce (4.4.16). ■

Lemma 4.4.7. *Assume that the boundary Γ of Ω is $C^{1,1}$, $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, $l \geq 1$ and that the multiplier control condition* **MCC** *on Γ_1 holds. Then, the solution $(u, v, w, z) \in D(\mathcal{A})$ of system (4.4.6) satisfies the following estimation:*

$$\int_{\Gamma_1} |\partial_\nu u|^2 d\Gamma = O(1). \tag{4.4.19}$$

Proof: First, we define the cut off function $\eta \in C^2(\overline{\Omega})$ by

$$\eta(x) = \begin{cases} 1 & x \in \Gamma_1, \\ 0 & x \in \Omega \backslash O_\alpha, \end{cases} \tag{4.4.20}$$

where O_α is a neighborhood of Γ_1 given by

$$O_\alpha = \left\{ x \in \Omega;\ \inf_{y \in \Gamma_1} |x - y| \leq \alpha \right\} \tag{4.4.21}$$

and where α is a positive constant small enough such that $\overline{\Gamma_0} \cap O_\alpha = \emptyset$. Next, multiplying the first equation of system (4.4.7) by $2\eta m \cdot \nabla \overline{u}$ we get

$$2\beta^2 \int_\Omega \eta u(m \cdot \nabla \overline{u}) dx + 2 \int_\Omega \eta \Delta u(m \cdot \nabla \overline{u}) dx = \frac{o(1)}{\beta^{l-1}}. \tag{4.4.22}$$

On the other hand, by integrating by parts we obtain

$$\begin{aligned} 2\beta^2 \Re \int_\Omega \eta u(m \cdot \nabla \overline{u}) dx = & - d \int_\Omega \eta |\beta u|^2 dx - \int_\Omega (m \cdot \nabla \eta) |\beta u|^2 dx \\ & + \int_{\Gamma_1} (m \cdot \nu) |\beta u|^2 d\Gamma. \end{aligned} \tag{4.4.23}$$

Moreover, since $U \in D(\mathcal{A})$ then using Proposition 4.2.2 we have $\eta u \in H^2(\Omega)$. Then, using Green's formula we can easily check that

$$\begin{aligned} 2\Re \int_\Omega \eta \Delta u(m \cdot \nabla \overline{u}) dx = & (d-2) \int_\Omega \eta |\nabla u|^2 dx \\ & - 2\Re \int_\Omega (\nabla u \cdot \nabla \eta)(m \cdot \nabla \overline{u}) dx \\ & + 2\Re \int_{\Gamma_1} \partial_\nu u(m \cdot \nabla \overline{u}) d\Gamma \\ & - \int_{\Gamma_1} (m \cdot \nu) |\nabla u|^2 d\Gamma + \int_\Omega (m \cdot \nabla \eta) |\nabla u|^2 dx. \end{aligned} \tag{4.4.24}$$

The fact that $\overrightarrow{\nabla u} = \partial_\nu u \overrightarrow{\nu} + \nabla_T u$ on Γ_1, then by taking the real part of

(4.4.22) and using (4.4.23)-(4.4.24) we obtain

$$
\begin{aligned}
&\int_{\Gamma_1}(m\cdot\nu)|\partial_\nu u|^2 d\Gamma + \int_{\Gamma_1}(m\cdot\nu)|\beta u|^2 d\Gamma + (d-2)\int_\Omega \eta|\nabla u|^2 dx = \\
&d\int_\Omega \eta|\beta u|^2 dx + \int_\Omega (m\cdot\nabla\eta)|\beta u|^2 dx + 2\Re\int_\Omega (\nabla u\cdot\nabla\eta)(m\cdot\nabla\overline{u})dx \\
&-2\Re\int_{\Gamma_1}\partial_\nu u(m\cdot\nabla_T\overline{u})d\Gamma + \int_{\Gamma_1}(m\cdot\nu)|\nabla_T u|^2 d\Gamma \\
&-\int_\Omega (m\cdot\nabla\eta)|\nabla u|^2 dx + \frac{o(1)}{\beta^{l-1}}.
\end{aligned}
$$

Later, using the multiplier control condition **MCC** on Γ_1 we get

$$
\begin{aligned}
m_0\int_{\Gamma_1}|\partial_\nu u|^2 d\Gamma + m_0\int_{\Gamma_1}|\beta u|^2 d\Gamma + (d-2)\int_\Omega \eta|\nabla u|^2 dx \le \\
d\int_\Omega \eta|\beta u|^2 dx + \int_\Omega (m\cdot\nabla\eta)|\beta u|^2 dx \\
+2\Re\int_\Omega (\nabla u\cdot\nabla\eta)(m\cdot\nabla\overline{u})dx \\
-2\Re\int_{\Gamma_1}\partial_\nu u(m\cdot\nabla_T\overline{u})d\Gamma + \int_{\Gamma_1}(m\cdot\nu)|\nabla_T u|^2 d\Gamma \\
-\int_\Omega (m\cdot\nabla\eta)|\nabla u|^2 dx + \frac{o(1)}{\beta^{l-1}}.
\end{aligned}
$$

Its follows that

$$
\begin{aligned}
m_0\int_{\Gamma_1}|\partial_\nu u|^2 d\Gamma \le & d\int_\Omega \eta|\beta u|^2 dx + \int_\Omega (m\cdot\nabla\eta)|\beta u|^2 dx \\
&+2\Re\int_\Omega (\nabla u\cdot\nabla\eta)(m\cdot\nabla\overline{u})dx - 2\Re\int_{\Gamma_1}\partial_\nu u(m\cdot\nabla_T\overline{u})d\Gamma \\
&+\int_{\Gamma_1}(m\cdot\nu)|\nabla_T u|^2 d\Gamma - \int_\Omega (m\cdot\nabla\eta)|\nabla u|^2 dx + \frac{o(1)}{\beta^{l-1}}.
\end{aligned}
$$

Thus, applying Cauchy-Schwarz's and Young's inequalities we obtain

$$
\begin{aligned}
(m_0-\epsilon)\int_{\Gamma_1}|\partial_\nu u|^2 d\Gamma \le \left(\frac{R^2}{\epsilon}+R\right)\int_{\Gamma_1}|\nabla_T u|^2 d\Gamma + C_1\int_\Omega |\beta u|^2 dx \\
+C_2\int_\Omega |\nabla u|^2 dx + \frac{o(1)}{\beta^{l-1}},
\end{aligned}
\tag{4.4.25}
$$

where ϵ is a positive constant, $R = \|m\|_\infty$, $C_1 = C(R, \|\eta\|_\infty)$ and $C_2 = C(\|\eta\|_\infty, \|\nabla\eta\|_\infty, R)$. Now, since $U \in D(\mathcal{A})$, we have $u = w$ and therefore $\nabla_T u = \nabla_T w$ on Γ_1. Thus, from (4.4.4) we deduce that $\nabla_T u$ (respectively ∇u) is uniformly bounded on Γ_1 (respectively in Ω). Further, using the second equation of system (4.4.6) we deduce that βu is uniformly bounded in Ω. Finally, setting $\epsilon = \dfrac{m_0}{2}$ in (4.4.25) and taking $l \geq 1$, we get directly (4.4.19). ■

Lemma 4.4.8. *Assume that $\Gamma_0 \neq \emptyset$, the boundary Γ of Ω is Lipschitz and l=8. Then, the solution $(u, v, w, z) \in D(\mathcal{A})$ of system (4.4.6) satisfies the following estimation:*

$$\int_{\Gamma_1} |\nabla_T u|^2 d\Gamma = \frac{o(1)}{\beta^4}. \tag{4.4.26}$$

On the other hand, assume that the boundary Γ of Ω is $C^{1,1}$, $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, the multiplier control condition **MCC** *on Γ_1 holds and $l = 2$. Then, the solution $(u, v, w, z) \in D(\mathcal{A})$ of system (4.4.6) satisfies the following estimation:*

$$\int_{\Gamma_1} |\nabla_T u|^2 d\Gamma = \frac{o(1)}{\beta^2}. \tag{4.4.27}$$

Proof: Multiplying the second equality of system (4.4.7) by $\overline{u}$ and integrating by parts and using (4.4.8) we obtain

$$\int_{\Gamma_1} |\nabla_T u|^2 d\Gamma + \int_{\Gamma_1} \partial_\nu u \overline{u} d\Gamma + i\beta \int_{\Gamma_1} |u|^2 d\Gamma - \int_{\Gamma_1} |\beta u|^2 d\Gamma = \frac{o(1)}{\beta^{\frac{3l}{2}}}. \tag{4.4.28}$$

First, if $\Gamma_0 \neq 0$, the boundary Γ of Ω is Lipschitz and $l = 8$, then using (4.4.8) and (4.4.16) we get

$$\begin{cases} \displaystyle\int_{\Gamma_1} |\beta u|^2 d\Gamma = \frac{o(1)}{\beta^8}, \\ \displaystyle\int_{\Gamma_1} \partial_\nu u \overline{u} d\Gamma = \frac{o(1)}{\beta^4}, \\ \displaystyle i\beta \int_{\Gamma_1} |u|^2 d\Gamma = \frac{o(1)}{\beta^9}. \end{cases} \tag{4.4.29}$$

Thus, substituting (4.4.29) into (4.4.28) with $l = 8$ we directly get (4.4.26). Next, if $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, the multiplier control condition **MCC** on Γ_1 holds and if $l = 2$, then using (4.4.8) and (4.4.19) we obtain

$$\begin{cases} \displaystyle\int_{\Gamma_1} |\beta u|^2 d\Gamma = \frac{o(1)}{\beta^2}, \\ \displaystyle\int_{\Gamma_1} \partial_\nu u \overline{u} d\Gamma = \frac{o(1)}{\beta^2}, \\ \displaystyle i\beta \int_{\Gamma_1} |u|^2 d\Gamma = \frac{o(1)}{\beta^3}. \end{cases} \tag{4.4.30}$$

Finally, substituting (4.4.30) into (4.4.28) with $l = 2$ we strictly get (4.4.27). ■

Now, we consider the following auxiliary problem:

$$\begin{cases} -(\beta^2+\Delta)\varphi_u & = & u, & \text{in} & \Omega, \\ \varphi_u & = & 0, & \text{on} & \Gamma_0, \\ \partial_\nu \varphi_u + i\beta\varphi_u & = & 0, & \text{on} & \Gamma_1, \end{cases} \tag{4.4.31}$$

where u is solution of system (4.4.7).

Lemma 4.4.9. *Assume that the conditions (H_1) holds. Then, the solution φ_u of problem (4.4.31) satisfies the following estimation:*

$$\beta\|\varphi_u\|_\Omega + \|\nabla\varphi_u\|_\Omega + \beta\|\varphi_u\|_{\Gamma_1} \lesssim \|u\|_\Omega. \tag{4.4.32}$$

Proof: The proof is same as in [1], it is based on a result for Huang and Prüss in [33, 46, 76]. Since the problem (4.4.1) is uniformly stable, then the resolvent of its operator is bounded on the imaginary axis. We omit the details here. ■

Lemma 4.4.10. *Assume that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and the condition (H_1) holds. Then, the solution φ_u of system (4.4.31) satisfies the following estimation:*

$$\int_{\Gamma_1} |\nabla_T \varphi_u|^2 d\Gamma = O(1). \tag{4.4.33}$$

Proof: First, let $h = \Delta(\eta\varphi_u) = \eta\Delta\varphi_u + 2\nabla\eta\cdot\nabla\varphi_u + \Delta\eta\varphi_u$ where η is defined in (4.4.20)-(4.4.21). Next, it is easy to check that

$$\partial_\nu(\eta\varphi_u) = \begin{cases} \partial_\nu\varphi_u & \text{on} \quad \Gamma_1, \\ 0 & \text{on} \quad \Gamma_0. \end{cases} \tag{4.4.34}$$

Thus, using the first equation of (4.4.31) and using (4.4.32), we obtain

$$\partial_\nu(\eta\varphi_u) \in L^2(\Gamma). \tag{4.4.35}$$

Later, we can assume that the boundary ∂O_α of O_α defined in (4.4.21) is Lipschitz. Then, using (4.4.35) and applying (4.4.12) we obtain

$$\begin{aligned} \int_\Gamma |\nabla_T\varphi_u|^2 d\Gamma &\lesssim \int_\Omega |\Delta(\eta\varphi_u)|^2 dx + \int_\Gamma |\partial_\nu(\eta\varphi_u)|^2 d\Gamma \\ &= \int_\Omega |h|^2 dx + \int_{\Gamma_1} |\partial_\nu\varphi_u|^2 d\Gamma. \end{aligned} \tag{4.4.36}$$

On the other hand, using (4.4.32) and the third equation of system (4.4.31) we get

$$\begin{aligned}\int_\Omega |h|^2 dx \leq& \|\eta\|_\infty^2 \int_\Omega |\Delta\varphi_u|^2 dx + \|\nabla\eta\|_\infty^2 \int_\Omega |\nabla\varphi_u|^2 dx \\ &+ \|\Delta\eta\|_\infty^2 \int_\Omega |\varphi_u|^2 dx \\ \lesssim& \beta^2 \int_\Omega |u|^2 dx \end{aligned}\tag{4.4.37}$$

and

$$\int_{\Gamma_1} |\partial_\nu \varphi_u|^2 d\Gamma = \beta^2 \int_{\Gamma_1} |\varphi_u|^2 dx \lesssim \int_\Omega |u|^2 dx. \tag{4.4.38}$$

Finally, since βu is uniformly bounded in Ω, combining (4.4.36)-(4.4.38), we deduce (4.4.33). ■

Lemma 4.4.11. *Assume that the boundary Γ of Ω is $C^{1,1}$, $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and that the multiplier control condition* **MCC** *on Γ_1 holds. Then, the solution φ_u of system (4.4.31) verifies the following estimation:*

$$\int_{\Gamma_1} |\nabla_T \varphi_u|^2 d\Gamma = \frac{O(1)}{\beta^2}. \tag{4.4.39}$$

Proof: First, multiplying the first equation of system (4.4.31) by $2\eta m \cdot \nabla\overline{\varphi_u}$ where η is the cut off function define in (4.4.20)-(4.4.21), we get

$$-2\beta^2 \int_\Omega \varphi_u \eta (m \cdot \nabla\overline{\varphi_u}) dx - 2\int_\Omega \Delta\varphi_u \eta (m \cdot \nabla\overline{\varphi_u}) dx = 2\int_\Omega u\eta (m \cdot \nabla\overline{\varphi_u}) dx.$$

Then, by taking the real part of the above equation and using (4.4.23)-(4.4.24) for $u = \varphi_u$ we obtain

$$\begin{aligned} d\int_\Omega \eta|\beta\varphi_u|^2 dx &+ \int_\Omega (m\cdot\nabla\eta)|\beta\varphi_u|^2 dx - \int_{\Gamma_1} (m\cdot\nu)|\beta\varphi_u|^2 d\Gamma \\ &- (d-2)\int_\Omega \eta|\nabla\varphi_u|^2 dx + 2\Re\int_\Omega (\nabla\varphi_u\cdot\nabla\eta)(m\cdot\nabla\overline{\varphi_u}) dx \\ &- 2\Re\int_{\Gamma_1} \partial_\nu\varphi_u (m\cdot\nabla\overline{\varphi_u}) d\Gamma + \int_{\Gamma_1} (m\cdot\nu)|\nabla\varphi_u|^2 d\Gamma \\ &- \int_\Omega (m\cdot\nabla\eta)|\nabla\varphi_u|^2 dx = 2\Re\int_\Omega u\eta(m\cdot\nabla\overline{\varphi_u}) dx. \end{aligned}\tag{4.4.40}$$

Next, using the first equation of system (4.4.6) we get $\|u\|_\Omega^2 = \dfrac{O(1)}{\beta^2}$. Since by remark 4.4.2, we claim that the multiplier control condition

MCC implies that the condition (H_1) holds. Using (4.4.32), we obtain

$$\int_{\Gamma_1}(m\cdot\nabla\eta)|\beta\varphi_u|^2d\Gamma \leq R\|\nabla\eta\|_\infty\int_{\Gamma_1}|\beta\varphi_u|^2d\Gamma \lesssim R\|\nabla\eta\|_\infty\int_\Omega|u|^2d\Gamma = \frac{O(1)}{\beta^2}$$

where $R=\|m\|_\infty$ and therefore

$$\int_{\Gamma_1}(m\cdot\nabla\eta)|\beta\varphi_u|^2d\Gamma = \frac{O(1)}{\beta^2}. \tag{4.4.41}$$

Similarly, we get

$$2\Re\int_\Omega(\nabla\varphi_u\cdot\nabla\eta)(m\cdot\nabla\overline{\varphi_u})dx = \frac{O(1)}{\beta^2}, \tag{4.4.42}$$

$$(d-2)\int_\Omega\eta|\nabla\varphi_u|^2dx = \frac{O(1)}{\beta^2}, \tag{4.4.43}$$

$$\int_{\Gamma_1}(m\cdot\nu)|\beta\varphi_u|^2d\Gamma = \frac{O(1)}{\beta^2}, \tag{4.4.44}$$

$$\int_\Omega(m\cdot\nabla\eta)|\nabla\varphi_u|^2dx = \frac{O(1)}{\beta^2} \tag{4.4.45}$$

and

$$2\Re\int_\Omega u\eta(m\cdot\nabla\overline{\varphi_u})dx = \frac{O(1)}{\beta^2}. \tag{4.4.46}$$

Later, inserting (4.4.41)-(4.4.46) into (4.4.40) we obtain

$$\int_{\Gamma_1}(m\cdot\nu)|\nabla\varphi_u|^2d\Gamma - 2\Re\int_{\Gamma_1}\partial_\nu\varphi_u(m\cdot\nabla\overline{\varphi_u})d\Gamma = \frac{O(1)}{\beta^2}.$$

Which implies that

$$\begin{aligned}\int_{\Gamma_1}(m\cdot\nu)|\nabla_T\varphi_u|^2d\Gamma \leq& \int_{\Gamma_1}(m\cdot\nu)|\partial_\nu\varphi_u|^2d\Gamma \\ &+2\int_{\Gamma_1}\partial_\nu\varphi_u(m\cdot\nabla_T\overline{\varphi_u})d\Gamma + \frac{O(1)}{\beta^2}.\end{aligned} \tag{4.4.47}$$

Now, using the multiplier control condition **MCC** on Γ_1 and the third equation of system (4.4.31) we get

$$\begin{aligned}m_0\int_{\Gamma_1}|\nabla_T\varphi_u|^2d\Gamma \leq& R\int_{\Gamma_1}|\beta\varphi_u|^2d\Gamma \\ &+2R\int_{\Gamma_1}|\nabla_T\varphi_u||\beta\varphi_u|d\Gamma + \frac{O(1)}{\beta^2}.\end{aligned} \tag{4.4.48}$$

Finally, by applying Cauchy-Schwarz's and Young's inequalities and using (4.4.32), we directly deduce (4.4.39). ∎

Proof of Theorem 4.4.3:

1. First, multiplying the first equation of system (4.4.7) by $\overline{\varphi_u}$ and applying Green's formula we obtain

$$\begin{aligned}&\int_\Omega u(\beta^2+\Delta)\overline{\varphi_u}dx+\int_{\Gamma_1}\left(\partial_\nu u\overline{\varphi_u}-u\partial_\nu\overline{\varphi_u}\right)d\Gamma \\ &=-\int_\Omega\left(\frac{f_2+i\beta f_1}{\beta^l}\right)\overline{\varphi_u}dx.\end{aligned} \tag{4.4.49}$$

Moreover, using the second equation of system (4.4.7) we have

$$\partial_\nu u=\frac{f_4+(1+i\beta)f_3}{\beta^l}+\beta^2u+\Delta_T u-i\beta u. \tag{4.4.50}$$

Then, substituting (4.4.50) into (4.4.49) and using the first equation of problem (4.4.31) and integrating by parts yields

$$\begin{aligned}\int_\Omega|\beta u|^2dx=&\int_\Omega\left(\frac{f_2+i\beta f_1}{\beta^{l-2}}\right)\overline{\varphi_u}dx \\ &+\int_{\Gamma_1}\left(\frac{f_4+(1+i\beta)f_3}{\beta^{l-2}}\right)\overline{\varphi_u}d\Gamma \\ &+\int_{\Gamma_1}\beta^4u\overline{\varphi_u}d\Gamma \\ &-\int_{\Gamma_1}(\beta\nabla_T u)(\beta\nabla_T\overline{\varphi_u})d\Gamma.\end{aligned} \tag{4.4.51}$$

On the other hand, multiplying the first equation of system (4.4.7) by $\overline{u}$ and applying Green's formula and using (4.4.50), we get

$$\begin{aligned}\int_\Omega|\nabla u|^2d\Omega=&\beta^2\left(\int_\Omega|u|^2d\Omega-\int_{\Gamma_1}|u|^2d\Gamma\right) \\ &-\int_{\Gamma_1}|\nabla_T u|^2d\Gamma-i\beta\int_{\Gamma_1}|u|^2d\Gamma \\ &+\int_{\Gamma_1}\left(\tilde{f}_4-(1+i\beta)\tilde{f}_3\right)\overline{u}d\Gamma+\int_\Omega\left(\tilde{f}_2+i\beta\tilde{f}_1\right)\overline{u}d\Omega.\end{aligned} \tag{4.4.52}$$

Next, under the assumptions $\Gamma_0\neq\emptyset$, $\overline{\Gamma_0}\cap\overline{\Gamma_1}=\emptyset$, the conditions (H_1) holds and $l=8$, using Lemmas 4.4.4, 4.4.8, 4.4.9 and 4.4.10, then from (4.4.51) and (4.4.52) we obtain

$$\int_\Omega|\beta u|^2dx=\int_\Omega|\nabla u|^2dx=o(1). \tag{4.4.53}$$

This implies from the first equation of (4.4.6) that

$$\int_{\Omega} |v|^2 d\Omega = o(1) \tag{4.4.54}$$

and therefore

$$\|U\|_{\mathcal{H}} = o(1). \tag{4.4.55}$$

Which is a contradiction with (4.4.4).

2. If the boundary Γ of Ω is $C^{1,1}$ and under the assumptions $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, the geometric control condition **GC** hold and $l = 2$, using Lemmas 4.4.4, 4.4.8, 4.4.9 and 4.4.11, then from (4.4.51) and (4.4.52) we get (4.4.53), (4.4.54) and (4.4.55). Which is a contradiction with (4.4.4).

■

4.5 Non-uniform stability on the unit square

In this section we prove that the uniform stability (*i.e.* exponential stability) of (4.2.9) does not hold in the unit square domain $\Omega = (0,1)^2$ with $\Gamma_1 = \{(0,y),\ y \in (0,1)\}$ and $\Gamma_0 = \Gamma \backslash \overline{\Gamma_1}$. This outcome is due to the existence of a subsequence of eigenvalues of $\mathcal{A}$ which is close to the imaginary axis. First, let λ be an eigenvalue of $\mathcal{A}$ and $U = (u, v, w, z)$ be an associated eigenfunction, then we obtain $\mathcal{A}U = \lambda U$. Equivalently, we have the following system:

$$\left\{\begin{array}{ccccc} \Delta u & = & \lambda^2 u, & \text{in} & \Omega, \\ u & = & 0, & \text{on} & \Gamma_0, \\ u & = & w, & \text{on} & \Gamma_1, \\ w_{yy} + u_x & = & (\lambda^2 + \lambda) w, & \text{on} & \Gamma_1. \end{array}\right. \tag{4.5.1}$$

Next, using the separation of variables method and by a straightforward computation, we give a solution of system (4.5.1) by the following proposition:

Proposition 4.5.1. *A solution (u, w) of system (4.5.1) is given as follows:*

$$\left\{\begin{array}{rc} u(x,y) = & 2ab \sinh(\sqrt{\lambda^2 + l^2\pi^2}(1-x)) \sin(l\pi y), \\ w(y) = & 2ab \sinh(\sqrt{\lambda^2 + l^2\pi^2}) \sin(l\pi y), \end{array}\right. \tag{4.5.2}$$

where $a, b \in \mathbb{C}$ are two constants and $l \in \mathbb{N}^*$. Moreover, the eigenvalue λ

associated to $\mathcal{A}$ verify the following characteristic equation:

$$\lambda^2+\lambda+\sqrt{\lambda^2+l^2\pi^2}\coth(\sqrt{\lambda^2+l^2\pi^2})+l^2\pi^2=0. \tag{4.5.3}$$

Proof: First, using the separation of variables method and the boundary conditions of system (4.5.1), we find that u and w are given as follows:

$$\begin{aligned} u(x,y) =& X(x)Y(y), && (4.5.4)\\ w(y) =& X(0)Y(y), && (4.5.5)\\ X(1)=Y(0)=Y(1)=& \quad 0. && (4.5.6)\end{aligned}$$

Then, substituting (4.5.4)-(4.5.5) in the first equation of system (4.5.1) and dividing by $X(x)Y(y)$ yields

$$\frac{X_{xx}(x)}{X(x)}+\frac{Y_{yy}(y)}{Y(y)}=\lambda^2. \tag{4.5.7}$$

Therefore, X and Y are the solutions of the following problems:

$$\begin{cases} X_{xx}(x) &=& (\lambda^2+\tilde{l}^2)X(x),\\ X(1) &=& 0 \end{cases} \tag{4.5.8}$$

and

$$\begin{cases} Y_{yy}(y) &=& -\tilde{l}^2Y(y),\\ Y(0)=Y(1) &=& 0, \end{cases} \tag{4.5.9}$$

where $\tilde{l}\geq 0$ denotes the constant of separation. Next, using the boundary condition of (4.5.8), we can easily prove that the solution X system (4.5.8) is given by

$$X(x)=2a\sinh(\sqrt{\lambda^2+\tilde{l}^2}(1-x)), \tag{4.5.10}$$

where $a\in\mathbb{C}$ is a constant. Similarly, the solution Y of the first equation of (4.5.9) is given by $Y(y)=c_1e^{i\tilde{l}y}+c_2e^{-i\tilde{l}y}$ where $c_1,c_2\in\mathbb{C}$ are two constants. The boundary conditions in (4.5.9) imply that $c_1=-c_2$ and $\tilde{l}=l\pi$ where $l\in\mathbb{N}^*$. Then, setting $b=2ic_1$, we claim that the solutions X and Y of the systems (4.5.8)-(4.5.9) are given by

$$X(x)=2a\sinh(\sqrt{\lambda^2+l^2\pi^2}(1-x)) \quad \text{and} \quad Y(y)=b\sin(l\pi y). \tag{4.5.11}$$

Now, inserting (4.5.11) into (4.5.4) and (4.5.5) we directly get (4.5.2). Finally, using the expression of u and w in (4.5.2) and the last equality of system (4.5.1) we get (4.5.3). ■

Now, since $\mathcal{A}$ is closed with compact resolvent (Proposition 4.2.4), the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ consists entirely of isolated eigenvalues with finite multiplicities. Moreover, as the coefficients of $\mathcal{A}$ are real then the eigenvalues appears by conjugate pairs. Finally, we study the spectrum of $\sigma(\mathcal{A})$ of $\mathcal{A}$ by the following proposition:

Proposition 4.5.2. *There exists $k_1 \in \mathbb{N}^*$ sufficiently large such that the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ in the unit square is given by*

$$\sigma(\mathcal{A}) = \widetilde{\sigma}_0 \cup \widetilde{\sigma}_1, \tag{4.5.12}$$

where

$$\widetilde{\sigma}_0 = \{\kappa_{l,j}\}_{j\in J}, \quad \widetilde{\sigma}_1 = \{\lambda_{l,k}\}_{\substack{k\in\mathbb{Z}\\ |k|\geq k_1}}, \quad \widetilde{\sigma}_0 \cap \widetilde{\sigma}_1 = \emptyset \tag{4.5.13}$$

and where J is a finite set. Moreover, $\lambda_{l,k}$ is simple and satisfies the following asymptotic behavior:

$$\lambda_{l,k} = i\left(k\pi + \frac{l^2\pi}{2k}\right) - \frac{1}{\pi^2 k^2} + o(\frac{1}{k^2}). \tag{4.5.14}$$

Proof: For clarity, we divided the proof into several steps.

Step 1. Roots of characteristic equation. First, we set

$$\xi = \lambda\sqrt{1 + \frac{l^2\pi^2}{\lambda^2}}. \tag{4.5.15}$$

Then $\xi^2 = \lambda^2 + l^2\pi^2$ and $\lambda = \xi\sqrt{1 - \frac{l^2\pi^2}{\xi^2}}$. Using the characteristic equation (4.5.3) we get

$$\xi^2 + \xi\sqrt{1 - \frac{l^2\pi^2}{\xi^2}} + \xi\frac{e^{2\xi}+1}{e^{2\xi}-1} = 0. \tag{4.5.16}$$

Multiplying (4.5.16) by $f_0(\xi) = e^{2\xi} - 1$, then (4.5.16) is equivalent to

$$
\begin{aligned}
0 = f(\xi) = f_0(\xi) + \frac{f_1(\xi)}{\xi} & \qquad (4.5.17)\\
= e^{2\xi} - 1 + \frac{1}{\xi}\left((e^{2\xi}+1) + (e^{2\xi}-1)\sqrt{1 - \frac{l^2\pi^2}{\xi^2}}\right).
\end{aligned}
$$

The real part of ξ is bounded. This is due to the fact that if $\Re(\xi) \longrightarrow -\infty$ then $f(\xi) \longrightarrow -1$. Then, with the help of Rouché's theorem, there exists k_1 large enough such that for all $|k| \geq k_1$ the large roots of f (denoted by ξ_k) are simple and close to ξ_k^0 roots of $f_0(\xi)$. More precisely we have

$$\xi_k = \xi_k^0 + \epsilon_k \quad \text{and} \quad \lim_{|k|\to\infty} \epsilon_k = 0, \qquad (4.5.18)$$

where

$$\xi_k^0 = ik\pi, \quad k \in \mathbb{Z}. \qquad (4.5.19)$$

Step 2. Asymptotic behavior of ϵ_k and $\lambda_{l,k}$. Using equation (4.5.18), we get

$$
\begin{aligned}
e^{2\xi_k} =& 1 + 2\epsilon_k + o(\epsilon_k), & (4.5.20)\\
\frac{1}{\xi_k} =& \frac{1}{ik\pi} + o(\frac{1}{k^2}), & (4.5.21)\\
\frac{1}{\xi_k^2} =& -\frac{1}{k^2\pi^2} + o(\frac{1}{k^3}) & (4.5.22)
\end{aligned}
$$

and

$$\sqrt{1 - \frac{l^2\pi^2}{\xi_k^2}} = 1 + \frac{l^2}{2k^2} + o(\frac{1}{k^3}). \qquad (4.5.23)$$

Substituting equations (4.5.20)-(4.5.23) into (4.5.17) and after some computations yields

$$\epsilon_k = \frac{i}{k\pi} + o(\frac{1}{k}) = \frac{i}{k\pi} + \frac{\tilde{\epsilon_k}}{k} \text{ with } \tilde{\epsilon_k} \longrightarrow 0. \qquad (4.5.24)$$

Then, using equation (4.5.24) we get

$$e^{2\xi_k} - 1 = \frac{2i}{k\pi} - \frac{2}{k^2\pi^2} + \frac{2\tilde{\epsilon_k}}{k} + o(\frac{1}{k^2}). \qquad (4.5.25)$$

Substituting equations (4.5.21)-(4.5.23) and (4.5.25) into (4.5.17) and after some computations yields

$$\tilde{\epsilon_k} = \frac{-1}{k\pi^2} + o(\frac{1}{k}). \tag{4.5.26}$$

Inserting equation (4.5.26) into (4.5.24) and (4.5.18) we obtain (4.5.14). ∎

Numerical validation. The asymptotic behavior of $\lambda_{l,k}$ in (4.5.14) can be numerically validated. For instance, with $l = 1$, then from (4.5.14) we have

$$-\lim_{k\to+\infty} k^2\pi^2\Re(\lambda_{1,k}) = 1.$$

The table below confirms this behavior.

k	100	150	200	250	300	350	400	450	500
$-\pi^2k^2\Re(\lambda_{1,k})$	0.999949	0.999977	0.999987	0.999992	0.999994	0.999996	0.999997	0.999998	0.999998

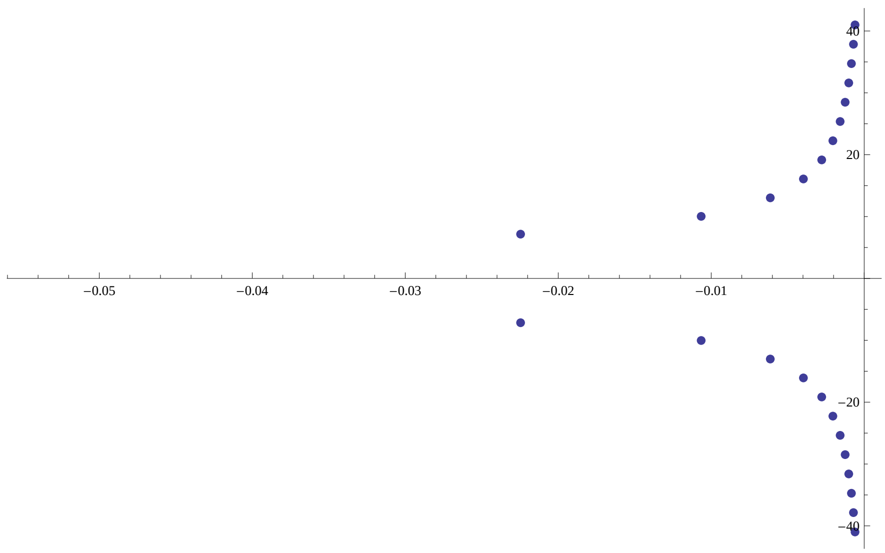

Figure 4.1: Eigenvalues of $\mathcal{A}$ with $l = 1$

In addition, figure 4.1 represents some eigenvalues in this case.

4.6 Polynomial energy decay rate of 1-d model with a parameter

The aim of this section is to establish a polynomial energy decay rate of 1-d model with a parameter associated with problem (4.1.2) on the unit square domain $\Omega = (0,1)^2$ with $\Gamma_1 = \{(0,y),\ y \in (0,1)\}$ and $\Gamma_0 = \Gamma \backslash \overline{\Gamma_1}$. First, we fixed a real parameter $L = p\pi$ with $p \in \mathbb{N}^*$ and we consider the solution (u^L, w^L) of the following wave equation in (4.1.2) with damping at 0:

$$\begin{cases} u^L_{tt} - u^L_{xx} + L^2 u^L & = & 0, & \text{in} \quad (0,1), \forall t > 0, \\ u^L(1,t) & = & 0, & \forall t > 0, \\ u^L(0,t) & = & w^L, & \forall t > 0, \\ w^L_{tt} + L^2 w^L - u_x(0,t) + w^L_t & = & 0, & \forall t > 0, \\ u^L(\cdot,0) = u^L_0, \qquad u^L_t(\cdot,0) = u^L_1, & & \forall x \in (0,1), & \\ w^L(0) = w^L_0, \qquad w^L_t(0) = w^L_1. & & & \end{cases} \tag{4.6.1}$$

Next, we introduce the energy associated to (4.6.1) by

$$\begin{aligned} E_L(t) = & \frac{1}{2} \int_0^1 \left(|u^L_t(x,t)|^2 + |u^L_x(x,t)|^2 + L^2 |u^L(x,t)|^2 \right) dx \\ & + \frac{1}{2} |w^L_t(t)|^2 + \frac{L^2}{2} |w^L(t)|^2. \end{aligned} \tag{4.6.2}$$

A simple integration by parts gives

$$\frac{d}{dt} E_L(t) = -|w^L_t(t)|^2. \tag{4.6.3}$$

Later, we split the solution $U^L = (u^L, w^L)$ of system (4.6.1) as follows:

$$U^L = U_1 + U_2, \tag{4.6.4}$$

where $U_1 = (u_1, w_1)$ and where $U_2 = (u_2, w_2)$, (u_1, w_1) is solution of the same problem that (u^L, w^L) but without damping and (u_2, w_2) is the remainder (for shortness we do not write the dependence of (u_i, w_i), $i = 1,2$ with respect to L). This means that they are respective solutions of

$$\begin{cases} u_{1,tt} - u_{1,xx} + L^2 u_1 & = & 0, & \text{in } (0,1), & \forall t > 0, \\ u_1(1,t) & = & 0, & & \forall t > 0, \\ u_1(0,t) & = & w_1, & & \forall t > 0, \\ w_{1,tt} + L^2 w_1 - u_{1,x}(0,t) & = & 0, & & \forall t > 0, \\ u_1(\cdot,0) = u^L_0, \qquad u_{1,t}(\cdot,0) = u^L_1, & & & & \\ w_1(0) = w^L_0, \qquad w_{1,t}(0) = w^L_1 & & & & \end{cases} \tag{4.6.5}$$

and

$$\begin{cases} u_{2,tt} - u_{2,xx} + L^2 u_2 & = & 0, & \text{in} & (0,1), & \forall t > 0, \\ u_2(1,t) & = & 0, & & & \forall t > 0, \\ u_2(0,t) & = & w_2, & & & \forall t > 0, \\ w_{2,tt} + L^2 w_2 - u_{2,x}(0,t) + w_t^L & = & 0, & & & \forall t > 0, \\ u_2(\cdot,0) = 0, \qquad u_{2,t}(\cdot,0) = 0, \\ w_2(0) = 0, \qquad w_{2,t}(0) = 0. \end{cases} \tag{4.6.6}$$

The above splitting is quite standard and it is based on the following idea: First, for the problem (4.6.5), we prove an observability inequality for the solution via spectral analysis and Ingham's inequality. Next, by a perturbation argument based on the dependence of the constants with respect to the time T and L, we find the requested observability estimate for the starting problem (4.6.1).

First, the problem (4.6.5) is related to the positive self-adjoint operator A_L from $H = L^2(0,1) \times \mathbb{C}$ into itself (with a compact inverse) with domain

$$D(A_L) = \left\{ \begin{array}{c} U_1 = (u_1, w_1) \in H^2(0,1) \times \mathbb{C}; \\ u_1(1) = 0, \\ u_1(0) = w_1 \end{array} \right\} \tag{4.6.7}$$

and defined by

$$A_L U_1 = (-u_{1,xx} + L^2 u_1, L^2 w_1 - u_{1,x}(0)). \tag{4.6.8}$$

Therefore, we can formulate problem (4.6.5) into a second order evolution equation

$$\begin{cases} U_{1,tt}(t) + A_L U_1(t) & = & 0, \\ U_1(0) & = & U_0^L, \\ U_{1,t}(0) & = & U_1^L, \end{cases} \tag{4.6.9}$$

where $U_0^L = (u_0^L, w_0^L)$ and $U_1^L = (u_1^L, w_1^L)$. The spectrum of A_L is characterized as follows:

Theorem 4.6.1. *The eigenvalues λ^2 of A_L are strictly larger then L^2 and are the roots of the transcendental equation*

$$\tan(\theta) = \frac{1}{\theta}, \tag{4.6.10}$$

with $\theta = \sqrt{\lambda^2 - L^2}$. Writing $\{\lambda_k^2\}_{k=0}^{\infty}$ the sequence of these roots in in-

creasing order, it forms the set of eigenvalues of A_L which are simple and of associated normalized eigenvectors given by

$$U_{1,k} = \frac{1}{\delta_k}\left(\sin(\theta_k(1-x)), \sin(\theta_k)\right), \quad \delta_k = \frac{\sqrt{1+\sin(\theta_k)^2}}{\sqrt{2}} \tag{4.6.11}$$

and $\theta_k = \sqrt{\lambda_k^2 - L^2}$. Furthermore, the next gap condition holds:

$$\lambda_{k+1} - \lambda_k \geq \frac{\widetilde{\gamma}}{L}, \quad \forall k \in \mathbb{N}, \tag{4.6.12}$$

with $\widetilde{\gamma}$ is a constant independent on k.

Proof: First, let λ^2 be an eigenvalue of A_L and $U_1 = (u_1, w_1)$ an associated eigenvector. Then, using Green's formula and the boundary conditions of system (4.6.5) we obtain

$$\begin{aligned}\langle A_L U_1, U_1\rangle_H &= \int_0^1 |u_{1,x}|^2 dx + L^2 \int_0^1 |u_1|^2 dx + L^2 |w_1|^2 \\ &\geq L^2 \|U_1\|_H^2,\end{aligned} \tag{4.6.13}$$

which clearly implies that the eigenvalues of A_L are larger than L^2. Next, for $\lambda^2 \geq L^2$, we look for (u_1, w_1) solution of

$$\begin{cases} -u_{1,xx} + L^2 u_1 &= \lambda^2 u_1, \quad \text{in} \quad (0,1), \\ L^2 w_1 - u_{1,x}(0) &= \lambda^2 w_1, \\ u_1(0) &= w_1, \\ u_1(1) &= 0. \end{cases} \tag{4.6.14}$$

We easily check that if $\lambda^2 = L^2$, the only solution of problem (4.6.14) is $u_1 = w_1 = 0$, hence $\lambda^2 = L^2$ cannot be an eigenvalue of A_L. Now for $\lambda^2 > L^2$, there exists $\alpha \in \mathbb{R}$ such that the solution of the first equation of system (4.6.14) is given by

$$u_1(x) = \alpha \sin(\theta(1-x)), \tag{4.6.15}$$

with $\theta = \sqrt{\lambda^2 - L^2}$. For convenience, we set $\alpha = 1$. Then, the second boundary condition of (4.6.14) becomes

$$\theta \sin(\theta) = \cos(\theta). \tag{4.6.16}$$

Therefore a nontrivial solution (u_1, w_1) exists if and only if (4.6.16) holds.

The form of the eigenvectors (4.6.11) also follows from this consideration. As (4.6.16) is equivalent to

$$\tan(\theta) = \frac{1}{\theta},$$

we deduce that its roots are simple and verify

$$0 < \theta_0 < \frac{\pi}{2}, \qquad \frac{\pi}{2} + (k-1)\pi < \theta_k < \frac{\pi}{2} + k\pi, \quad \forall k \in \mathbb{N}^*, \tag{4.6.17}$$

with $\theta_k = \sqrt{\lambda_k^2 - L^2}$. Later, we check the gap between λ_k. We have

$$\lambda_{k+1} - \lambda_k = \sqrt{\theta_{k+1}^2 + L^2} - \sqrt{\theta_k^2 + L^2}. \tag{4.6.18}$$

Then, setting $\varphi(t, L) = \sqrt{t^2 + L^2}$ and using the mean value's theorem, we deduce that there exists $\theta_c \in (\theta_k, \theta_{k+1})$ such that:

$$\begin{aligned} \varphi(\theta_{k+1}, L) - \varphi(\theta_k, L) &= \partial_t \varphi(\theta_c, L)(\theta_{k+1} - \theta_k) \\ &= \frac{\theta_c}{\sqrt{\theta_c^2 + L^2}}(\theta_{k+1} - \theta_k). \end{aligned} \tag{4.6.19}$$

Since $\dfrac{t}{\sqrt{\frac{t^2}{L^2} + 1}}$ is an increasing function of the time variable t we obtain

$$\frac{\theta_c}{\sqrt{\theta_c^2 + L^2}} \geq \frac{\theta_0}{L\sqrt{\frac{\theta_0^2}{L^2} + 1}} \geq \frac{C}{L}, \tag{4.6.20}$$

with $C = \dfrac{\theta_0}{\sqrt{\frac{\theta_0}{\pi} + 1}}$. Finally, setting $\widetilde{\gamma} = C \min\limits_{k \in \mathbb{N}}(\theta_{k+1} - \theta_k)$, then from (4.6.18)-(4.6.20) we obtain (4.6.12). ■

Before going on, we recall Lemma 3.3 from [71] which give a variant version of Ingham's inequality [48] (see also [44]), where the dependence of the constants of equivalence are given with respect to the gape condition.

Lemma 4.6.2. *Let ξ_n, $n \in \mathbb{Z}$, be a sequence of real numbers and a positive real number γ such that the following gap condition:*

$$\xi_{n+1} - \xi_n \geq \gamma, \quad \forall k \in \mathbb{Z}$$

holds. Then, there exists two positive constants c, C independent of γ

such that for all function f in the form

$$f(t) = \sum_{n \in \mathbb{Z}} a_n e^{i\xi_n t}$$

with $a_n \in \mathbb{C}$, we have

$$\frac{c}{\gamma} \sum_{n \in \mathbb{Z}} |a_n|^2 \leq \int_0^{\frac{4\pi}{\gamma}} |f(t)|^2 dt \leq \frac{C}{\gamma} \sum_{n \in \mathbb{Z}} |a_n|^2.$$

■

Now, we set $V_L = D(A_L^{\frac{1}{2}})$. We will bound a weak energy of system (4.6.9) with respect to an appropriate boundary term by the following theorem:

Theorem 4.6.3. *Let $\widetilde{E}_{U_1}$ be a weak energy of $U_1(x,t) = (u_1(x,t), w_1(t))$ solution of (4.6.9) defined by*

$$\widetilde{E}_{U_1}(t) = \frac{1}{2}\|U_1(x,t)\|_H^2 + \frac{1}{2}\|U_{1,t}(x,t)\|_{V_L'}^2, \quad t \geq 0. \tag{4.6.21}$$

Then, there exists two positive constants C_1, C_2 independent on L such that for all $T \geq C_1 L$ we have

$$C_2 L \widetilde{E}_{U_1}(0) \leq \int_0^T |w_{1,t}(t)|^2 dt. \tag{4.6.22}$$

Proof: First, let λ_k^2 be an eigenvalue of A_L and $U_{1,k} = (u_{1,k}, w_{1,k})$ the associated eigenvectors already determined in (4.6.11). By the spectral theorem, the solution U_1 of (4.6.9) is given by

$$U_1(\cdot,t) = \sum_{k=0}^{+\infty} \left(u_0^k \cos(\lambda_k t) + u_1^k \frac{\sin(\lambda_k t)}{\lambda_k} \right) U_{1,k}, \tag{4.6.23}$$

where u_0^k (resp. u_1^k) is the Fourier coefficients of U_0^L (resp. U_1^L), *i.e.*

$$U_0^L = \sum_{k=0}^{+\infty} u_0^k U_{1,k} \quad \text{and} \quad U_1^L = \sum_{k=0}^{+\infty} u_1^k U_{1,k}.$$

Writing $U_{1,k} = (u_{1,k}, w_{1,k})$, this implies that

$$w_1(t) = \sum_{k=0}^{+\infty} \left(u_0^k \cos(\lambda_k t) + u_1^k \frac{\sin(\lambda_k t)}{\lambda_k} \right) w_{1,k}. \tag{4.6.24}$$

Thus we obtain

$$w_{1,t}(t) = \sum_{k=0}^{+\infty} \left(-u_0^k \lambda_k \sin(\lambda_k t) + u_1^k \cos(\lambda_k t) \right) w_{1,k}. \tag{4.6.25}$$

Then according to the gap condition (4.6.12) and using Lemma 4.6.2, we deduce that there exists C_1, C_3 independent of L such that for all $T \geq C_1 L$ we have

$$C_3 L \sum_{k=0}^{+\infty} \left(\lambda_k^2 |u_0^k|^2 + |u_1^k|^2 \right) |w_{1,k}|^2 \leq \int_0^T |w_{1,t}(t)|^2 dt. \tag{4.6.26}$$

Next, using (4.6.10) and (4.6.11) we get

$$|w_{1,k}|^2 = \frac{1}{\delta_k^2} \times \sin^2(\theta_k) = \frac{2}{(1+\sin^2(\theta_k))} \times \frac{\cos^2(\theta_k)}{\theta_k^2} \sim \frac{C}{\theta_k^2} \sim \frac{C}{\lambda_k^2}, \tag{4.6.27}$$

as $\theta_k \sim k\pi$ as k goes to infinity. This equivalence in (4.6.26) yields the existence of a positive constant C_2 independent of L such that for $T \geq C_1 L$ we have

$$C_2 L \sum_{k=0}^{+\infty} \left(|u_0^k|^2 + \frac{|u_1^k|^2}{\lambda_k^2} \right) \leq \int_0^T |w_{1,t}(t)|^2 dt. \tag{4.6.28}$$

We now conclude by identity

$$\sum_{k=0}^{+\infty} \left(|u_0^k|^2 + \frac{|u_1^k|^2}{\lambda_k^2} \right) = \|U_0^L\|_H^2 + \|U_1^L\|_{V_L'}^2.$$

■

We go on with an estimate on w_2.

Theorem 4.6.4. *There exists a positive constant C_4 independent of L and $T > 0$ such that*

$$\int_0^T |w_{2,t}(t)|^2 dt \leq C_4^2 T^2 \int_0^T |w_t^L(t)|^2 dt. \tag{4.6.29}$$

Proof: First, we start by rewriting problem (4.6.6) as follows:

$$\begin{cases} U_{2,tt}(t) + A_L U_2(t) &= K(t) H_0, \\ U_2(0) &= 0, \\ U_{2,t}(0) &= 0, \end{cases} \tag{4.6.30}$$

with $H_0 = (0,1) \in H$ and $K(t) = w_t^L(t)$. Remark that H_0 is given by

$$H_0 = \sum_{k=0}^{\infty} w_{1,k} U_{1,k}.$$

Indeed, we have

$$< H_0, U_{1,k} >_H = < (0,1), (u_{1,k}, w_{1,k}) >_H = w_{1,k}. \tag{4.6.31}$$

Next, using the orthonormal basis $\{U_{1,k}\}_{k=0}^{+\infty}$ of H, we can write the solution $U_2 = (u_2, w_2)$ of problem (4.6.30) as follows:

$$U_2(t) = \sum_{k=0}^{+\infty} u_{2,k}(t) U_{1,k}. \tag{4.6.32}$$

From (4.6.30)-(4.6.32) we deduce that for every fixed $k \in \mathbb{N}$, $u_{2,k}$ is solution of the following problem:

$$\begin{cases} u_{2,k,tt}(t) + \lambda_k^2 u_{2,k}(t) & = & K(t) w_{1,k}, \\ u_{2,k}(0) & = & 0, \\ u_{2,k,t}(0) & = & 0. \end{cases} \tag{4.6.33}$$

It is easy to verify that $u_{2,k}$ is given by

$$u_{2,k}(t) = w_{1,k} \int_0^t \frac{\sin(\lambda_k s)}{\lambda_k} K(t-s) ds. \tag{4.6.34}$$

Thus, we obtain

$$U_2(t) = \int_0^t u(s) K(t-s) ds, \tag{4.6.35}$$

where $u(s) = \sum_{k=0}^{\infty} \frac{\sin(\lambda_k s)}{\lambda_k} w_{1,k} U_{1,k}$. It is follows that

$$w_2(t) = \int_0^t \psi(s) K(t-s) ds, \tag{4.6.36}$$

where $\psi(s) = \sum_{k=0}^{\infty} \frac{\sin(\lambda_k s)}{\lambda_k} w_{1,k}^2$, which implies that

$$w_{2,t}(t) = \int_0^t \psi_t(s) K(t-s) ds. \tag{4.6.37}$$

On the other hand, using (4.6.27) we have

$$|\psi_t(s)| = \left|\sum_{k=0}^{+\infty} \cos(\lambda_k s) w_{1,k}^2\right| \le \sum_{k=0}^{+\infty} |w_{1,k}|^2 \le C_4 < \infty. \tag{4.6.38}$$

Later, using (4.6.37)-(4.6.38) and applying Cauchy-Schwarz's inequality we obtain

$$|w_{2,t}(t)|^2 \le C_4^2 t \int_0^t |K(t-s)|^2 ds. \tag{4.6.39}$$

Finally, by integrating (4.6.39) between 0 and T and by a change of variable we deduce that

$$\int_0^T |w_{2,t}(t)|^2 dt \le C_4^2 T^2 \int_0^T |K(t)|^2 dt = C_4^2 T^2 \int_0^T |w_t^L(t)|^2 dt. \tag{4.6.40}$$

■

We are ready to prove our main results of this section.

Theorem 4.6.5. *There exists a positive constant C_9 independent on L such that for all initial data $(U_0^L, U_1^L) \in D(A_L) \times V_L$ we have*

$$E_L(t) \le \frac{C_9 L^2}{t} E_L^1(0), \tag{4.6.41}$$

where E_L^1 is given by

$$E_L^1(t) = \|U^L(t)\|_{D(A_L)}^2 + \|U_t^L(t)\|_{V_L}^2, \quad t \ge 0. \tag{4.6.42}$$

Proof: According to Theorem (4.6.3), we fix $T = C_1 L$. Now, using the splitting (4.6.4) we obtain

$$|w_{1,t}(t)|^2 \le 2\left(|w_t^L(t)|^2 + |w_{2,t}(t)|^2\right). \tag{4.6.43}$$

Then, integrating (4.6.43) between 0 and T and using the inequalities (4.6.22) and (4.6.29) we get

$$\int_0^T |w_t^L(t)|^2 dt \ge \frac{C_6 L}{T^2} \widetilde{E}_{U_1}(0), \tag{4.6.44}$$

for $T \ge C_1 L$ and $C_6 = \dfrac{C_2}{2C_4^2}$. Next, since $\widetilde{E}_{U_1}(0) = \widetilde{E}_{U^L}(0)$ and using

(4.6.3) the inequalities (4.6.44) becomes

$$E_L(0) - E_L(T) \geq \frac{C_6 L}{T^2} \widetilde{E}_{U^L}(0). \tag{4.6.45}$$

On another hand, using interpolation theory we can show that

$$\|U_0^L\|_H^2 \geq \frac{\|U_0^L\|_{V_L}^4}{\|U_0^L\|_{D(A_L)}^2} \quad \text{and} \quad \|U_1^L\|_{V_L'}^2 \geq \frac{\|U_1^L\|_H^4}{\|U_1^L\|_{V_L}^2}. \tag{4.6.46}$$

Thus, combining (4.6.21) and (4.6.46) we obtain

$$\widetilde{E}_{U^L}(0) \geq \frac{1}{2} \frac{\|U_0^L\|_{V_L}^4 + \|U_1^L\|_H^4}{\|U_0^L\|_{D(A_L)}^2 + \|U_1^L\|_{V_L}^2} = \frac{E_L^2(0)}{2E_L^1(0)} \tag{4.6.47}$$

where $E_L^1(t)$ is defined in (4.6.42). Later, substituting (4.6.47) into (4.6.45) and using the fact that $E_L(t)$ is a decreasing function of variable t we get

$$E_L(T) \leq E_L(0) - C_7 \frac{E_L^2(T)}{E_L^1(0)}, \tag{4.6.48}$$

where $C_7 = \dfrac{C_6 L}{2T^2}$. Now, we introduce the sequence $\xi_k = \dfrac{E_L(kT)}{E_L^1(0)}$ for $k \in \mathbb{N}$. Then, since $E_L(t)$ is a decreasing function of variable t, then dividing (4.6.48) by $E_L^1(0)$ we can easily check that ξ_k verify the following inequality:

$$\xi_{k+1} \leq \xi_k - C_7 \xi_{k+1}^2, \quad \forall k \in \mathbb{N}. \tag{4.6.49}$$

Our goal is to determine a constant M such that $\xi_k \leq \dfrac{M}{k+1}$. For this aim, we introduce the sequence F_k as follows:

$$F_k = \frac{M}{k+1}, \quad k \in \mathbb{N}.$$

First, we notice that

$$F_k - F_{k+1} = \frac{M}{(k+1)(k+2)} \leq \frac{2}{M} F_{k+1}^2. \tag{4.6.50}$$

Next, if we assume that

$$\frac{2}{C_7} \leq M \quad \text{and} \quad \xi_0 \leq F_0, \tag{4.6.51}$$

then we can prove by induction that

$$\xi_k \leq F_k, \quad \forall k \in \mathbb{N}. \tag{4.6.52}$$

Hence (4.6.52) holds as soon as $M = \max\{\frac{2}{C_7}, \xi_0\}$. Clearly, (4.6.52) is equivalent to

$$E_L(kT) \leq \frac{M}{k+1} E_L^1(0), \tag{4.6.53}$$

and therefore, for any $t > 0$, as there exists $k \in \mathbb{N}$ such that $kT \leq t \leq (k+1)T$, we deduce that

$$E_L(t) \leq \frac{MC_1L}{t} E_L^1(0) = \begin{cases} \dfrac{C_1 L E_L(0)}{t} & \text{if} \quad M = \xi_0, \\ \text{or} & \\ \dfrac{C_8 L^2 E_L^1(0)}{t} & \text{if} \quad M = \dfrac{2}{C_7}, \end{cases} \tag{4.6.54}$$

where $C_8 = \dfrac{4C_1^3}{C_6}$. On the other hand, from Theorem 4.6.1 we have $\lambda_k^2 \geq L^2$, then we can easily prove the following inequality:

$$E_L(0) \leq \frac{1}{L^2} E_L^1(0). \tag{4.6.55}$$

Finally, combining (4.6.54) and (4.6.55), we conclude that (4.6.41) holds. ∎

4.7 Polynomial energy decay rate on the unit square

In this section, we establish a polynomial decay rate of the energy of system (4.1.2) when our domain Ω is the unit square of the plane in $\mathbb{R}^2$ with $\Gamma_1 = \{0\} \times (0,1)$. This case does not satisfies the assumptions of Theorem (4.4.3) since neither the condition (H_1) holds nor $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ holds. Nevertheless, combining a Fourier analysis and the results from the previous section we obtain a polynomial decay rate (compare with [71]). Consequently, we perform the partial Fourier analysis of the solution $U = (u, w)$ of system (4.1.2)

$$U(x, y, t) = \sum_{p=1}^{+\infty} U^{p\pi}(x, t) \sin(p\pi y), \tag{4.7.1}$$

where $U^{p\pi}(x,t) = (u^{p\pi}(x,t), w^{p\pi}(t))$ is solution of system (4.6.1) with $L = p\pi$. Recalling that the energy of system (4.6.1) is given by

$$
\begin{aligned}
E_{p\pi}(t) =& \frac{1}{2}\int_0^1 \left(|u_t^{p\pi}(x,t)|^2 + |u_x^{p\pi}(x,t)|^2 + p^2\pi^2|u^{p\pi}(x,t)|^2\right) dx \\
&+ \frac{1}{2}|w_t^{p\pi}(t)|^2 + \frac{p^2\pi^2}{2}|w^{p\pi}(t)|^2.
\end{aligned}
\tag{4.7.2}
$$

We clearly have

$$
E(t) = \sum_{p=1}^{+\infty} E_{p\pi}(t). \tag{4.7.3}
$$

Using a Fourier synthesis and the result of Theorem 4.6.5, we obtain the following polynomial decay of energy of system (4.1.2) on the unit square:

Theorem 4.7.1. *There exists a positive constant $C > 0$, such that for all initial data $U_0 = (u_0, u_1, w_0, w_1) \in D(\mathcal{A}^2)$, the energy of system (4.1.2) in the unit square of $\mathbb{R}^2$ with $\Gamma_1 = \{0\} \times (0,1)$, satisfies*

$$
E(t) \leq \frac{C}{t}\|U_0\|^2_{D(\mathcal{A}^2)}. \tag{4.7.4}
$$

Proof: First, combining (4.6.41) and (4.7.3) we obtain

$$
E(t) = \sum_{p=1}^{+\infty} E_{p\pi}(t) \leq \frac{C_9}{t}\sum_{p=1}^{+\infty} p^2\pi^2 E^1_{p\pi}(0), \tag{4.7.5}
$$

where

$$
E^1_{p\pi}(0) = \frac{1}{2}\|U^{p\pi}(x,0)\|_{D(A_L)} + \frac{1}{2}\|U_t^{p\pi}(x,0)\|_{V_L}, \tag{4.7.6}
$$

$U^{p\pi}(x,0) = (u_0^{p\pi}(x), w_0^{p\pi})$, $\quad U_t^{p\pi}(0) = (u_1^{p\pi}(x), w_1^{p\pi})$ and where $(u_i^{p\pi}(x), w_i^{p\pi})$, $i \in \{0,1\}$ are the initial data of system (4.6.1). By integrating by parts and by using the boundary conditions of system (4.6.1) we obtain

$$
\begin{aligned}
\|A_L U^{p\pi}(x,0)\|_H^2 =& \int_0^1 |u_{0,xx}^{p\pi}(x)|^2 dx + p^4\pi^4\int_0^1 |u_0^{p\pi}(x)|^2 dx \\
&+ 2p^2\pi^2\int_0^1 |u_{0,x}^{p\pi}(x)|^2 dx + p^4\pi^4|w_0^{p\pi}|^2 + |u_{0,x}^{p\pi}(0)|^2,
\end{aligned}
\tag{4.7.7}
$$

while $\|\cdot\|_{D(A_L)} \lesssim \|A_L \cdot\|_H$, while by definition we have

$$\begin{aligned}\|U_t^{p\pi}(x,0)\|_{V_L}^2 &= \langle A_L U_t^{p\pi}(x,0), U_t^{p\pi}(x,0)\rangle \\ &= \int_0^1 |u_{1,x}^{p\pi}(x)|^2 dx + p^2\pi^2 \int_0^1 |u_1^{p\pi}(x)|^2 dx \\ &\quad + p^2\pi^2 |w_1^{p\pi}|^2.\end{aligned} \tag{4.7.8}$$

Now, combining (4.7.5), (4.7.6), (4.7.7) and (4.7.8) we get

$$E(t) \leq \frac{C_9}{t} E_2(0) \tag{4.7.9}$$

where

$$\begin{aligned}E_2(0) =& \sum_{p=1}^{+\infty} \int_0^1 \left(p^2\pi^2 |u_{0,xx}^{p\pi}(x)|^2 + p^6\pi^6 |u_0^{p\pi}(x)|^2 + 2p^4\pi^4 |u_{0,x}^{p\pi}(x)|^2\right) dx \\ &+ \sum_{p=1}^{+\infty} p^6\pi^6 |w_0^{p\pi}|^2 + \sum_{p=1}^{+\infty} p^2\pi^2 |u_{0,x}^{p\pi}(0)|^2 \\ &+ \sum_{p=1}^{+\infty} \int_0^1 \left(p^2\pi^2 |u_{1,x}^{p\pi}(x)|^2 + p^4\pi^4 |u_1^{p\pi}(x)|^2\right) dx + \sum_{p=1}^{+\infty} p^4\pi^4 |w_1^{p\pi}|^2.\end{aligned} \tag{4.7.10}$$

Finally, by Parseval's inequality and the result of Proposition 4.2.4 we deduce that

$$\begin{aligned}E_2(0) \leq& \|u_0\|_{H^3(\Omega)}^2 + \|w_0\|_{H^3(\Gamma_1)}^2 + \|u_1\|_{H^2(\Omega)}^2 + \|w_1\|_{H^2(\Gamma_1)}^2 \\ &\sim \|U_0\|_{D(\mathcal{A}^2)}.\end{aligned} \tag{4.7.11}$$

■

5

Polynomial stabilization of the finite difference space discretization of the 1-d wave equation with dynamic boundary control

Abstract: In [91], Wehbe has shown that the energy of the wave equation with dynamic boundary control decays polynomially in $\frac{1}{t}$ to zero as time goes to infinity. In this chapter, we consider its finite difference space discretization scheme and we analyze whether the decay rate is independent of the mesh size. First, we show that the polynomial decay in $\frac{1}{t}$ of the energy of the classical semi-discrete system is not uniform with respect to the mesh size. Next, we add a suitable vanishing numerical viscosity term which leads to a uniform (with respect to the mesh size) polynomially decay in $\frac{1}{t}$ of the energy. Finally, we prove the convergence of the scheme towards the original damped wave equation. Our method is essentially based on discrete multiplier techniques.

5.1 Introduction

In this chapter, we consider the finite difference space discretization scheme of the following 1-d damped wave equation with dynamic boundary control:

$$\begin{cases} y''(x,t) - y_{xx}(x,t) &= 0, & (x,t) \in]0,1[\times \mathbb{R}_*^+, \\ y(0,t) &= 0, & t \in \mathbb{R}_*^+, \\ y_x(1,t) + \eta(t) &= 0, & t \in \mathbb{R}_*^+, \\ \eta'(t) - y'(1,t) + \beta\eta(t) &= 0, & t \in \mathbb{R}_*^+, \\ y(x,0) &= y^0(x), & x \in]0,1[, \\ y'(x,0) &= y^1(x), & x \in]0,1[, \\ \eta(0) &= \eta^0, & \end{cases} \tag{5.1.1}$$

where $(y^0, y^1, \eta^0) \in \mathcal{H} := V \times L^2(0,1) \times \mathbb{C}$,

$$V = H_L^1(0,1) = \left\{ y \in H^1(0,1);\ y(0) = 0 \right\},$$

$\beta > 0$ is a constant and $'$ denotes the partial derivative with respect to the time variable t. The concept of dynamical control in the infinite dimensional case is very close to the one of indirect damping proposed by Russel [85]. System (5.1.1) arises in many areas of mechanics, engineering and technology. This model may be viewed as a model for describing the vibrations of structures, the propagation of acoustic or seismic waves, etc.

The stabilization of the wave equation retains the attention of many authors. In this regard, different types of wave equation with diverse dampings and in various domains was studied: semilinear wave equation with localized damping in unbounded domains [27, 94], wave equation with a nonlinear internal damping [61] and wave equation on general 1-d networks [8, 9, 10, 11, 12, 73, 89, 93].

The energy of the damped wave equation (5.1.1) is given by

$$E(t) = \frac{1}{2}\left(\int_0^1 |y'(x,t)|^2 dx + \int_0^1 |y_x(x,t)|^2 dx + |\eta(t)|^2 \right),\ t \geq 0 \tag{5.1.2}$$

and pursues the following dissipation law:

$$E^{'}(t) = -\beta |\eta(t)|^2, \quad t \geq 0. \tag{5.1.3}$$

Equation (5.1.3) shows that the energy decreases as time increases. In [91], Wehbe started by formulate system (5.1.1) into a first order evolution equation $U'(t) = (\mathcal{A}+C)U(t)$ for $t>0$, with $U(0)=U_0 \in \mathcal{H}$, where $U=(y, y_t, \eta)$,

$$D(\mathcal{A}) = \{U=(y,z,\eta) \in \mathcal{H};\ z \in V \quad \text{and} \quad y_x(1)+\eta = 0\}, \tag{5.1.4}$$

$$\mathcal{A}U = (z, y_{xx}, z(1)), \quad \forall U=(y,z,\eta) \in D(\mathcal{A})$$

and where

$$CU = (0,0,\beta\eta), \quad \forall U \in \mathcal{H}.$$

Next, using a multiplier method, he has shown a polynomial decay in $\dfrac{1}{t}$ of the energy of system (5.1.1), for smooth initial data. Roughly speaking, he showed that the energy of (5.1.1) satisfies the following estimation:

$$E(t) \leq \frac{M_1}{M_1+t}E(0), \qquad \forall t>0, \tag{5.1.5}$$

where

$$M_1 = 2\left(\frac{E_1(0)}{\beta E(0)} + \beta + \frac{1}{2\beta} + 3\right) \tag{5.1.6}$$

and where E_1 denotes the energy of higher order system associated to (5.1.1), *i.e.*

$$E_1(t) = \frac{1}{2}\left(\int_0^1 |y''(x,t)|^2 dx + \int_0^1 |y_x'(x,t)|^2 dx + |\eta'(t)|^2\right),\ t \geq 0. \tag{5.1.7}$$

Later, using a spectral method and a Riesz basis approach, Wehbe proved that the obtained decay rate is optimal in the sense that for any $\epsilon>0$, we cannot expect a decay rate of type $\dfrac{1}{t^{1+\epsilon}}$.

However as far as numerical approximation schemes are concerned, little is known about the uniform (with respect to the mesh size) decay of the discretized energy. In many applications, although the continuous system is exponentially or polynomially stable, the discrete ones does not inherit the same property uniformly with respect to the mesh size. In [88], Tebou and Zuazua considered the approximation scheme of the wave equation with static boundary condition

$$\begin{cases} y''(x,t) - y_{xx}(x,t) &= & 0, & (x,t) \in]0,1[\times \mathbb{R}_*^+, \\ y(0,t) &= & 0, & t \in \mathbb{R}_*^+, \\ y_x(1,t) + \alpha y'(1,t) &= & 0, & t \in \mathbb{R}_*^+, \\ y(x,0) &= & y^0(x), & x \in]0,1[, \\ y'(x,0) &= & y^1(x), & x \in]0,1[. \end{cases} \tag{5.1.8}$$

It is surely understood (see [16, 25, 51, 52, 53, 54, 77, 84, 95]) that the energy of (5.1.8) satisfies the exponential decay to zero. First, Tebou and Zuazua shown that the decay rate of the energy of the classical semi-discrete system associated to (5.1.8) is not uniform with respect to the net-spacing size. This is due to the existence of high frequency spurious solutions of the semi-discrete model that propagate very slowly (with group velocity of the order of the mesh size). By adding a suitable vanishing numerical viscosity term, they next proved the uniform (with respect to the mesh size) exponential decay of the energy. Finally, they shown the convergence of the scheme towards the original damped wave equation (without viscosity term). Due to the presence of the dynamic term, the method used in [88] does not work for our system. In [3] , Abdallah *and al.* considered the approximation of second order evolution equations with a bounded damping. First, they damped the spurious high frequency modes by introducing a numerical viscosity term in the approximation scheme. Next, with this viscosity term, they showed the exponential or polynomial decay of the discrete scheme when the continuous problem has such a decay and when the spectrum of the spatial operator associated with the undamped problem satisfies the generalized gap condition. Finally, using the Trotter-Kato's theorem [49], they showed the convergence of the discrete solution to the continuous one. In our work, the damping is not bounded and for this reason there method cannot be applied for our system. Then, the stabilization of the discrete scheme of the wave equation with dynamical boundary control remains to be an open problem. For more details about the stabilization of the discretization scheme of wave equation we refer to [28, 64, 87, 96].

In this chapter, we consider the finite-difference space semi-discretization scheme of (5.1.1) and we analyze whether the decay rate is independent of the mesh size. Our main purpose in this work is twofold:

1) To scrupulously prove that for the classical finite difference scheme,

the polynomial decay of the discretized energy is not uniform with respect to the mesh size.

2) To add a correctly numerical viscous term in the equation in order to achieve an uniformly (with respect to the mesh size) energy decay rate of type (5.1.5).

We now introduce the finite difference scheme we will work on. For this aim, let N be an non-negative integer, set $h = \dfrac{1}{N+1}$ and consider the subdivision of $]0,1[$ given by

$$0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1, \quad x_j = jh.$$

The finite-difference space semi-discretization of system (5.1.1) that we consider is

$$\begin{cases} y_j''(t) - \dfrac{y_{j+1}(t) - 2y_j(t) + y_{j-1}(t)}{h^2} & = & 0, & t > 0,\ j = 1, .., N, \\ y_0(t) & = & 0, & t > 0, \\ \dfrac{y_{N+1}(t) - y_N(t)}{h} + \eta(t) & = & 0, & t > 0, \\ \eta'(t) - y_{N+1}'(t) + \beta\eta(t) & = & 0, & t > 0, \\ y_j(0) & = & y_j^0, & j = 1, ..., N, \\ y_j'(0) & = & y_j^1, & j = 1, ..., N, \\ \eta(0) & = & \eta^0, & \end{cases} \tag{5.1.9}$$

where $(y_0^i, y_1^i, .., y_N^i, y_{N+1}^i)$ for $i = 0, 1$ provides an approximation of the function $y^i(x)$ for $i = 0, 1$ at point x_j for $j = 0, .., N+1$ and η^0 is the third component of the initial data (y^0, y^1, η^0) of system (5.1.1).
The energy of system (5.1.9) is given by

$$\mathcal{E}_h(t) = \frac{h}{2}\left(\sum_{j=1}^{N} |y_j'(t)|^2 + \sum_{j=0}^{N}\left(\frac{y_{j+1}(t) - y_j(t)}{h}\right)^2\right) + \frac{1}{2}|\eta(t)|^2, \tag{5.1.10}$$

for $t \geq 0$ and it is a non-increasing function with respect to the time variable t since its derivative is given by

$$\mathcal{E}_h'(t) = -\beta|\eta(t)|^2, \quad t \geq 0. \tag{5.1.11}$$

For simplicity, here and below we eliminate t, *i.e.* we denote $y_j(t)$ (respectively $\eta(t)$) by y_j for $j = 0, .., N+1$ (respectively η).
Note that $\mathcal{E}_h$ is a natural semi-discrete version of the energy E of system (5.1.1) and that (5.1.11) is the semi-discrete analogue of the energy dis-

sipation (5.1.3). It is logic to ask if the energy $\mathcal{E}_h$ decays polynomially, and uniformly with respect to h, to zero as the time t tends to infinity. For any fixed $h > 0$, it is easy to see that the energy of (5.1.9) tends exponentially to zero as time goes to infinity. However, as we already said, earlier results obtained by Banks et al. [15], Tebou and Zuazua [88] and others [3, 28, 64, 87, 96] lead us to think that the decay rates degenerates as h tends to zero. Before presenting our first result which confirms this issue, we need to introduce some operators. First, using the third equation of (5.1.9) we have

$$y_{N+1} = y_N - h\eta. \tag{5.1.12}$$

Eliminating y_{N+1} from (5.1.9), we get the following system:

$$\begin{cases} y_j'' - \dfrac{y_{j+1} - 2y_j + y_{j-1}}{h^2} & = & 0, & t > 0, j = 1, 2, .., N-1, \\ y_N'' - \dfrac{y_{N-1} - y_N - h\eta}{h^2} & = & 0, & t > 0, \\ y_0 & = & 0, & t > 0, \\ (1+h)\eta' - y_N' + \beta\eta & = & 0, & t > 0, \\ y_j(0) & = & y_j^0, & j = 1, ..., N, \\ y_j'(0) & = & y_j^1, & j = 1, ..., N, \\ \eta(0) & = & \eta^0. & \end{cases} \tag{5.1.13}$$

The energy of system (5.1.13) is given by

$$\begin{aligned} \widetilde{\mathcal{E}}_h(t) =& \frac{h}{2}\left(\sum_{j=1}^{N} |y_j'(t)|^2 + \sum_{j=1}^{N}\left(\frac{y_j(t) - y_{j-1}(t)}{h}\right)^2\right) \\ &+ \frac{(1+h)}{2}|\eta(t)|^2, \qquad t \geq 0 \end{aligned} \tag{5.1.14}$$

and it is a non-increasing function with respect to the time variable t since

$$\widetilde{\mathcal{E}}_h'(t) = -\beta|\eta(t)|^2, \quad t \geq 0. \tag{5.1.15}$$

Remark 5.1.1. *We can check that the solution of system (5.1.13) and the one of (5.1.9) are equivalent via equation (5.1.12). Consequently, the energy of system (5.1.13) given in (5.1.14) is equal to the one of system (5.1.9) given in (5.1.10) via the same equation.* ■

Next, we define $C_h := \mathbb{C}^N$ the subspace which contains the discretized vectors $y_h = (y_1, .., y_N)$ of y and we introduce the operator A_h and its bilinear form a_h as follows:

$$(A_h y_h, \widetilde{y}_h)_{C_h\times C_h} = a_h(y_h, \widetilde{y}_h) = h\sum_{j=1}^{N}\left(\frac{y_j - y_{j-1}}{h}\right)\left(\frac{\overline{\widetilde{y}_j} - \overline{\widetilde{y}_{j-1}}}{h}\right) \quad (5.1.16)$$
$$= \frac{1}{h} y_h K_h^0 \overline{\widetilde{y}_h^t}, \quad \forall y_h, \widetilde{y}_h \in C_h,$$

where

$$K_h^0 = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & -1 & 2 & -1 \\ 0 & 0 & \dots & -1 & 1 \end{pmatrix} \in M_{N\times N}(\mathbb{R}).$$

Moreover, for all $\eta \in \mathbb{C}$, we define the operator $B_h \in \mathcal{L}(\mathbb{C}, C_h)$ by

$$B_h \eta = \frac{\eta}{h} V_0, \quad (5.1.17)$$

where $V_0 = (0, .., 1) \in C_h$. We can easily check that the adjoint operator $B_h^* \in \mathcal{L}(C_h, \mathbb{C})$ associated to B_h is given by

$$B_h^* y_h = y_N, \quad \forall y_h \in C_h. \quad (5.1.18)$$

Further, we introduce the Hilbert space $\mathcal{H}_h = C_h \times C_h \times \mathbb{C}$ endowed with the norm

$$\|U_h\|_{\mathcal{H}_h}^2 = a_h(y_h, y_h) + (z_h, z_h)_h + (1+h)|\eta|^2, \quad (5.1.19)$$
$$\forall U_h = (y_h, z_h, \eta) \in \mathcal{H}_h,$$

where

$$(z_h, z_h)_h = h\sum_{j=1}^{N} |z_j|^2$$

and we define the bounded operator $\mathcal{A}_{\beta,h}$ in the Hilbert space $\mathcal{H}_h$ by

$$\mathcal{A}_{\beta,h} U_h = \left(z_h, -A_h y_h - B_h \eta, \frac{1}{1+h}\left(B_h^* z_h - \beta\eta\right)\right) \quad (5.1.20)$$
$$\forall U_h = (y_h, z_h, \eta) \in \mathcal{H}_h.$$

Let $U_h = (y_h, y_h^{'}, \eta) \in D(\mathcal{A}_{\beta,h})$ be a solution of system (5.1.13). Then we have

$$\begin{cases} U_h'(t) &= \mathcal{A}_{\beta,h} U_h(t), \quad t \geq 0, \\ U_h(0) &= U_h^0 \in \mathcal{H}_h. \end{cases} \tag{5.1.21}$$

A direct computation gives

$$\Re(\mathcal{A}_{\beta,h} U_h, U_h)_{\mathcal{H}_h} = -\beta |\eta(t)|^2.$$

Furthermore, we introduce the norm in the domain of $\mathcal{A}_{\beta,h}$ by

$$\|U_h\|^2_{D(\mathcal{A}_{\beta,h})} = \|U_h\|^2_{\mathcal{H}_h} + \|\mathcal{A}_{\beta,h} U_h\|^2_{\mathcal{H}_h}, \qquad U_h \in \mathcal{H}_h. \tag{5.1.22}$$

In the following theorem, we show that the discrete energy $\widetilde{\mathcal{E}}_h$ does not inherit the polynomial decay type of the continuous one E, uniformly with respect to the mesh size h. More precisely, we have the next result:

Theorem 5.1.2. *For any $h_0 > 0$, there do not exists a positive constant M_2 independent of $h \in]0, h_0[$ and of $U_h^0 \in \mathcal{H}_h$, such that the energy $\widetilde{\mathcal{E}}_h$ of system (5.1.21) satisfies the following estimation:*

$$\widetilde{\mathcal{E}}_h(t) \leq \frac{M_2}{t} \|U_h^0\|^2_{D(\mathcal{A}_{\beta,h})}, \quad \forall t > 0. \tag{5.1.23}$$

■

Result of Theorem 5.1.2 is due to the existence of a sequence of eigenvalues associated to the operator $\mathcal{A}_{\beta,h}$, for $h = \frac{1}{N+1}$, N large enough, which do not satisfy a necessary condition for (5.1.23). Several remedies have been proposed and analyzed to overcome this difficulties like the Tychonoff regularization [38, 39, 78, 87], a bi-grid algorithm [36, 70], a mixed finite element method [15, 22, 23, 37, 68], or filtering the high frequencies [47, 57]. As in [3, 88], our goal is to damp the spurious high frequency modes by introducing a numerical viscosity in the approximation scheme. For this aim, we consider the new system with the extra numerical viscosity

$$\begin{cases} y_j'' - \dfrac{y_{j+1} - 2y_j + y_{j-1}}{h^2} - (y_{j+1}' - 2y_j' + y_{j-1}') &= 0, \quad t > 0, j = 1, .., N, \\ \dfrac{y_{N+1} - y_N}{h} + \eta &= 0, \quad t > 0, \\ \eta' - y_{N+1}' + \beta\eta &= 0, \quad t > 0, \\ y_0 &= 0, \quad t > 0, \\ y_j(0) &= y_j^0, \quad j = 1, ..., N, \\ y_j'(0) &= y_j^1, \quad j = 1, ..., N, \\ \eta(0) &= \eta^0. \end{cases} \tag{5.1.24}$$

where y_j^0 and y_j^1 (for $j = 0, .., N+1$) are given by

$$y_j^0 = y^0(jh), \quad \text{for } j = 0, .., N+1, \tag{5.1.25}$$

$$y_0^1 = 0, \quad y_j^1 = \frac{1}{h}\int_{(j-1)h}^{jh} y^1(x)dx, \quad \text{for } j = 1, .., N+1 \tag{5.1.26}$$

and where (y^0, y^1, η^0) designates the initial data of system (5.1.1). The natural energy of system (5.1.24) is given by

$$\begin{aligned} E_h(t) =& \frac{h}{2}\sum_{j=1}^{N}|y_j^{'}(t)|^2 + \frac{h}{2}\sum_{j=0}^{N}\left(\frac{y_{j+1}(t) - y_j(t)}{h}\right)^2 \\ &+ \frac{(1+h^2\beta)}{2}|\eta(t)|^2, \qquad t \geq 0 \end{aligned} \tag{5.1.27}$$

and a direct computation gives

$$\begin{aligned} E_h^{'}(t) =& -h^3\sum_{j=0}^{N}\left(\frac{y_{j+1}^{'}(t) - y_j^{'}(t)}{h}\right)^2 - \beta|\eta(t)|^2 \\ &- h^2|\eta^{'}(t)|^2, \qquad t \geq 0. \end{aligned} \tag{5.1.28}$$

Hence, E_h is a nonincreasing function with respect to the time variable t. Note that, the first equation of system (5.1.24) is the semi-discrete analogue of

$$y'' - y_{xx} - h^2 y_{xxt} = 0.$$

Now, for system (5.1.24), we prove:

i) A decay rate of type (5.1.5) which is uniform with respect to the net-spacing h.

ii) The convergence of its solution towards the one of the original wave equation (5.1.1) as $h \to 0$.

These two results show that the discretization (5.1.24) of system (5.1.1) is a good approximate scheme because it not only warranties the convergence of solutions as $h \to 0$ but it also furnishes a uniform (with respect to the mesh size h) polynomial decay rate of the energy as $t \to \infty$. The second fact proves that the viscous damping term added in (5.1.24) captures the long time asymptotic properties of system (5.1.1). The suitability of this numerical damping mechanism to restore the uniform polynomial decay is closely connected to the efficiency of the Tychonoff

regularization techniques.
Before going, we give the following natural convergence result which is a consequence of our choice of discretization in (5.1.25)-(5.1.26):

Lemma 5.1.3. *Assume the initial data (y^0, y^1, η^0) of system (5.1.1) belongs to $D(\mathcal{A})$ where $D(\mathcal{A})$ is given in (5.1.4). Then, the initial discrete energy $E_h(0)$ of system (5.1.24) tends to the initial continuous one $E(0)$ as $h \to 0$.* ■

For the uniform polynomial decay result of energy, we introduce the discrete energy $E_{h,1}$ of higher order system associated to (5.1.24) given by

$$
\begin{aligned}
E_{h,1}(t) =& \frac{h}{2}\sum_{j=1}^{N} |y_j^{''}(t)|^2 + \frac{h}{2}\sum_{j=0}^{N}\left(\frac{y_{j+1}^{'}(t) - y_j^{'}(t)}{h}\right)^2 \\
&+ \frac{(1+h^2\beta)}{2}|\eta^{'}(t)|^2 \qquad t \geq 0
\end{aligned}
\tag{5.1.29}
$$

and we need the following result:

Lemma 5.1.4. *Assume the initial data (y^0, y^1, η^0) of system (5.1.1) belongs to $D(\mathcal{A})$ where $D(\mathcal{A})$ is given in (5.1.4). Then, the energy $E_{h,1}$ given in (5.1.29) is uniformly bounded with respect to h at $t = 0$.* ■

We are ready to state our uniform polynomial decay of the energy (5.1.27):

Theorem 5.1.5. *Assume the initial data (y^0, y^1, η^0) of system (5.1.1) belongs to $D(\mathcal{A})$ where $D(\mathcal{A})$ is given in (5.1.4). Then, for h_0 small enough, for all $(y_j^0)_j$, $(y_j^1)_j$, $j = 1, .., N+1$, in $\mathbb{C}^{N+1}$ and $\eta \in \mathbb{C}$, the energy E_h of system (5.1.24) satisfies*

$$
E_h(t) \leq \frac{M}{M+t} E_h(0), \qquad \forall t > 0, \tag{5.1.30}
$$

where $M = \sup\limits_{h \in]0,h_0[} M_h$ and where M_h is given by

$$
M_h = 22 + \beta + \frac{3}{2\beta} + \frac{1}{2\beta}(6+\beta)\frac{E_{h,1}(0)}{E_h(0)}. \tag{5.1.31}
$$

■

Remark 5.1.6. *The constant M of the inequality (5.1.30) exists. Indeed, since the initial data (y^0, y^1, η^0) of system (5.1.1) belongs to $D(\mathcal{A})$,* *then from Lemma 5.1.3 we have $E_h(0) \to E(0) > 0$ as h tends to zero. Therefore, $E_h(0)$ is uniformly bounded with respect to the mesh size h and there exists a constant $\alpha > 0$ such that $E_h(0) \geq \alpha$. Moreover, from Lemma 5.1.4, we know that $E_{h,1}(0)$ is uniformly bounded with respect to the mesh size h. Thus, the supremum of M_h given in (5.1.31) exists.* ■

Theorem 5.1.5 proves that the numerical viscosity term $-(y'_{j+1} - 2y'_j + y'_{j-1})$ added in system (5.1.24) is enough to restore the uniform polynomial decay in $\frac{1}{t}$ of the energy with respect to h. This was already showed for the uniform (with respect to the mesh size h) exponential decay of the energy, in [87] in the case where the damping term is locally distributed in the domain, and in [88] where the damping term is static and localized on the boundary. In addition, we will use the discretization of the multiplier $xy_xE(t)$ used by Wehbe in [91] and then we will deduce estimation (5.1.30) by using a classical result of Haraux in [44] (see also [52]). Before showing our convergence results, we require some complementary notations. We set $\vec{y}_h = (y_j)_j$, for $j = 1, .., N$ and we introduce the extension operators defined by

$$p_h\vec{y}_h(x) = \frac{y_{j+1} - y_j}{h}(x - jh) + y_j, \quad x \in [jh, (j+1)h], \quad j = 0, .., N \tag{5.1.32}$$

and

$$q_h\vec{y}_h(x) = \begin{cases} 0 & \text{if} \quad x \in]0, h[, \\ y_j & \text{if} \quad x \in]jh, (j+1)j[, \quad j = 1, .., N. \end{cases} \tag{5.1.33}$$

Further, we can check that

$$\int_0^1 (p_h\vec{y}_h(x))_x(p_h\vec{z}_h(x))_x dx = h\sum_{j=0}^{N}\left(\frac{y_{j+1} - y_j}{h}\right)\left(\frac{z_{j+1} - z_j}{h}\right) \tag{5.1.34}$$

and

$$\int_0^1 q_h\vec{y}_h(x)q_h\vec{z}_h(x)dx = h\sum_{j=0}^{N} y_j z_j. \tag{5.1.35}$$

We are now ready to give our convergence results.

Theorem 5.1.7. *Let $(\vec{y}_h, \vec{y}_h{}', \eta)$ denotes the solution of (5.1.24) and assume the initial data (y^0, y^1, η^0) of system (5.1.1) belongs to $D(\mathcal{A})$ where*

$D(\mathcal{A})$ is given in (5.1.4). Then as $h \to 0$, we have

$$\begin{cases} p_h \vec{y}_h & \rightarrow & y & \text{weakly* in} & L^\infty(0,\infty;V), \\ q_h \vec{y}_h{}' & \rightarrow & y' & \text{weakly* in} & L^\infty(0,\infty;L^2(0,1)), \end{cases} \tag{5.1.36}$$

where (y, y', η) is the solution of system (5.1.1).

Moreover, the following convergence holds:

$$\begin{cases} p_h \vec{y}_h & \rightarrow & y & \text{strongly in} & L^2_{loc}(0,\infty;V), \\ q_h \vec{y}_h{}' & \rightarrow & y' & \text{strongly in} & L^2_{loc}(0,\infty;L^2(0,1)), \\ p_h \vec{y}_h & \rightarrow & y & \text{strongly in} & C([0,T];L^2(0,1)), \end{cases} \tag{5.1.37}$$

where $T > 0$ and

$$\lim_{h\to 0} \|E_h - E\|_{C([0,\infty[)} = 0. \tag{5.1.38}$$

■

Let us briefly outline the content of this chapter:
Section 5.2 is devoted to prove Theorem 5.1.2. In section 5.3, we give the proofs of Lemma 5.1.3, Lemma 5.1.4 and the uniform polynomially energy decay result given by Theorem 5.1.5. Section 5.4 deals with the proof of the convergence results of Theorem 5.1.7.

5.2 Non uniform polynomial energy decay

In this section, we give the proof of Theorem 5.1.2. This proof relies on the following lemma:

Lemma 5.2.1. *Assume that there exists a positive constant M_2 independent of h and of $U_h^0 \in \mathcal{H}_h$ such that the energy $\widetilde{\mathcal{E}}_h$ of system (5.1.21) given in (5.1.14) satisfies (5.1.23). Then, there exists another constant M_3 independent of h such that any eigenvalue λ of $\mathcal{A}_{\beta,h}$ with $|\lambda| > 1$, must satisfies the following estimation:*

$$-\Re(\lambda)|\lambda|^2 > M_3. \tag{5.2.1}$$

Proof: Let λ be an eigenvalue of $\mathcal{A}_{\beta,h}$ such that $|\lambda| > 1$ and $U_h^0 \in \mathcal{H}$ be an associated eigenvector such that $\|U_h^0\|_{\mathcal{H}_h} = 1$. Since $\mathcal{A}_{\beta,h}$ is dissipative then $\Re(\lambda) \leq 0$ and we have

$$\begin{aligned}
\widetilde{\mathcal{E}}_h(t) &= \frac{1}{2}\|U_h(t)\|^2_{\mathcal{H}_h} \\
&= \frac{1}{2}\|e^{\lambda t}U_h^0\|^2_{\mathcal{H}_h} \\
&= \frac{1}{2}e^{-2|\Re(\lambda)|t}\|U_h^0\|^2_{\mathcal{H}_h} \\
&= \frac{1}{2}e^{-2|\Re(\lambda)|t}.
\end{aligned} \tag{5.2.2}$$

The fact that $\|U_h^0\|_{\mathcal{H}_h} = 1$ and $|\lambda| > 1$, from (5.1.22) we obtain

$$\|U_h^0\|^2_{D(\mathcal{A}_{\beta,h})} = (1+|\lambda|^2)\|U_h^0\|^2_{\mathcal{H}_h} = (1+|\lambda|^2) < 2|\lambda|^2. \tag{5.2.3}$$

Therefore, inserting (5.2.2) and (5.2.3) in (5.1.23), we get

$$\frac{1}{2}e^{-2|\Re(\lambda)|t} < \frac{2M_2|\lambda|^2}{t}, \qquad \forall t > 0.$$

Taking $t = -\dfrac{1}{2\Re(\lambda)}$ in the above inequality we obtain

$$\frac{1}{2}e^{-1} < -4M_2\Re(\lambda)|\lambda|^2,$$

or equivalently

$$\frac{1}{8eM_2} < -\Re(\lambda)|\lambda|^2.$$

Finally, setting

$$M_3 = \frac{1}{8eM_2}, \tag{5.2.4}$$

we deduce (5.2.1).

In order to proof Theorem 5.1.2, thanks to Lemma 5.2.1, we need to find a sequence $(\lambda_N)_N \in \sigma(\mathcal{A}_{\beta,h})$ (with $h = \frac{1}{N+1}$) such that $|\lambda_N| > 1$ and

$$\Re(\lambda_N)|\lambda_N|^2 \to 0.$$

For this aim, let $\lambda \in \sigma(\mathcal{A}_{\beta,h})$ be an eigenvalue of $\mathcal{A}_{\beta,h}$ and $U_h = (y_h, z_h, \eta) \in \mathcal{H}_h$ be an associated eigenvector, for $h = \frac{1}{N+1}$, N large enough. Equivalently, λ and U_h verify the following system:

$$\begin{cases} z_h & = & \lambda y_h, \\ -A_h y_h - B_h \eta & = & \lambda z_h, \\ B_h^* z_h - \beta \eta & = & (1+h)\lambda \eta, \end{cases}$$

where the operators A_h, B_h and B_h^* are given in (5.1.16), (5.1.17) and (5.1.18) respectively. Eliminating z_h and η from above system, we obtain

$$A_h y_h + \alpha_h y_N B1 = -\lambda^2 y_h, \tag{5.2.5}$$

where

$$\alpha_h = \frac{d_h \lambda}{\lambda + d_h \beta} \quad \text{and} \quad d_h = \frac{1}{1+h}. \tag{5.2.6}$$

From (5.1.16) we have $A_h := \dfrac{1}{h^2} K_h^0$. Multiplying (5.2.5) by h^2 yields

$$K_h^0 y_h^t + h\alpha_h y_N V_0^t = -h^2 \lambda^2 y_h^t, \tag{5.2.7}$$

where V_0 is given in (5.1.17). Next, setting $\widetilde{K}_h^0 = 2I - K_h^0$, from (5.2.7) we get

$$\widetilde{K}_h^0 y_h^t - h\alpha_h y_N V_0^t = (2 + h^2\lambda^2) y_h^t. \tag{5.2.8}$$

Equivalently

$$\widetilde{K}_h y_h^t = (2 + h^2\lambda^2) y_h^t, \tag{5.2.9}$$

where

$$\widetilde{K}_h = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & 1 & 0 & 1 \\ 0 & 0 & \dots & 1 & 1 - h\alpha_h \end{pmatrix} \in M_{N\times N}(\mathbb{R}).$$

We have found that λ is an eigenvalue of $\mathcal{A}_{\beta,h}$ (equivalently $2 + h^2\lambda^2$ is an eigenvalue of $\widetilde{K}_h$) if and only if there is a non trivial solution y_h of (5.2.7) (equivalently of (5.2.9)). The following lemma give us the eigenvalue which does not satisfy condition (5.2.1):

Lemma 5.2.2. *There exists a sequence of eigenvalues of* $\mathcal{A}_{\beta,h}$*,* $h = \frac{1}{N+1}$*,* *which has the following expansion:*

$$\lambda_N = i\left(2+2N-\frac{\pi^2}{4N}+\frac{3\pi^2}{16N^3}+\frac{\pi^4}{192N^3}-\frac{\pi^4}{192N^4}-\frac{5\pi^2}{16N^4}\right) - \frac{\beta\pi^2}{16N^4}+o(\frac{1}{N^4}), \quad N\to+\infty. \tag{5.2.10}$$

Proof: First, in order to solve (5.2.9), we set $y_h = (y_1, .., y_N)$ and we introduce

$$y_{N+1} = (1-h\alpha_h)y_N \tag{5.2.11}$$

and

$$\delta = 2+h^2\lambda^2. \tag{5.2.12}$$

From (5.2.9) we obtain the following system:

$$\begin{cases} \delta y_j &= y_{j+1}+y_{j-1}, \quad j=1,..,N, \\ y_0 &= 0. \end{cases} \tag{5.2.13}$$

We can check that $\delta = 0$ cannot be an eigenvalue of $\widetilde{K}_h$. Next, let $\xi \in \mathbb{C}$, we look for a solution of system (5.2.13) such that $y_j = \xi^j$. Thus

$$\xi^2-\delta\xi+1=0.$$

The above equation has two complex roots ξ_1 and ξ_2 which can be writed as $\xi_1 = e^{\alpha_1+i\theta_1}$ and $\xi_2 = e^{\alpha_2+i\theta_2}$, where $\alpha_i,\ \theta_i \in \mathbb{R}$ for $i=1,2$. Since the product $P = \xi_1\xi_2 = 1$, we have $\alpha_1 = -\alpha_2$ and $\theta_1 = -\theta_2$. Introducing θ such that $i\theta = \alpha_1 + i\theta_1$, then the general solution y_j of the first equation of (5.2.13) has the form

$$y_j = c_1e^{ji\theta}+c_2e^{-ij\theta}, \qquad j=1,..,N+1, \tag{5.2.14}$$

with $c_1, c_2 \in \mathbb{C}$. The fact that $y_0 = 0$ and taking $c_1 = \dfrac{1}{2i}$, from above equation we obtain

$$y_j = \sin(j\theta), \qquad j=1,..,N+1. \tag{5.2.15}$$

Moreover, inserting the above equation into the first one of (5.2.13) for $j=1$ we get

$$\delta = 2\cos(\theta). \tag{5.2.16}$$

Now, inserting (5.2.15) in (5.2.11) we obtain the following characteristic equation:

$$\sin((N+1)\theta) - \sin(N\theta) + h\alpha_h \sin(N\theta) = 0. \tag{5.2.17}$$

We rewrite (5.2.17) as

$$f_N(\theta) = f_N^0(\theta) + f_N^1(\theta) = 0,$$

where $f_N^0(\theta) = \sin((N+1)\theta) - \sin(N\theta)$ and where $f_N^1(\theta) = h\alpha_h \sin(N\theta)$. For clarity, we divide the rest of the proof into three steps.
Step 1. A root of f_N. The roots of f_N^0 are given by

$$\theta_{N,k}^0 = \frac{2\pi}{2N+1}(k+\frac{1}{2}), \quad k \in \mathbb{Z}$$

and we are interested to a root of f_N which is close to

$$\theta_{N,N-1}^0 = \pi - \frac{2\pi}{2N+1}. \tag{5.2.18}$$

For this aim, let $\theta \in \partial D(\theta_{N,N-1}^0, \frac{1}{N^2})$, *i.e.* $\theta = \theta_{N,N-1}^0 + \frac{1}{N^2}e^{it}$ where $t \in [0, 2\pi]$. For $N \in \mathbb{N}^*$ large enough, we can check that

$$|f_N^0(\theta)| \gtrsim \frac{1}{N} \quad \text{and} \quad |f_N^1(\theta)| \lesssim \frac{1}{N^2}.$$

Consequently, there exists $N_0 \in \mathbb{N}^*$, sufficiently large, such that for $N \geq N_0$ we have

$$|f_N(\theta) - f_N^0(\theta)| = |f_N^1(\theta)| < |f_N^0(\theta)|, \quad \forall \theta \in \partial D(\theta_{N,N-1}^0, \frac{1}{N^2}).$$

Then, with the help of Rouché's theorem, we deduce that f_N has a root (denoted by θ_N) which is simple and close to $\theta_{N,N-1}^0$ for $N \geq N_0$. More precisely, for $N \geq N_0$, θ_N is given by

$$\theta_N = \theta_{N,N-1}^0 + \zeta_N, \quad \zeta_N = O(\frac{1}{N^2}). \tag{5.2.19}$$

Step 2. Asymptotic behavior of θ_N. First, using (5.2.17) we have

$$\sin((N+1)\theta_N) - \sin(N\theta_N) + h\alpha_h \sin(N\theta_N) = 0. \tag{5.2.20}$$

Next, using (5.2.18) and (5.2.19) we obtain

$$\begin{aligned}\sin((N+1)\theta_N) =&(-1)^N \sin\left(-\frac{\pi}{2N+1}+(N+1)\zeta_N\right) \\ =&(-1)^N \sin\left(-\frac{\pi}{2N+1}\right)\cos((N+1)\zeta_N) \\ &+(-1)^N \cos\left(\frac{\pi}{2N+1}\right)\sin((N+1)\zeta_N).\end{aligned} \tag{5.2.21}$$

On the other hand, using the asymptotic behavior when $N \to +\infty$ we get

$$\sin\left(-\frac{\pi}{2N+1}\right) = -\frac{\pi}{2N}+\frac{\pi}{4N^2}+\frac{\pi^3-6\pi}{48N^3}+O(\frac{1}{N^4}), \tag{5.2.22}$$

$$\cos((N+1)\zeta_N) = 1-\frac{N^2\zeta_N^2}{2}-N\zeta_N^2+O(\frac{1}{N^4}), \tag{5.2.23}$$

$$\cos\left(-\frac{\pi}{2N+1}\right) = 1-\frac{\pi^2}{8N^2}+\frac{\pi^2}{8N^3}+O(\frac{1}{N^4}) \tag{5.2.24}$$

and

$$\sin((N+1)\zeta_N) = N\zeta_N+\zeta_N-\frac{N^3\zeta_N^3}{6}+O(\frac{1}{N^4}). \tag{5.2.25}$$

Substituting (5.2.22)-(5.2.25) into (5.2.21) yields

$$\begin{aligned}\sin((N+1)\theta_N) =&(-1)^N\left(-\frac{\pi}{2N}+N\zeta_N+\frac{\pi}{4N^2}+\zeta_N+\frac{\pi^3-6\pi}{48N^3}\right) \\ &-(-1)^N\left(\frac{N^3\zeta_N^3}{6}+\frac{\pi^2\zeta_N}{8N}-\frac{N\pi\zeta_N^2}{4}\right)+O(\frac{1}{N^4}).\end{aligned} \tag{5.2.26}$$

Similarly, we get

$$\begin{aligned}\sin(N\theta_N) =&(-1)^{N-1}\left(\frac{\pi}{2N}+N\zeta_N-\frac{\pi}{4N^2}+\frac{\pi}{8N^3}\right) \\ &-(-1)^{N-1}\left(\frac{\pi^3}{48N^3}+\frac{\pi N\zeta_N^2}{4}+\frac{N^3\zeta_N^3}{6}+\frac{\pi^2\zeta_N}{8N}\right)+O(\frac{1}{N^4}).\end{aligned} \tag{5.2.27}$$

Now, using (5.2.6) and (5.2.12) we have

$$h\alpha_h = \frac{h\lambda_N d_h}{\lambda_N+d_h\beta} = \frac{id_h\sqrt{2-2\cos(\theta_N)}}{i\sqrt{2-2\cos(\theta_N)}+d_hh\beta}. \tag{5.2.28}$$

Applying the same strategy used to obtain (5.2.21)-(5.2.27), we get

$$\cos(\theta_N) = -1 + \frac{\pi^2}{2N^2} - \frac{\pi\zeta_N}{N} - \frac{\pi^2}{2N^3} + O(\frac{1}{N^4}) \tag{5.2.29}$$

and therefore

$$\sqrt{2 - 2\cos(\theta_N)} = 2 - \frac{\pi^2}{4N^2} + \frac{\pi\zeta_N}{2N} + \frac{\pi^2}{4N^3} + O(\frac{1}{N^4}). \tag{5.2.30}$$

Thus, inserting (5.2.30) into (5.2.28) and after some computations we obtain

$$h\alpha_h = \frac{1}{N} - \frac{2}{N^2} + \frac{1}{4N^3}(16 - \beta^2) + i\left(\frac{\beta}{2N^2} - \frac{2\beta}{N^3}\right) + O(\frac{1}{N^4}). \tag{5.2.31}$$

Using (5.2.27) and (5.2.31) we get

$$\begin{aligned} h\alpha_h \sin(N\theta_N) =& (-1)^{N-1}\left(\zeta_N + \frac{\pi}{2N^2} - \frac{2\zeta_N}{N} - \frac{5\pi}{4N^3}\right) \\ &+ i(-1)^{N-1}\left(\frac{\beta\zeta_N}{2N} + \frac{\beta\pi}{4N^3}\right) + O(\frac{1}{N^4}). \end{aligned} \tag{5.2.32}$$

Substituting (5.2.26), (5.2.27) and (5.2.32) into (5.2.20) and after a straightforward computation we get

$$\begin{aligned} \zeta_N =& \frac{N^2\zeta_N^3}{6} + \frac{\pi^2\zeta_N}{2N^2} - \frac{\zeta_N}{N^2} - \frac{5\pi}{8N^4} + \frac{\pi}{4N^3} \\ &+ i\left(\frac{\beta\zeta_N}{4N^2} + \frac{\beta\pi}{8N^4}\right) + O(\frac{1}{N^5}). \end{aligned} \tag{5.2.33}$$

From (5.2.19) we have $\zeta_N \sim \dfrac{1}{N^2}$ and therefore

$$\zeta_N = \frac{\pi}{4N^3} - \frac{5\pi}{8N^4} + i\frac{\beta\pi}{8N^4} + o(\frac{1}{N^4}). \tag{5.2.34}$$

Finally, inserting (5.2.34) into (5.2.19) we deduce

$$\theta_N = \pi - \frac{2\pi}{2N+1} + \frac{\pi}{4N^3} - \frac{5\pi}{8N^4} + i\frac{\pi\beta}{8N^4} + o(\frac{1}{N^4}). \tag{5.2.35}$$

Step 3. Asymptotic behavior of λ_N. First, using (5.2.35) we obtain

$$\cos(\theta_N) = -1 + \frac{\pi^2}{2N^2} - \frac{\pi^2}{2N^3} + \frac{\pi^2}{8N^4} - \frac{\pi^4}{24N^4} + o(\frac{1}{N^4}).$$

Next, using the above equation and after some computations we get

$$\sqrt{2-2\cos(\theta_N)} = 2 - \frac{\pi^2}{4N^2} + \frac{\pi^2}{4N^3} - \frac{\pi^2}{16N^4} + \frac{\pi^4}{192N^4} + o(\frac{1}{N^4}). \quad (5.2.36)$$

Finally, inserting (5.2.36) into (5.2.12), after some calculations we deduce (5.2.10).

Proof of Theorem 5.1.2: First, assume that there exists a positive constant M_2 independent of h such that for all $U_0^h \in \mathcal{H}_h$ inequality (5.1.23) holds. Thanks to Lemma 5.2.1, there exists another constant M_3 independent of h such that equation (5.2.1) holds for any eigenvalue λ of $\mathcal{A}_{\beta,h}$ with $|\lambda| > 1$. On the other hand, using (5.2.10) we have $|\lambda_N| > 1$ and

$$-\Re(\lambda)|\lambda|^2 \sim \frac{1}{N^2},$$

which leads to a contradiction with (5.2.1). ■

5.3 Uniform polynomial energy decay rate

In this section, we give the proofs of Lemma 5.1.4 and Theorem 5.1.5. The proof of Lemma 5.1.3 is left to the reader. It is based on the discretization choice in (5.1.25)-(5.1.26) and the fact that the initial data (y^0, y^1, η^0) of system (5.1.1) belongs to $D(\mathcal{A})$.

Proof of Lemma 5.1.4: We recall that the energy $E_{h,1}$ at $t = 0$ is given by

$$\begin{aligned}
E_{h,1}(0) =& \frac{h}{2}\sum_{j=1}^{N} |y_j^{''}(0)|^2 + \frac{h}{2}\sum_{j=0}^{N}\left(\frac{y_{j+1}^{'}(0) - y_j^{'}(0)}{h}\right)^2 + \frac{(1+h^2\beta)}{2}|\eta^{'}(0)|^2 \\
=& \frac{h}{2}\sum_{j=1}^{N} |y_j^{''}(0)|^2 + \frac{h}{2}\sum_{j=0}^{N}\left(\frac{y_{j+1}^{1} - y_j^{1}}{h}\right)^2 \\
&+ \frac{(1+h^2\beta)}{2}|\eta^{'}(0)|^2
\end{aligned} \quad (5.3.1)$$

with y_j^0, y_j^1, η^0 are given in (5.1.25)-(5.1.26). For $x \in [0, 1-h]$, we introduce the function $\tau_h y$ defined by $\tau_h y(x) = y(x+h) - y(x)$ and we set $I_j = [(j-1)h, jh]$, for $j = 1, .., N+1$. Moreover, we notice that since that the initial data (y^0, y^1, η^0) of system (5.1.24) belongs to $D(\mathcal{A})$ then we have $y^0 \in V \cap H^2(0,1)$ and $y^1 \in V$. For clarity, we divide the proof into three steps.

Step 1. Uniformly boundedness of the first term of $E_{h,1}(0)$. First, from the first equation of system (5.1.24) we have

$$\begin{aligned} y_j^{''}(0) &= \frac{y_{j+1}(0) - 2y_j(0) + y_{j-1}(0)}{h^2} + (y_{j+1}^{'}(0) - 2y_j^{'}(0) + y_{j-1}^{'}(0)) \\ &= \frac{y_{j+1}^0 - 2y_j^0 + y_{j-1}^0}{h^2} + (y_{j+1}^1 - 2y_j^1 + y_{j-1}^1),\ j = 1, .., N. \end{aligned} \tag{5.3.2}$$

Taking the square of above equation, the sum over j and multiplying by $\frac{h}{2}$ we obtain

$$\begin{aligned} \frac{h}{2}\sum_{j=1}^{N} |y_j^{''}(0)|^2 \leq & h\sum_{j=1}^{N}\left(\frac{y_{j+1}^0 - 2y_j^0 + y_{j-1}^0}{h^2}\right)^2 \\ & + h\sum_{j=1}^{N}\left|y_{j+1}^1 - 2y_j^1 + y_{j-1}^1\right|^2. \end{aligned} \tag{5.3.3}$$

Next, using the definition of τ_h given at the beginning of this proof and since $(y^0, y^1) \in V \cap H^2(0,1) \times V$, applying Cauchy-Schwarz's inequality we get

$$\begin{aligned} |y_{j+1}^0 - 2y_j^0 + y_{j-1}^0|^2 &= \left|\int_{I_j} \tau_h y_x^0(x)dx\right|^2 \\ &\leq h\int_{I_j} |\tau_h y_x^0(x)|^2 dx \\ &\leq h^3 \int_{I_j \cup I_{j+1}} |y_{xx}^0(x)|^2 dx, \qquad j = 1, .., N \end{aligned}$$

and

$$\left|y_{j+1}^1 - 2y_j^1 + y_{j-1}^1\right|^2 = \left|\int_{I_j} \tau_h y_x^1(x)dx\right|^2 \leq h\int_{I_j} |\tau_h y_x^1(x)|^2 dx, \qquad j = 1, .., N.$$

Now, multiplying by h and taking the sum over j, we can check that

$$\begin{aligned} h\sum_{j=1}^{N}\left(\frac{y_{j+1}^0 - 2y_j^0 + y_{j-1}^0}{h^2}\right)^2 &\leq \sum_{j=1}^{N}\int_{I_j \cup I_{j+1}} |y_{xx}^0(x)|^2 dx \\ &\leq 2\int_0^1 |y_{xx}^0(x)|^2 dx \end{aligned} \tag{5.3.4}$$

and

$$\begin{aligned} h\sum_{j=1}^{N}\left|y_{j+1}^1-2y_j^1+y_{j-1}^1\right|^2 &\leq h^2\sum_{j=1}^{N}\int_{I_j}|\tau_h y_x^1(x)|^2dx \\ &\leq h^2\sum_{j=1}^{N}\int_{I_j\cup I_{j+1}}|y_x^1(x)|^2dx \\ &\leq 2h^2\int_0^1|y_x^1(x)|^2dx. \end{aligned} \tag{5.3.5}$$

Finally, combining (5.3.3), (5.3.4) and (5.3.5), we deduce that the first term of $E_{h,1}(0)$ is uniformly bounded with respect to the mesh size h.
Step 2. Uniformly boundedness of the second term of $E_{h,1}(0)$. Since $y^1 \in V$, using Cauchy-Schwarz's inequality we get

$$\left|y_{j+1}^1-y_j^1\right|^2=\left|\int_{I_{j+1}}y_x^1(x)dx\right|^2\leq h\int_{I_{j+1}}|y_x^1(x)|^2dx.$$

Then multiplying by $\frac{h}{2}$ and taking the sum over j, we obtain

$$\frac{h}{2}\sum_{j=0}^{N}\left(\frac{y_{j+1}^1-y_j^1}{h}\right)^2\leq\frac{1}{2}\sum_{j=0}^{N}\int_{I_{j+1}}|y_x^1(x)|^2dx\leq\frac{1}{2}\int_0^1|y_x^1(x)|^2dx. \tag{5.3.6}$$

Hence, we deduce the uniformly boundedness of the second term of $E_{h,1}(0)$ with respect to the mesh size h.
Step 3. Uniformly boundedness of $E_{h,1}(0)$. From the third and the sixth equations of (5.1.24), we have

$$\eta^{'}(0)=y_{N+1}^{'}(0)-\beta\eta(0)=y_{N+1}^1-\beta\eta^0. \tag{5.3.7}$$

Next, since $y^1 \in V \subset C([0,1])$ and using (5.1.26) for $j = N+1$ we get

$$\left|y_{N+1}^1\right|=\left|\frac{1}{h}\int_{I_{N+1}}y^1(x)dx\right|\leq\max_{x\in[0,1]}|y^1(x)|\lesssim\|y^1\|_V. \tag{5.3.8}$$

Therefore y_{N+1}^1 is uniformly bounded with respect to the mesh size h. On the other hand, since η^0 is independent of h, from (5.3.7) we conclude the uniformly boundedness of the third term of $E_{h,1}(0)$ with respect to the mesh size h. Finally, combining the results of the previous steps with equation (5.3.1), we deduce that $E_{h,1}(0)$ is uniformly bounded with respect to the mesh size h. ■

Next, we prove Theorem 5.1.5 using a multiplier method inspired from [91]. Indeed, we need to find a constant M uniformly bounded with respect to the mesh size h and $T>0$ such that

$$\int_0^T E_h^2(t)dt \leq M E_h^2(0). \tag{5.3.9}$$

In this case, applying Theorem 9.1 in [52] we deduce directly (5.1.30). We now prove (5.3.9) by the following lemma:

Lemma 5.3.1. *Let $T>0$ and assume the initial data (y^0, y^1, η^0) of system (5.1.1) belongs to $D(\mathcal{A})$ where $D(\mathcal{A})$ is given in (5.1.4). Then there exists $h_0>0$ small enough such that the energy E_h of system (5.1.24) given in (5.1.27) satisfies the following estimation:*

$$\int_0^T E_h^2(t)dt \leq M E_h^2(0), \tag{5.3.10}$$

where $M = \sup\limits_{h\in]0,h_0[} M_h$, for h_0 small enough and where M_h is given in (5.1.31).

Proof: For simplicity, in this proof we denote $\eta(t)$ by η. First, we set $T>0$ and we define the following scalars:

$$\begin{cases} \Psi_j & = & j\dfrac{y_{j+1}-y_{j-1}}{2}, \qquad \text{for} \quad j=1,...,N, \\ \Psi_{N+1} & = & \dfrac{(N+1)}{2}(y_{N+1}-y_N). \end{cases}$$

We notice that the multiplier Ψ_j is no more then the discrete form of xy_x used in [91] to prove Theorem 2.1. Next, multiplying the first equation of (5.1.24) by $\Psi_j E_h(t)$, integrating from 0 to T and taking the sum over j we obtain

$$I_1 - I_2 - I_3 = 0, \tag{5.3.11}$$

where

$$I_1 = \frac{1}{2}\int_0^T \left(\sum_{j=1}^N j y_j^{''}(y_{j+1}-y_{j-1}) \right) E_h(t)dt, \tag{5.3.12}$$

$$I_2 = \int_0^T \left(\sum_{j=1}^N j \left(\frac{y_{j+1}-2y_j+y_{j-1}}{h^2} \right) \left(\frac{y_{j+1}-y_{j-1}}{2} \right) \right) E_h(t)dt \tag{5.3.13}$$

and where

$$I_3 = \frac{1}{2}\int_0^T \left(\sum_{j=1}^{N} j(y'_{j+1} - 2y'_j + y'_{j-1})(y_{j+1} - y_{j-1})\right) E_h(t)dt. \tag{5.3.14}$$

For clarity, we split the rest of the poof into five steps.

Step 1. Elementary calculations for I_1 and I_2.

First, integrating by parts in (5.3.12) we obtain

$$I_1 = \frac{1}{2}\sum_{j=1}^{N} jy'_j(y_{j+1} - y_{j-1})E_h(t)\Big|_0^T - \frac{1}{2}\int_0^T \left(\sum_{j=1}^{N} jy'_j(y'_{j+1} - y'_{j-1})\right) E_h(t)dt \tag{5.3.15}$$
$$- \frac{1}{2}\int_0^T \left(\sum_{j=1}^{N} jy'_j(y_{j+1} - y_{j-1})\right) E'_h(t)dt.$$

Next, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{N} jy'_j(y'_{j+1} - y'_{j-1}) = -\frac{1}{2}\sum_{j=0}^{N} y'_j y'_{j+1} + \frac{(N+1)}{2} y'_N y'_{N+1} = \\
&\frac{1}{4}\left(-2\sum_{j=0}^{N} y'_j y'_{j+1} + \sum_{j=0}^{N} |y'_j|^2 + \sum_{j=0}^{N} |y'_{j+1}|^2 - \sum_{j=0}^{N} |y'_j|^2 - \sum_{j=0}^{N} |y'_{j+1}|^2\right) \\
&\quad + \frac{(N+1)}{2} y'_N y'_{N+1} \\
&= \frac{h^2}{4}\sum_{j=0}^{N}\left(\frac{y'_{j+1} - y'_j}{h}\right)^2 - \frac{1}{2}\sum_{j=0}^{N} |y'_j|^2 - \frac{1}{4}|y'_{N+1}|^2 \\
&\qquad + \frac{(N+1)}{2} y'_N y'_{N+1}.
\end{aligned} \tag{5.3.16}$$

On the other hand, using the second equation of (5.1.24) we have $y'_N = h\eta' + y'_{N+1}$. Thus, from the third one we get

$$\begin{aligned}
\frac{(N+1)}{2} y'_N y'_{N+1} &= \frac{1}{2}\eta' y'_{N+1} + \frac{(N+1)}{2}|y'_{N+1}|^2 \\
&= \frac{(N+1)}{2}|y'_{N+1}|^2 + \frac{1}{2}|\eta'|^2 + \frac{\beta}{2}\eta\eta'.
\end{aligned} \tag{5.3.17}$$

Then, substituting (5.3.17) into (5.3.16) yields

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^{N} jy_j^{'}(y_{j+1}^{'}-y_{j-1}^{'}) =& \frac{h^2}{4}\sum_{j=0}^{N}\left(\frac{y_{j+1}^{'}-y_j^{'}}{h}\right)^2-\frac{1}{2}\sum_{j=0}^{N}|y_j^{'}|^2 \\ &+\frac{(N+1)}{2}|y_{N+1}^{'}|^2-\frac{1}{4}|y_{N+1}^{'}|^2 \\ &+\frac{1}{2}|\eta^{'}|^2+\frac{\beta}{2}\eta\eta^{'}.\end{aligned} \tag{5.3.18}$$

Inserting (5.3.18) in (5.3.15), we obtain

$$\begin{aligned}I_1 =& \frac{1}{2}\sum_{j=1}^{N} jy_j^{'}(y_{j+1}-y_{j-1})E_h(t)\Bigg|_0^T \\ &-\frac{h^2}{4}\int_0^T\left(\sum_{j=0}^{N}\left(\frac{y_{j+1}^{'}-y_j^{'}}{h}\right)^2\right)E_h(t)dt \\ &+\frac{1}{2}\int_0^T\left(\sum_{j=0}^{N}|y_j^{'}|^2\right)E_h(t)dt-\frac{(N+1)}{2}\int_0^T|y_{N+1}^{'}|^2E_h(t)dt \\ &+\frac{1}{4}\int_0^T|y_{N+1}^{'}|^2E_h(t)dt-\frac{1}{2}\int_0^T|\eta^{'}|^2E_h(t)dt-\frac{\beta}{2}\int_0^T\eta\eta^{'}E_h(t)dt \\ &-\frac{1}{2}\int_0^T\left(\sum_{j=1}^{N} jy_j^{'}(y_{j+1}-y_{j-1})\right)E_h^{'}(t)dt.\end{aligned} \tag{5.3.19}$$

Now, we have

$$\begin{aligned}&\frac{1}{2h^2}\sum_{j=1}^{N} j\left(y_{j+1}-2y_j+y_{j-1}\right)\left(y_{j+1}-y_{j-1}\right) \\ &\quad=\frac{1}{2h^2}\sum_{j=1}^{N} j((y_{j+1}-y_j)-(y_j-y_{j-1}))((y_{j+1}-y_j)+(y_j-y_{j-1})) \\ &\quad=-\frac{1}{2}\sum_{j=0}^{N}\left(\frac{y_{j+1}-y_j}{h}\right)^2+\frac{(N+1)}{2}\left(\frac{y_{N+1}-y_N}{h}\right)^2.\end{aligned} \tag{5.3.20}$$

Finally, substituting (5.3.20) into (5.3.13) yields

$$\begin{aligned}I_2 =& -\frac{1}{2}\int_0^T\left(\sum_{j=0}^{N}\left(\frac{y_{j+1}-y_j}{h}\right)^2\right)E_h(t)dt \\ &+\frac{(N+1)}{2}\int_0^T\left(\frac{y_{N+1}-y_N}{h}\right)^2E_h(t)dt.\end{aligned} \tag{5.3.21}$$

Step 2. First estimation of the energy. Substituting (5.3.14), (5.3.19) and (5.3.21) into (5.3.11) and multiplying by h yields

$$
\begin{aligned}
&\frac{h}{2}\int_0^T\left(\sum_{j=0}^{N}|y_j^{'}|^2\right)E_h(t)dt+\frac{h}{2}\int_0^T\left(\sum_{j=0}^{N}\left(\frac{y_{j+1}-y_j}{h}\right)^2\right)E_h(t)dt=\\
&\quad\frac{h^3}{4}\int_0^T\left(\sum_{j=0}^{N}\left(\frac{y_{j+1}^{'}-y_j^{'}}{h}\right)^2\right)E_h(t)dt+\frac{1}{2}\int_0^T|y_{N+1}^{'}|^2E_h(t)dt \qquad (5.3.22)\\
&\quad-\frac{h}{4}\int_0^T|y_{N+1}^{'}|^2E_h(t)dt+\frac{h}{2}\int_0^T|\eta^{'}|^2E_h(t)dt+\frac{h\beta}{2}\int_0^T\eta\eta^{'}E_h(t)dt\\
&\quad+\frac{1}{2}\int_0^T\left(\frac{y_{N+1}-y_N}{h}\right)^2E_h(t)dt+\frac{h}{2}\int_0^T\left(\sum_{j=1}^{N}jy_j^{'}(y_{j+1}-y_{j-1})\right)E_h^{'}(t)dt\\
&\quad-\frac{h}{2}\sum_{j=1}^{N}jy_j^{'}(y_{j+1}-y_{j-1})E_h(t)\Big|_0^T+hI_3.
\end{aligned}
$$

Adding $\dfrac{(1+h^2\beta)}{2}\displaystyle\int_0^T|\eta|^2E_h(t)dt$ on the left and right sides of above identity, taking in mind that for h small enough we have $\dfrac{1+h^2\beta}{2}\leq 1$ and since

$$-\frac{h}{4}\int_0^T|y_{N+1}^{'}|^2E_h(t)dt\leq 0,$$

we obtain

$$
\begin{aligned}
\int_0^T E_h^2(t)dt\leq&\frac{h^3}{4}\int_0^T\left(\sum_{j=0}^{N}\left(\frac{y_{j+1}^{'}-y_j^{'}}{h}\right)^2\right)E_h(t)dt \qquad (5.3.23)\\
&+\frac{1}{2}\int_0^T|y_{N+1}^{'}|^2E_h(t)dt+\frac{h}{2}\int_0^T|\eta^{'}|^2E_h(t)dt\\
&+\int_0^T|\eta|^2E_h(t)dt+\frac{h\beta}{2}\int_0^T\eta\eta^{'}E_h(t)dt\\
&+\frac{h}{2}\int_0^T\left(\sum_{j=1}^{N}jy_j^{'}(y_{j+1}-y_{j-1})\right)E_h^{'}(t)dt\\
&+\frac{1}{2}\int_0^T\left(\frac{y_{N+1}-y_N}{h}\right)^2E_h(t)dt\\
&-\frac{h}{2}\sum_{j=1}^{N}jy_j^{'}(y_{j+1}-y_{j-1})E_h(t)\Big|_0^T+hI_3.
\end{aligned}
$$

Step 3. Estimations of the terms of (5.3.23).

First, since E_h is a decreasing function with respect to time t and using

(5.1.28) we get

$$\begin{aligned}\frac{h^3}{4}\int_0^T\left(\sum_{j=0}^{N}\left(\frac{y_{j+1}^{'}-y_j^{'}}{h}\right)^2\right)E_h(t)dt &\leq -\frac{1}{4}\int_0^T E_h^{'}(t)E_h(t)dt\\ &=\frac{1}{8}\left(E_h^2(0)-E_h^2(T)\right)\\ &\leq \frac{1}{8}E_h^2(0). \end{aligned}\tag{5.3.24}$$

We notice that the energy $E_{h,1}$ is a decreasing function with respect to time t since its derivative of $E_{h,1}$ is given by

$$E_{h,1}^{'}(t)=-h^3\sum_{j=0}^{N}\left(\frac{y_{j+1}^{''}-y_j^{''}}{h}\right)^2-\beta|\eta^{'}(t)|^2-h^2|\eta^{''}(t)|^2. \tag{5.3.25}$$

Thus, from (5.1.28), (5.3.25) and the third equation of (5.1.24) we obtain

$$\begin{aligned}&\frac{1}{2}\int_0^T|y_{N+1}^{'}|^2E_h(t)dt+\frac{h}{2}\int_0^T|\eta^{'}|^2E_h(t)dt+\int_0^T|\eta|^2E_h(t)dt\\ &\leq(1+\frac{h}{2})\int_0^T|\eta^{'}|^2E_h(t)dt+(1+\beta^2)\int_0^T|\eta|^2E_h(t)dt\\ &\leq-\frac{1}{\beta}(1+\frac{h}{2})\int_0^T E_{h,1}^{'}(t)E_h(t)dt-\frac{(1+\beta^2)}{\beta}\int_0^T E_h^{'}(t)E_h(t)dt\\ &\leq-\frac{1}{\beta}(1+\frac{h}{2})E_h(0)\int_0^T E_{h,1}^{'}(t)dt+\frac{(1+\beta^2)}{2\beta}E_h^2(0)\\ &\leq(1+\frac{h}{2})\frac{E_{h,1}(0)}{\beta E_h(0)}E_h^2(0)+\frac{(1+\beta^2)}{2\beta}E_h^2(0)\\ &=\frac{1}{2\beta}\left(1+\beta^2+(2+h)\frac{E_{h,1}(0)}{E_h(0)}\right)E_h^2(0). \end{aligned}\tag{5.3.26}$$

Moreover, using Young's inequality, (5.1.28) and (5.3.25) we get

$$\begin{aligned}\frac{h\beta}{2}\int_0^T\eta\eta^{'}E_h(t)dt &\leq \frac{h\beta}{4}\int_0^T|\eta|^2E_h(t)dt+\frac{h\beta}{4}\int_0^T|\eta^{'}|^2E_h(t)dt\\ &\leq -\frac{h}{4}\int_0^T E_h^{'}(t)E_h(t)dt-\frac{h}{4}E_h(0)\int_0^T E_{h,1}^{'}(t)dt\\ &\leq -\frac{h}{8}E_h^2(t)\Big|_0^T+\frac{hE_{h,1}(0)}{4E_h(0)}E_h^2(0)\\ &\leq \frac{h}{8}\left(1+2\frac{E_{h,1}(0)}{E_h(0)}\right)E_h^2(0). \end{aligned}\tag{5.3.27}$$

Further, using Young's inequality and (5.1.27), we check that

$$\begin{aligned}
-\frac{h}{2}\sum_{j=1}^{N} j y_j^{'}(y_{j+1}-y_{j-1}) =& -\frac{h}{2}\sum_{j=1}^{N} h j y_j^{'}\frac{((y_{j+1}-y_j)+(y_j-y_{j-1}))}{h}\\
\leq& \frac{h^2}{2}\sum_{j=1}^{N} j|y_j^{'}|^2+\frac{h^2}{4}\sum_{j=1}^{N} j\left(\frac{y_{j+1}-y_j}{h}\right)^2\\
&+\frac{h^2}{4}\sum_{j=1}^{N} j\left(\frac{y_j-y_{j-1}}{h}\right)^2\\
\leq& \frac{h^2N}{2}\sum_{j=1}^{N}|y_j^{'}|^2+\frac{3h^2N}{4}\sum_{j=0}^{N}\left(\frac{y_{j+1}-y_j}{h}\right)^2\\
\leq& \frac{5}{2}E_h(t). \qquad (5.3.28)
\end{aligned}$$

Its follows that

$$\begin{aligned}
\frac{h}{2}\int_0^T\left(\sum_{j=1}^{N} j y_j^{'}(y_{j+1}-y_{j-1})\right)(E_h^{'}(t))dt \leq& -\frac{5}{2}\int_0^T E_h(t)E_h^{'}(t)dt\\
\leq& \frac{5}{4}E_h^2(0). \qquad (5.3.29)
\end{aligned}$$

Now, using the second equation of (5.1.24) and (5.1.28) we get

$$\begin{aligned}
\frac{1}{2}\int_0^T\left(\frac{y_{N+1}-y_N}{h}\right)^2 E_h(t)dt =& \frac{1}{2}\int_0^T|\eta|^2E_h(t)dt\\
\leq& -\frac{1}{2\beta}\int_0^T E_h^{'}(t)E_h(t)dt\\
\leq& \frac{1}{4\beta}E_h^2(0). \qquad (5.3.30)
\end{aligned}$$

Finally, using (5.3.28) we obtain

$$\begin{aligned}
-\frac{h}{2}\sum_{j=1}^{N} j y_j^{'}(y_{j+1}-y_{j-1})E_h(t)\Big|_0^T \leq& \left|\frac{h}{2}\sum_{j=1}^{N} j y_j^{'}(y_{j+1}-y_{j-1})E_h(t)\Big|_0^T\right|\\
\leq& 5E_h^2(0). \qquad (5.3.31)
\end{aligned}$$

Step 4. Estimation of hI_3. Setting $\epsilon>0$ and using Young's inequality, (5.1.27) and (5.1.29) we get

$$\begin{aligned}
&\frac{h}{2}\sum_{j=1}^{N} j(y'_{j+1}-2y'_j+y'_{j-1})(y_{j+1}-y_{j-1})\\
\leq&\frac{h^2}{4\epsilon}\sum_{j=1}^{N} j(y'_{j+1}-2y'_j+y'_{j-1})^2+\frac{\epsilon}{4}\sum_{j=1}^{N} j(y_{j+1}-y_{j-1})^2\\
\leq&\frac{h^4}{2\epsilon}\sum_{j=0}^{N} j\left(\frac{y'_{j+1}-y'_j}{h}\right)^2+\frac{h^4}{2\epsilon}\sum_{j=1}^{N} j\left(\frac{y'_{j}-y'_{j-1}}{h}\right)^2\\
&+\frac{h^2\epsilon}{2}\sum_{j=0}^{N} j\left(\frac{y_{j+1}-y_j}{h}\right)^2+\frac{h^2\epsilon}{2}\sum_{j=1}^{N} j\left(\frac{y_{j}-y_{j-1}}{h}\right)^2\\
\leq&\frac{3h^4N}{2\epsilon}\sum_{j=0}^{N}\left(\frac{y'_{j+1}-y'_j}{h}\right)^2+\frac{3h^2N\epsilon}{2}\sum_{j=0}^{N}\left(\frac{y_{j+1}-y_j}{h}\right)^2\\
\leq&\frac{3h^3}{2\epsilon}\sum_{j=0}^{N}\left(\frac{y'_{j+1}-y'_j}{h}\right)^2+\frac{3h\epsilon}{2}\sum_{j=0}^{N}\left(\frac{y_{j+1}-y_j}{h}\right)^2\\
\leq&-\frac{3}{2\epsilon}E'_h(t)+3\epsilon E_h(t).
\end{aligned}$$

Combining the above inequality together with (5.3.14) we obtain

$$\begin{aligned}
hI_3 &\leq -\frac{3}{2\epsilon}\int_0^T E'_h(t)E_h(t)dt+3\epsilon\int_0^T E_h^2(t)dt\\
&\leq \frac{3}{4\epsilon}E_h^2(0)+3\epsilon\int_0^T E_h^2(t)dt. \qquad (5.3.32)
\end{aligned}$$

Step 5. Second estimation of the energy. Inserting (5.3.24), (5.3.26), (5.3.27), (5.3.29), (5.3.30), (5.3.31) and (5.3.32) into (5.3.23) with $\epsilon=\dfrac{1}{6}$ we get

$$\begin{aligned}
\int_0^T E_h^2(t)dt &\leq \left[\frac{87}{4}+\beta+\frac{3}{2\beta}+\frac{h}{4}+\frac{1}{2\beta}\left(4+2h+h\beta\right)\frac{E_{h,1}(0)}{E_h(0)}\right]E_h^2(0)\\
&\leq \left[22+\beta+\frac{3}{2\beta}+\frac{1}{2\beta}\left(6+\beta\right)\frac{E_{h,1}(0)}{E_h(0)}\right]E_h^2(0)\\
&= M_hE_h^2(0). \qquad (5.3.33)
\end{aligned}$$

Since the initial data (y^0, y^1, η^0) of system (5.1.1) belongs to $D(\mathcal{A})$, thanks to Remark 5.1.6, from above equation we deduce (5.3.9) with $M=\sup\limits_{h\in]0,h_0[} M_h$, for h_0 small enough.

5.4 Convergence results: proof of Theorem 5.1.7

Following ZuaZua in [88], from (5.1.27)-(5.1.28) and the definitions of p_h and q_h in (5.1.32)-(5.1.33), we first rewrite the energies E_h, $E_{h,1}$ and their derivatives as follows:

$$\begin{aligned} E_h(t) =& \frac{1}{2}\|p_h\vec{y_h}(t)\|_V^2 + \frac{1}{2}\|q_h\vec{y_h}'(t)\|_{L^2(0,1)}^2 \qquad (5.4.1)\\ &+ \frac{1}{2}(1+\beta h^2)|\eta(t)|^2, \quad t \geq 0, \\ E_h'(t) =& -h^2\|p_h\vec{y_h}'\|_V^2 - \beta|\eta(t)|^2 - h^2|\eta'(t)|^2, \quad t \geq 0 \qquad (5.4.2) \end{aligned}$$

and

$$\begin{aligned} E_{h,1}(t) =& \frac{1}{2}\|p_h\vec{y_h}'(t)\|_V^2 + \frac{1}{2}\|q_h\vec{y_h}''(t)\|_{L^2(0,1)}^2 \qquad (5.4.3)\\ &+ \frac{1}{2}(1+\beta h^2)|\eta'(t)|^2, \quad t \geq 0, \\ E_{h,1}'(t) =& -h^2\|p_h\vec{y_h}''\|_V^2 - \beta|\eta'(t)|^2 - h^2|\eta''(t)|^2, \quad \forall t \geq 0. \qquad (5.4.4) \end{aligned}$$

Moreover, we denote by $y^{(n)}$, $n \in \mathbb{N}$, the derivative of y of order n with respect to time t, *i.e.* $y^{(n)} = \dfrac{\partial^n y}{\partial^n t}$. Before going on, we give a relation between $p_h\vec{y_h}^{(n)}$ and $q_h\vec{y_h}^{(n)}$ in the Hilbert space $L^2(0,1)$ by the following lemma:

Lemma 5.4.1. *The functions $p_h\vec{y_h}^{(n)}$ and $q_h\vec{y_h}^{(n)}$ satisfy the following relation:*

$$\|p_h\vec{y_h}^{(n)} - q_h\vec{y_h}^{(n)}\|_{L^2(0,1)}^2 = \frac{h^3}{3}\sum_{j=0}^{N}\left(\frac{y_{j+1}^{(n)} - y_j^{(n)}}{h}\right)^2. \qquad (5.4.5)$$

Proof: Using (5.1.32)-(5.1.33), we can check that

$$\begin{aligned} \|p_h\vec{y_h}^{(n)} - q_h\vec{y_h}^{(n)}\|_{L^2(0,1)}^2 &= \int_0^1 |p_h\vec{y_h}^{(n)}(x) - q_h\vec{y_h}^{(n)}(x)|^2 dx \\ &= \sum_{j=0}^{N}\int_{jh}^{(j+1)h}\left|\left(\frac{y_{j+1}-y_j}{j}\right)(x-jh)\right|^2 dx \\ &= \frac{h^3}{3}\sum_{j=0}^{N}\left(\frac{y_{j+1}^{(n)} - y_j^{(n)}}{h}\right)^2. \end{aligned}$$

Next, we prove the convergence results (5.1.36) by the following two lemmas:

Lemma 5.4.2. *Assume the initial data* (y^0, y^1, η^0) *of system* (5.1.1) *belongs to* $D(\mathcal{A})$ *where* $D(\mathcal{A})$ *is given in* (5.1.4). *Then, the convergence results in* (5.1.36) *hold.*

Proof: First, since E_h is a decreasing function as time increase and using (5.4.1) we obtain

$$\|p_h \vec{y}_h(t)\|_V^2 \leq 2E_h(0), \qquad \forall t \geq 0.$$

Since the initial data (y^0, y^1, η^0) of system (5.1.1) belongs to $D(\mathcal{A})$, we claim from Lemma 5.1.3 that $E_h(0)$ is uniformly bounded with respect to the mesh size h. Therefore from above equation we get

$$\sup_{t\in[0,+\infty[} \|p_h \vec{y}_h(t)\|_V^2 < +\infty,$$

i.e. $p_h \vec{y}_h$ is bounded in $L^\infty(0,\infty; V)$. Thus, $p_h \vec{y}_h$ is bounded in $L^\infty(0,\infty; L^2(0,1))$. For the same reason, we conclude that $q_h \vec{y}_h^{\,'}$ is bounded in $L^\infty(0,\infty; L^2(0,1))$ and η is bounded in $L^\infty(0,\infty;\mathbb{C})$. Moreover, using (5.4.3) and (5.4.5) for $n = 1$, we have

$$\begin{aligned}
\|p_h \vec{y}_h^{\,'}(t)\|_{L^2(0,1)}^2 &\leq \|p_h \vec{y}_h^{\,'}(t) - q_h \vec{y}_h^{\,'}(t)\|_{L^2(0,1)}^2 + \|q_h \vec{y}_h^{\,'}(t)\|_{L^2(0,1)}^2 \\
&\leq \frac{h^3}{12}\sum_{j=0}^{N}\left(\frac{y_{j+1}^{'}(t) - y_j^{'}(t)}{h}\right)^2 + 2E_h(t) \\
&\leq \frac{h^2}{6}E_{h,1}(t) + 2E_h(0) \leq \frac{h^2}{6}E_{h,1}(0) + E_h(0).
\end{aligned}$$

Since the initial data (y^0, y^1, η^0) of system (5.1.1) belongs to $D(\mathcal{A})$, thanks to Lemma 5.1.4, we know $E_{h,1}(0)$ is uniformly bounded with respect to the mesh size h and therefore $p_h \vec{y}_h^{\,'}$ is bounded in $L^\infty(0,\infty; L^2(0,1))$. Moreover, since $E_{h,1}$ is a decreasing function as time increase, we deduce from (5.4.3) that $p_h \vec{y}_h^{\,'}$ is bounded in $L^\infty(0,\infty; V)$, $q_h \vec{y}_h^{\,''}$ is bounded in $L^\infty(0,\infty; L^2(0,1))$ and $\eta^{'}$ is bounded in $L^\infty(0,\infty;\mathbb{C})$. Thus, from the third equation of (5.1.24), we deduce that $y_{N+1}^{'}$ is bounded in $L^\infty(0,\infty;\mathbb{C})$. On the other hand, integrating (5.4.2) from 0 to s we get

$$E_h(0) = E_h(s) + \int_0^s \|h p_h \vec{y}_h^{\,'}(t)\|_V^2 dt + \beta\int_0^s |\eta(t)|^2 dt + h^2\int_0^s |\eta^{'}(t)|^2 dt,$$

$\forall s > 0$. From Theorem 5.1.5 we know that the energy E_h decrease to zero as s tends to ∞. Taking $s \to \infty$ we obtain

$$\begin{aligned} E_h(0) = &\int_0^\infty \|hp_h \vec{y_h}'(t)\|_V^2 dt + \beta \int_0^\infty |\eta(t)|^2 dt \\ &+ h^2 \int_0^\infty |\eta'(t)|^2 dt, \quad \forall s > 0. \end{aligned} \tag{5.4.6}$$

Since $E_h(0)$ is uniformly bounded with respect to h, we deduce from above equation that $hp_h\vec{y_h}'$ is bounded in $L^2(0,\infty;V)$, η is bounded in $L^2(0,\infty;\mathbb{C})$ and $h\eta'$ is bounded in $L^2(0,\infty;\mathbb{C})$. Furthermore, integrating (5.4.4) from 0 to s we get

$$\begin{aligned} E_{h,1}(0) - E_{h,1}(s) = &\int_0^s \|hp_h\vec{y_h}''(t)\|_V^2 dt + \beta \int_0^s |\eta'(t)|^2 dt \\ &+ \int_0^s |h\eta''(t)|^2 dt, \quad \forall s > 0. \end{aligned}$$

It follows that

$$E_{h,1}(0) \geq \int_0^s \|hp_h\vec{y_h}''(t)\|_V^2 dt + \beta \int_0^s |\eta'(t)|^2 dt + \int_0^s |h\eta''(t)|^2 dt, \quad \forall s > 0.$$

Taking $s \to \infty$ and since the energy $E_{h,1}$ at $t = 0$ is uniformly bounded with respect to h, we deduce from the above equation that $hp_h\vec{y_h}''$ is bounded in $L^2(0,\infty;V)$, η' and $h\eta''$ are bounded in $L^2(0,\infty;\mathbb{C})$. Finally, using the third equation of (5.1.24), we claim that y'_{N+1} is bounded in $L^2(0,\infty;\mathbb{C})$.
From the above boundedness results, we can extract the following subsequences:

$$\begin{cases} p_h\vec{y_h} \to y & \text{weakly* in} \quad L^\infty(0,\infty;V), \\ p_h\vec{y_h} \to y & \text{weakly* in} \quad L^\infty(0,\infty;L^2(0,1)), \\ p_h\vec{y_h}' \to y' & \text{weakly* in} \quad L^\infty(0,\infty;V), \\ p_h\vec{y_h}' \to y' & \text{weakly* in} \quad L^\infty(0,\infty;L^2(0,1)), \\ q_h\vec{y_h} \to y & \text{weakly* in} \quad L^\infty(0,\infty;L^2(0,1)), \\ q_h\vec{y_h}' \to y' & \text{weakly* in} \quad L^\infty(0,\infty;L^2(0,1)), \\ y'_{N+1} \to y'(1,t) & \text{weakly in} \quad L^2(0,\infty;\mathbb{C}), \\ hp_h\vec{y_h}' \to 0 & \text{weakly in} \quad L^2(0,\infty;V), \\ h\eta' \to 0 & \text{weakly in} \quad L^2(0,\infty;\mathbb{C}). \end{cases} \tag{5.4.7}$$

The eighth convergence in (5.4.7) follows from the fourth one and the boundedness of the sequence of $hp_h\vec{y_h}'$ in $L^2(0,\infty;V)$. As for the seventh convergence, it follows from the first one and the boundedness of y'_{N+1} in $L^2(0,\infty;\mathbb{C})$. Note that in (5.4.7), using (5.4.5) we implicitly claim that the limits of $p_h\vec{y_h}^{(n)}$ and $q_h\vec{y_h}^{(n)}$ are the same for $n=0$ and $n=1$. The proof is complete.

Lemma 5.4.3. *Assume the initial data (y^0, y^1, η^0) of system (5.1.1) belongs to $D(\mathcal{A})$ where $D(\mathcal{A})$ is given in (5.1.4). Then, the weakly* limit (y, y', η) of the previous lemma is the unique solution of system (5.1.1).*

Proof: For clarity, we divide the proof into several steps.
Step 1. y verifies the first equation of (5.1.1). Applying the same strategy used in section 3.2 in [88], we show that y satisfies

$$y'' - y_{xx} = 0. \tag{5.4.8}$$

But for calculations reason needed in the following steps of this proof, we will give the details. First, let $w \in \mathcal{D}([0,1]\times]0,\infty[)$ with $w(0,\cdot)\equiv 0$ and we set $\vec{w}_h = (w_j)_j$ where $w_j = w(jh,\cdot)$. Multiplying the first equation of (5.1.24) by hw_j, integrating by parts on $(0,\infty)$ and taking the sum over j, we obtain

$$\begin{aligned} 0 =& h\int_0^\infty \sum_{j=1}^N y_j w_j^{''} dt - h\int_0^\infty \sum_{j=1}^N \left(\frac{y_{j+1}-2y_j+y_{j-1}}{h^2}\right) w_j dt \\ & - h\int_0^\infty \sum_{j=1}^N (y_{j+1}^{'} - 2y_j^{'} + y_{j-1}^{'}) w_j dt. \end{aligned} \tag{5.4.9}$$

Using the second equation of (5.1.24) and since $w_0 = 0$, we have

$$\begin{aligned} h\sum_{j=1}^N \left(\frac{y_{j+1}-2y_j+y_{j-1}}{h^2}\right) w_j =& h\sum_{j=1}^N \left(\frac{y_{j+1}-y_j}{h^2}\right) w_j - h\sum_{j=1}^N \left(\frac{y_j-y_{j-1}}{h^2}\right) w_j \\ =& h\sum_{j=0}^N \left(\frac{y_{j+1}-y_j}{h^2}\right) w_j - h\sum_{j=0}^{N-1} \left(\frac{y_{j+1}-y_j}{h^2}\right) w_{j+1} \\ =& h\sum_{j=0}^N \left(\frac{y_{j+1}-y_j}{h}\right)\left(\frac{w_j-w_{j+1}}{h}\right) \\ & + \left(\frac{y_{N+1}-y_N}{h}\right) w_{N+1} \\ =& -h\sum_{j=0}^N \left(\frac{y_{j+1}-y_j}{h}\right)\left(\frac{w_{j+1}-w_j}{h}\right) \\ & - \eta w_{N+1} \end{aligned} \tag{5.4.10}$$

and

$$
\begin{aligned}
h\sum_{j=1}^{N}(y_{j+1}^{'}-2y_j^{'}+y_{j-1}^{'})w_j =& h\sum_{j=1}^{N}(y_{j+1}^{'}-y_j^{'})w_j - h\sum_{j=1}^{N}(y_j^{'}-y_{j-1}^{'})w_j \\
=& h\sum_{j=0}^{N}(y_{j+1}^{'}-y_j^{'})w_j - h\sum_{j=0}^{N-1}(y_{j+1}^{'}-y_j^{'})w_{j+1} \\
=& h^3\sum_{j=0}^{N}\left(\frac{y_{j+1}^{'}-y_j^{'}}{h}\right)\left(\frac{w_j-w_{j+1}}{h}\right) \\
&+ h(y_{N+1}^{'}-y_N^{'})w_{N+1} \\
=& -h^3\sum_{j=0}^{N}\left(\frac{y_{j+1}^{'}-y_j^{'}}{h}\right)\left(\frac{w_{j+1}-w_j}{h}\right) \qquad (5.4.11) \\
&- h^2\eta^{'}w_{N+1}.
\end{aligned}
$$

Next, inserting (5.4.10)-(5.4.11) into (5.4.9) we obtain

$$
\begin{aligned}
0 =& h\int_0^\infty\sum_{j=1}^{N}y_jw_j^{''}dt + h\int_0^\infty\sum_{j=0}^{N}\left(\frac{y_{j+1}-y_j}{h}\right)\left(\frac{w_{j+1}-w_j}{h}\right)dt \qquad (5.4.12) \\
&+ \int_0^\infty\eta(t)w_{N+1}(t)dt + h^2\int_0^\infty\eta^{'}(t)w_{N+1}(t)dt \\
&+ h^3\int_0^\infty\sum_{j=0}^{N}\left(\frac{y_{j+1}^{'}-y_j^{'}}{h}\right)\left(\frac{w_{j+1}-w_j}{h}\right).
\end{aligned}
$$

It follows from the definitions of p_h and q_h in (5.1.32)-(5.1.33) that (5.4.12) is equivalent to

$$
\begin{aligned}
0 =& \int_0^\infty\int_0^1(q_h\vec{y}_h)(q_h\vec{w}_h^{''})dxdt + \int_0^\infty\int_0^1(p_h\vec{y}_h)_x(p_h\vec{w}_h)_xdxdt \qquad (5.4.13) \\
&+ \int_0^\infty\eta(t)w_{N+1}(t)dt + h^2\int_0^\infty\eta^{'}(t)w_{N+1}(t)dt \\
&+ h^2\int_0^\infty\int_0^1(p_h\vec{y}_h^{'})_x(p_h\vec{w}_h)_xdxdt.
\end{aligned}
$$

Now, we recall the following elementary convergence results: for every $w\in\mathcal{D}([0,1]\times(0,\infty))$ we have

$$
\begin{cases}
p_h\vec{w}_h\to w \quad \text{strongly in} \quad L^2(0,\infty;H^1(0,1)), \\
q_h\vec{w}_h\to w \quad \text{strongly in} \quad L^2(0,\infty;L^2(0,1)).
\end{cases} \qquad (5.4.14)
$$

Using the convergence results in (5.4.7) and (5.4.14), passing to the limit in (5.4.13) as $h \to 0$ yields

$$\int_0^\infty \int_0^1 y w^{''} dxdt + \int_0^\infty \int_0^1 y_x w_x dxdt + \int_0^\infty \eta(t) w(1,t) dt = 0. \quad (5.4.15)$$

Finally, choosing w such that we also have $w(1,\cdot) \equiv 0$, from above equation we easily derive (5.4.8).

Step 2. Regularity of y. In order to prove that y satisfies the boundary conditions of system (5.1.1), we need to show that y belongs to $H^2(0,1)$. To this end, we demonstrate that $y(0) = y^0$ and $y'(0) = y^1$ and therefore we obtain our goal since $(y^0, y^1) \in H^2(0,1) \times V$. First, for $T > 0$, we have from (5.4.7) that $y \in L^\infty(0,T;V)$ and $y_t \in L^\infty(0,T;L^2(0,1))$. It follows from Aubin and Simon's theorem (see [13, 86]) that $y \in C([0,T];L^2(0,1))$. On the other hand, we can see that $y_{xx} \in L^\infty(0,T;H^{-1}(0,1))$ where $H^{-1}(0,1)$ denotes the dual of $H_0^1(0,1)$. Thus, from (5.4.8) we obtain that $y_{tt} \in L^\infty(0,T;H^{-1}(0,1))$, which implies again from Aubin and Simon's theorem that $y_t \in C([0,T];H^{-1}(0,1))$. Next, let $v \in \mathcal{D}(0,1)$, $l \in \mathcal{D}([0,\infty[)$ and set $\vec{v}_h = (v_j)_j$ where $v_j = v(jh)$. Multiplying the first equation of (5.1.24) by $hv_j l$, integrating by parts on $[0,\infty[$, taking the sum over j and after some calculations we find

$$\begin{aligned}
0 = &- hl(0)\sum_{j=1}^N v_j y_j^1 + hl^{'}(0)\sum_{j=1}^N v_j y_j^0 + h\int_0^\infty \sum_{j=1}^N v_j y_j l^{''} dt \\
&h\int_0^\infty \sum_{j=0}^N \left(\frac{y_{j+1}-y_j}{h}\right)\left(\frac{v_{j+1}-v_j}{h}\right) l dt \\
&+ h^3 \int_0^\infty \sum_{j=0}^N \left(\frac{y_{j+1}^{'}-y_j^{'}}{h}\right)\left(\frac{v_{j+1}-v_j}{h}\right) l dt.
\end{aligned}$$

From the definitions of p_h and q_h given in (5.1.32)-(5.1.33), it is easy to check that the above equation is equivalent to

$$\begin{aligned}
0 = &- l(0)\int_0^1 (q_h \vec{y_h}^1)(q_h \vec{v}_h) dx + l^{'}(0)\int_0^1 (q_h \vec{y_h}^0)(q_h \vec{v}_h) dx \\
&+ \int_0^\infty \int_0^1 (q_h \vec{y}_h)(q_h \vec{v}_h) l^{''} dxdt + \int_0^\infty \int_0^1 (p_h \vec{y}_h)_x (p_h \vec{v}_h)_x l dxdt \\
&+ h^2 \int_0^\infty \int_0^1 (p_h \vec{y_h}^{'})_x (p_h \vec{v}_h)_x l dxdt.
\end{aligned}$$

Passing to the limit as $h \to 0$ we get

$$0 = -\,l(0)\int_0^1 y^1 v dx + l^{'}(0)\int_0^1 y^0 v dx + \int_0^\infty \int_0^1 y v l^{''} dx dt + \int_0^\infty \int_0^1 y_x v_x dx dt.$$

Integrating by parts over $[0,\infty[$ and over $(0,1)$ and using (5.4.8) we obtain

$$l^{'}(0)\int_0^1 (y^0 - y(0)) v dx + l(0)\int_0^1 (y^{'}(0) - y^1) v dx = 0,$$

$$\forall v \in D(0,1), \quad \forall l \in D([0,\infty[),$$

from which we derive that $y(0) = y^0$ and $y^{'}(0) = y^1$.

Step 3. The solution (y, y', η) verifies the boundary conditions of system (5.1.1). First, choosing $w(1,\cdot) \neq 0$, integrating by parts in (5.4.15) and using (5.4.8) we obtain

$$y_x(1,t) + \eta(t) = 0, \qquad t \in \mathbb{R}^+.$$

Next, using the third equation of (5.1.24) and the convergence results in (5.4.7), we get the third equation of (5.1.1). Finally, since system (5.1.1) has a unique solution, we conclude that the convergence results hold for the whole sequence $\{h\}$, and not only for an extracted subsequence.

Proof of Theorem 5.1.7: First, from the two Lemmas 5.4.2 and 5.4.3, we directly get the convergence results in (5.1.36). To complete the proof it remains to verify (5.1.37) and (5.1.38). For clarity, we divide the proof into three steps.

Step 1. Proof of (5.1.37). We begin by integrating (5.4.2) over 0 and s

$$E_h(0) = E_h(s) + \int_0^s \|h p_h \vec{y_h}^{'}(t)\|_V^2 dt + \beta \int_0^s |\eta(t)|^2 dt + \int_0^s |h\eta^{'}(t)|^2 dt. \tag{5.4.16}$$

Since $E_h(t)$ decreases to zero as $t \to \infty$, if follows from above equation that

$$E_h(0) = \int_0^\infty \|h p_h \vec{y_h}^{'}(t)\|_V^2 dt + \beta \int_0^\infty |\eta(t)|^2 dt + \int_0^\infty |h\eta^{'}(t)|^2 dt. \tag{5.4.17}$$

Next, the fact that $E'(t) = -\beta|\eta(t)|^2$ and since $E(t)$ decreases to zero as $t \to \infty$, we get

$$E(0) = \beta \int_0^\infty |\eta(t)|^2 dt, \tag{5.4.18}$$

which combined with the weak convergence results in (5.4.7) and the identities (5.4.17)-(5.4.18) we obtain

$$\begin{cases} hp_h\vec{y_h}' \to 0 & \text{strongly in} \quad L^2(0,\infty;V), \\ h\eta' \to 0 & \text{strongly in} \quad L^2(0,\infty;\mathbb{C}). \end{cases} \tag{5.4.19}$$

Now, using (5.1.3) and (5.4.16), we can check for all $s>0$ that

$$\begin{aligned} |E_h(s)-E(s)| \leq & |E_h(0)-E(0)| + \int_0^s\int_0^1 |hp_h\vec{y_h}'|^2 dxdt + \int_0^s |h\eta'(t)|^2 dt \\ \leq & |E_h(0)-E(0)| + \int_0^\infty\int_0^1 |hp_h\vec{y_h}'|^2 dxdt + \int_0^\infty |h\eta'(t)|^2 dt. \end{aligned}$$

So that, combining the above inequality with (5.4.19) and Lemma 5.1.3, we deduce that

$$\lim_{h\to 0} \|E_h - E\|_{C([0,\infty[)} = 0. \tag{5.4.20}$$

Step 2. Strongly convergence in $L^2_{Loc}(0,\infty;\mathcal{H})$. First, we have

$$\begin{aligned} \|p_h\vec{y}_h(\cdot,t)-y(\cdot,t)\|_V^2 = & \int_0^1 |(p_h\vec{y}_h(x,t))_x|^2 dx + \int_0^1 |y_x(x,t)|^2 dx \\ & - 2\int_0^1 (p_h\vec{y}_h(x,t))_x y_x(x,t) dx, \end{aligned} \tag{5.4.21}$$

$$\begin{aligned} \|q_h\vec{y_h}'(\cdot,t)-y'(\cdot,t)\|_{L^2(0,1)}^2 = & \int_0^1 |q_h\vec{y_h}'(x,t)|^2 dx + \int_0^1 |y'(x,t)|^2 dx \\ & - 2\int_0^1 q_h\vec{y_h}'(x,t) y'(x,t) dx \end{aligned} \tag{5.4.22}$$

and

$$\begin{aligned} |\sqrt{1+h^2\beta}\eta(t)-\eta(t)|^2 = & (1+h^2\beta)|\eta(t)|^2 + |\eta(t)|^2 \\ & - 2\sqrt{1+h^2\beta}|\eta(t)|^2. \end{aligned} \tag{5.4.23}$$

Combining (5.1.2), (5.4.1) and (5.4.21)-(5.4.23), we obtain

$$\begin{aligned} & \|p_h\vec{y}_h(\cdot,t)-y(\cdot,t)\|_V^2 + \|q_h\vec{y_h}'(\cdot,t)-y'(\cdot,t)\|_{L^2(0,1)}^2 + |\sqrt{1+h^2\beta}\eta(t)-\eta(t)|^2 \\ = & 2E_h(t) + 2E(t) - 2\int_0^1 (p_h\vec{y}_h(x,t))_x y_x(x,t) dx - 2\int_0^1 q_h\vec{y_h}'(x,t) y'(x,t) dx \\ & - 2\sqrt{1+h^2\beta}\eta(t)\eta(t). \end{aligned}$$

Next, let $s > 0$. Integrating the above equation between 0 and s, letting $h \to 0$ and using (5.4.20), we deduce that

$$\begin{cases} p_h\vec{y_h} \to y & \text{strongly in} \quad L^2_{Loc}(0,\infty;V), \\ q_h\vec{y_h}' \to y' & \text{strongly in} \quad L^2_{Loc}(0,\infty;L^2(0,1)), \\ \sqrt{1+h^2\beta}\eta \to \eta, & \text{strongly in} \quad L^2_{Loc}(0,\infty;\mathbb{C}). \end{cases} \tag{5.4.24}$$

Step 3. $p_h y_h \to y$ **strongly in** $C([0,T],L^2(0,1))$**.** First, we have

$$\begin{aligned} p_h\vec{y_h}(x,t) - y(x,t) &= \int_0^t (p_h\vec{y_h}'(x,s) - y'(x,s))ds + p_h\vec{y_h}^0(x) - y^0(x) \\ &= \int_0^t (p_h\vec{y_h}'(x,s) - q_h\vec{y_h}'(x,s))ds \\ &\quad + \int_0^t (q_h\vec{y_h}'(x,s) \\ &\quad - y'(x,s))ds + p_h\vec{y_h}^0(x) - y^0(x). \end{aligned} \tag{5.4.25}$$

Using (5.4.5) for $n = 1$, it follows from above equation that

$$\begin{aligned} \|p_h\vec{y_h}(t) - y(t)\|^2_{L^2(0,1)} \leq & 3\int_0^t \|p_h\vec{y_h}'(\cdot,s) - q_h\vec{y_h}'(\cdot,s)\|^2_{L^2(0,1)}ds \\ & + 3\int_0^t \|q_h\vec{y_h}'(\cdot,s) - y'(\cdot,s)\|^2_{L^2(0,1)}ds \\ & + 3\|p_h\vec{y_h}^0(\cdot) - y^0(\cdot)\|^2_{L^2(0,1)} \\ \leq & \frac{1}{4}\int_0^t \|hp_h\vec{y_h}'(\cdot,s)\|^2_{L^2(0,1)}ds \\ & + 3\int_0^t \|q_h\vec{y_h}'(\cdot,s) - y'(\cdot,s)\|^2_{L^2(0,1)}ds \\ & + 3\|p_h\vec{y_h}^0(\cdot) - y^0(\cdot)\|^2_{L^2(0,1)}. \end{aligned}$$

From (5.1.25), we can check that $p_h\vec{y_h}^0 \to y^0$ weakly in V, thus

$$3\|p_h\vec{y_h}^0(\cdot) - y^0(\cdot)\|^2_{L^2(0,1)} \to 0, \quad \text{as} \quad h \to 0.$$

Whence the desired convergence result from the first one of (5.4.19) and the second one in (5.4.24). ■

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Université Libanaise

Université de Valenciennes et du Hainaut-Cambrésis

Thèse de Doctorat

Mohamad Ali Hassan SAMMOURY

Etude théorique et numérique de la stabilité de certains systèmes distribués avec contrôle frontière de type dynamique

Résumé
Cette thèse est consacrée à l'étude de la stabilisation de certains systèmes distribués avec un contrôle frontière de type dynamique. Nous considérons d'abord la stabilisation de l'équation de la poutre de Rayleigh avec un seul contrôle frontière dynamique moment ou force. Ensuite, nous étudions la stabilisation indirecte de l'équation des ondes avec un amortissement frontière de type dynamique fractionnel. Nous montrons que le taux de décroissance de l'énergie dépendent de la nature géométrique du domaine et nous établissons plusieurs résultats de stabilité polynomiale. Enfin, nous considérons l'approximation de l'équation des ondes un-dimensionnelle avec un seul amortissement frontière de type dynamique par un schéma de différence finie. Par une méthode spectrale, nous montrons que l'énergie discrétisée ne décroit pas uniformément (par rapport au pas du maillage) polynomialement vers zéro comme l'énergie du système continu. Nous introduisons, alors, un terme de viscosité numérique et nous montrons la décroissance polynomiale uniforme de l'énergie de notre schéma discret avec ce terme de viscosité.

Mots clés
Contrôle frontière dynamique, Stabilité polynomiale, étude spectrale, Méthode de Différence fini.

Abstract
This thesis is devoted to the study of the stabilization of some distributed systems with dynamic boundary control. First, we consider the stabilization of the Rayleigh beam equation with only one dynamic boundary control moment or force. Next, we study the indirect stability of the wave equation with a fractional dynamic boundary control. We show that the decay rate of the energy depends on the nature of the geometry of the domain and we establish different polynomial stability results. Finally, we consider the finite difference space discretization of the 1-d wave equation with dynamic boundary control. First, using a spectral approach, we show that the polynomial decay of the discretized energy is not uniform with respect to the mesh size, as the energy of the continuous system. Next, we introduce a viscosity term and we establish the uniform (with respect to the mesh size) polynomial energy decay of our discrete scheme.

Key Words
Boundary dynamical control, Polynomial stability, Spectral study, Finite difference method.

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