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Résultats de stabilité pour certains systèmes hyperboliques avec amortissements locaux directs ou indirects

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Abstract

In this thesis, we study the indirect stability of some coupled systems with different kinds of local discontinuous dampings. We also study the stability and the instability results of the Kirchhoff plate equation with delay terms on the boundary or dynamical boundary controls.

First, we investigate the stabilization of locally coupled wave equations with non-smooth localized viscoelastic damping of Kelvin-Voigt type and localized time delay. Using a general criteria of Arendt-Batty, we show the strong stability of our system in the absence of the compactness of the resolvent. However, by combining the frequency domain approach with the multiplier method, we prove a polynomial energy decay rate.

Second, we investigate the stabilization of locally coupled wave equations with local viscoelastic damping of past history type acting only on one equation via non-smooth coefficients. We prove the strong stability of our system. Next, we establish the exponential stability of the solution if the two waves have the same speed of propagation. In the case of different propagation speeds, we prove that the energy of our system decays polynomially. Moreover, we show the lack of exponential stability if the speeds of wave propagation are different with a global damping and a global coupling.

Third, we investigate the stabilization of a linear Bresse system with one discontinuous local internal viscoelastic damping of Kelvin-Voigt type acting on the axial force, under fully Dirichlet boundary conditions. We prove the strong and polynomial stabilities of our system.

Finally, we consider two models of the Kirchhoff plate equation, the first one with delay terms on the dynamical boundary controls, and the second one where delay terms on the boundary control are added. For the first system, we prove its well-posedness, strong stability, non-exponential stability, and polynomial stability under a multiplier geometric control condition. For the second one, we prove its well-posedness, strong stability, and exponential stability under the same multiplier geometric control condition. Finally, we give some instability examples of the second system for some choices of delays.

Keywords

C_0-semigroup, frequency domain approach, coupled wave equations, Bresse system, Kirchhoff plate equation, Kelvin-Voigt damping, past history damping, dynamical control, time delay, strong stability, exponential stability, polynomial stability

Résumé

Dans cette thèse, nous étudions la stabilité indirecte de certains systèmes couplés avec différents types d'amortissements locaux discontinus. Nous étudions également des résultats de stabilité et d'instabilité de l'équation des plaques de Kirchhoff avec des termes de retard à la frontière ou des contrôles dynamiques à la frontière.

Tout d'abord, nous étudions la stabilisation des équations d'ondes localement couplées avec un amortissement viscoélastique localisé non régulier de type Kelvin-Voigt et un retard temporel localisé. En utilisant un critère général d'Arendt-Batty, nous montrons la stabilité forte de notre système en l'absence de la compacité la résolvante. Cependant, en combinant l'approche du domaine fréquentielle avec la méthode des multiplicateurs, nous prouvons un taux de décroissance polynomial de l'énergétique.

Deuxièmement, nous étudions la stabilisation d'équations d'ondes localement couplées avec un amortissement viscoélastique local de type histoire passée agissant seulement sur une équation via des coefficients non régulier. Nous prouvons la stabilité forte de notre système. Ensuite, nous établissons la stabilité exponentielle de la solution si les deux ondes ont la même vitesse de propagation. Dans le cas de vitesses de propagation différentes, nous prouvons que l'énergie de notre système décroît de façon polynomiale. De plus, nous montrons l'absence de stabilité exponentielle si les vitesses de propagation des ondes sont différentes avec un amortissement global et un couplage global.

Troisièmement, nous étudions la stabilisation d'un système linéaire de Bresse avec un amortissement viscoélastique interne local discontinu de type Kelvin-Voigt agissant sur la force axiale, sous des conditions aux limites entièrement de Dirichlet. Nous prouvons la stabilité forte et polynomiale de notre système.

Enfin, nous considérons deux modèles de l'équation des plaques de Kirchhoff, le premier avec des termes de retard sur les contrôles dynamiques aux bords, et le second où des termes de retard sur le contrôle aux bords sont ajoutés. Pour le premier système, nous prouvons son caractère bien posé, sa stabilité forte, sa stabilité non-exponentielle et sa stabilité polynomiale sous une condition de contrôle géométrique par multiplicateur. Pour le second système, nous prouvons son caractère bien posé, sa stabilité forte et sa stabilité exponentielle sous la même condition de contrôle géométrique par multiplicateur. Enfin, nous donnons quelques exemples d'instabilité du second système pour certains choix de délais.

Mots clés

C_0-semigroupe, methode fréquentielle, équations d'ondes couplées, système de Bresse, plate de Kirchhoff, amortissement de Kelvin-Voigt, amortissement du type mémoire, contrôle dynamique, délai de temps, stabilité forte, stabilité exponentielle, stabilité polynomiale.

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Symbols and Notations

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_+$	The set of non negative real numbers.
$\mathbb{R}^*$	The set of non zero real numbers.
$\mathbb{N}$	The set of natural numbers.
$\mathbb{N}^*$	The set of non zero natural numbers.
$\mathbb{Z}$	The set of integer numbers.
$\mathbb{C}$	The set of complex numbers.
i	The imaginary unit.
$\Re$	The real part.
$\Im$	The imaginary part.
L^p	The Lebesgue space.
H^m	The sobolev space.
C^0	The space of continuous function.
C^1	The space of continuously differentiable functions.
C^2	The space of twice continuously differentiable functions.
$\mathcal{D}(\Omega)$	The set of smooth functions in Ω
$\mathcal{D}'(\Omega)$	The space of distribution in Ω.
$\lvert\cdot\rvert$	The modulus.
$\lvert\cdot\rvert_X$	The semi-norm in X.
$\lVert\cdot\rVert_X$	The norm in X.
$(\cdot,\cdot)_X$	The inner product in X.
max	The maximum.
min	The minimum.
sup	The supreme.
inf	The infimum.
$f_y = \partial_y f$	The partial derivative of f with respect of y.
$f_{yy} = \partial_{yy} f$	The second partial derivative of f with respect of y.
$g'(s)$	The derivative of g with respect to s.
$A \lesssim B$	Means that there exists a constant $C > 0$ independent of A, B and a natural parameter n such that $A \leq CB$.

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Introduction

In this thesis, we study the indirect stability of some coupled systems with different kinds of local discontinuous dampings. We also study the stability and the instability results of the Kirchhoff plate equation with delay terms on the boundary or dynamical boundary controls. This thesis is divided into five chapters.

In the first chapter, we recall some well-known results about semigroups, including some theorems about strong, exponential, polynomial, and analytic stability of a C_0-semigroup. We also recall the definition of the multiplier geometric control condition denoted by **MGC**.

A wave is created when a vibrating source disturbs the medium. In order to restrain those vibrations, several dampings can be added such as frictional (viscous), Kelvin-Voigt, time delay, past history (infinite memory) dampings. However, time delays appear in several applications such as in physics, chemistry, biology, thermal phenomena not only depend on the present state but also on some past occurrences (see [44] and [72]) . In the last years, the control of partial differential equations with time delays have become popular among scientists, since in many cases time delays induce some instabilities see [36, 38, 39, 42].

The notion of indirect damping mechanisms has been introduced by Russell in [100] and since this time, it retains the attention of many authors. In particular, the fact that only one equation of the coupled system is damped refers to the so-called class of "indirect" stabilization problems initiated and studied in [10, 11, 12] and further studied by many authors, see for instance [13, 78, 109] and the rich references therein. The study of such systems is also motivated by several physical considerations like Timoshenko and Bresse systems (see for instance [1], [8], [84] and [86]). The Bresse system is a model for arched beams (see Fig. 1 for an illustration), see [74, Chap. 6]. It can be expressed by the equations of motion:

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + lN, \\ \rho_2\psi_{tt} = M_x - Q, \\ \rho_1 w_{tt} = N_x - lQ, \end{cases} \tag{Bresse System}$$

where $N = k_3(w_x - l\varphi)$ is the axial force, $Q = k_1(\varphi_x + \psi + lw)$ is the shear force, and $M = k_2\psi_x$ is the bending moment. The functions φ, ψ, and w are respectively the vertical, shear angle, and longitudinal displacements. Here $\rho_1 = \rho A$, $\rho_2 = \rho I$, $k_1 = kGA$, $k_3 = EA$, $k_2 = EI$ and $l = R^{-1}$, in which ρ is the density of the material, E the modulus of the elasticity, G the shear modulus, k the shear factor, A the cross-sectional area, I the second moment of area of the cross section, R the radius of the curvature, and l the curvature. We note that by neglecting w ($l \to 0$) in (Bresse System), the Bresse system reduces to the following

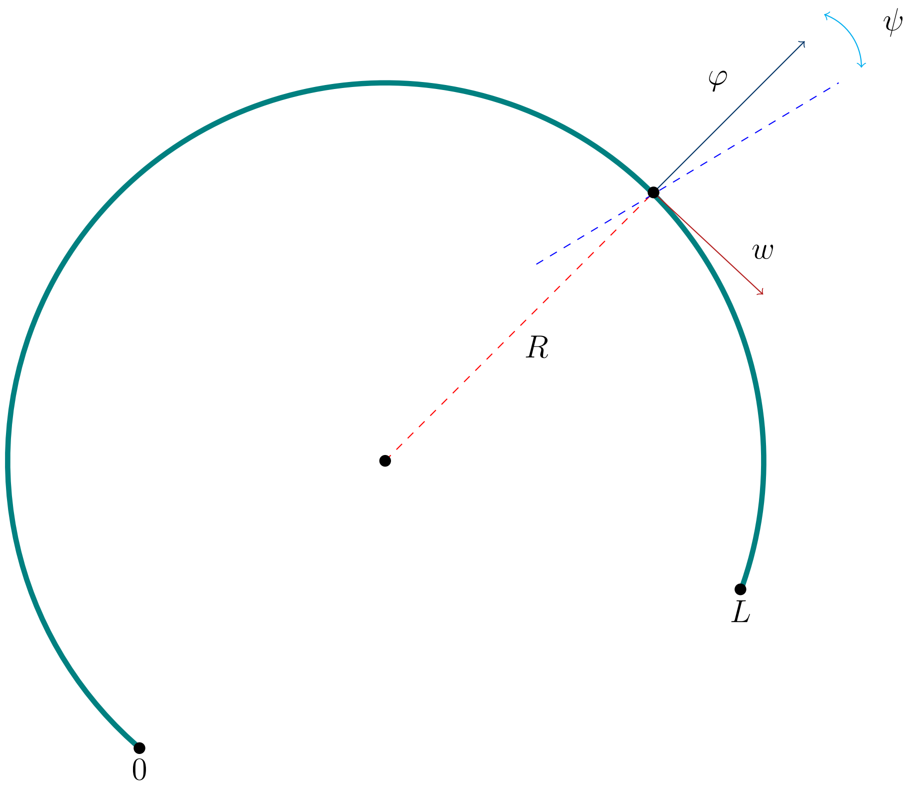

Figure 1: The circular arch

conservative Timoshenko system:

$$\rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x = 0,$$
$$\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) = 0.$$

In chapters two, three, and four, we focus on strongly coupled systems with different kinds of indirect local dampings and non-smooth coefficients at the interface.

In chapter two, we investigate the stability of local coupled wave equations with singular localized viscoelastic damping of Kelvin-Voigt type and localized time delay. More precisely, we consider the following system:

$$\begin{cases} u_{tt} - \left[au_x + b(x)(\kappa_1 u_{tx} + \kappa_2 u_{tx}(x, t-\tau))\right]_x + c(x)y_t & \\ = 0, & (x,t) \in (0,L)\times(0,\infty), \\ y_{tt} - y_{xx} - c(x)u_t = 0, & (x,t) \in (0,L)\times(0,\infty), \\ u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, & t > 0, \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in (0,L), \\ (y(x,0), y_t(x,0)) = (y_0(x), y_1(x)), & x \in (0,L), \\ u_t(x,t) = f_0(x,t), & (x,t) \in (0,L)\times(-\tau,0), \end{cases} \tag{Sys1}$$

where L, τ, a and κ_1 are positive real numbers, κ_2 is a non-zero real number and $(u_0, u_1, y_0, y_1, f_0)$ belongs to a suitable space. We suppose that there exist $0 < \alpha < \beta < \gamma < L$

and a non-zero constant c_0, such that

$$b(x) = \begin{cases} 1, & x \in (0,\beta), \\ 0, & x \in (\beta, L), \end{cases} \quad \text{and} \quad c(x) = \begin{cases} c_0, & x \in (\alpha,\gamma), \\ 0, & x \in (0,\alpha) \cup (\gamma, L). \end{cases}$$

In fact, there are few results concerning the stability of coupled wave equations with local Kelvin-Voigt damping and without time delay, especially in the absence of smoothness of the damping and coupling coefficients (see Subsection 2.1.2). This motivates our interest to study the stabilization of system (Sys1) in this chapter. As in [88], we introduce the following auxiliary change of variable

$$\eta(x,\rho,t) := u_t(x, t-\rho\tau), \quad x \in (0,\beta),\ \rho \in (0,1),\ t > 0.$$

Then, system (Sys1) becomes

$$\begin{cases} u_{tt} - (S_b(u,u_t,\eta))_x + c(x)y_t = 0, & (x,t) \in (0,L) \times (0,\infty), \\ y_{tt} - y_{xx} - c(x)u_t = 0, & (x,t) \in (0,L) \times (0,\infty), \\ \tau\eta_t(x,\rho,t) + \eta_\rho(x,\rho,t) = 0, & (x,\rho,t) \in (0,\beta) \times (0,1) \times (0,\infty), \\ u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, & t > 0, \\ \eta(0,\rho,t) = 0, & (\rho,t) \in (0,1) \times (0,\infty), \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in (0,L), \\ (y(x,0), y_t(x,0)) = (y_0(x), y_1(x)), & x \in (0,L), \\ \eta(x,\rho,0) = f_0(x,-\rho\tau), & (x,\rho) \in (0,\beta) \times (0,1), \end{cases} \tag{Sys2}$$

where $S_b(u,u_t,\eta) := au_x + b(x)(\kappa_1 u_{tx} + \kappa_2 u_{tx}(x,t-\tau))$. Moreover, from the definition of $b(x)$, we have

$$S_b(u,u_t,\eta) = \begin{cases} S_1(u,u_t,\eta) := au_x + \kappa_1 u_{tx} + \kappa_2 \eta_x(\cdot,1,t), & \text{in } (0,\beta), \\ au_x, & \text{in } (\beta,L). \end{cases}$$

The energy of system (Sys2) is given by

$$E_1(t) = E_{1,u}(t) + E_{1,y}(t) + E_{1,\eta}(t),$$

where

$$\begin{cases} E_{1,u}(t) = \dfrac{1}{2}\displaystyle\int_0^L \left(|u_t|^2 + a|u_x|^2\right) dx, \quad E_{1,y}(t) = \dfrac{1}{2}\displaystyle\int_0^L \left(|y_t|^2 + |y_x|^2\right) dx \quad \text{and} \\ E_{1,\eta}(t) = \dfrac{\tau|\kappa_2|}{2}\displaystyle\int_0^\beta \int_0^1 |\eta_x(\cdot,\rho,t)|^2 d\rho dx. \end{cases}$$

According to Lemma 2.2.1, we have

$$\frac{d}{dt}E_1(t) \leq -(\kappa_1 - |\kappa_2|)\int_0^\beta |u_{tx}|^2 dx.$$

In the sequel, we make the following assumptions

$$\kappa_1 > 0, \quad \kappa_2 \in \mathbb{R}^* \quad \text{and} \quad |\kappa_2| < \kappa_1.$$

Then, system (Sys2) is dissipative in the sense that its energy is non-increasing with respect to time (i.e. $E_1'(t) \leq 0$). Now, we write system (Sys2) as the following first order evolution equation

$$U_t = \mathcal{A}_1 U, \quad U(0) = U_0,$$

where $U_0 = (u_0, u_1, y_0, y_1, f_0(\cdot, -\rho\tau))^\top \in \mathcal{H}_1$. The Hilbert space $\mathcal{H}_1$ is defined by

$$\mathcal{H}_1 := \left(H_0^1(0,L) \times L^2(0,L)\right)^2 \times \mathcal{W},$$

where

$$\mathcal{W} := L^2((0,1); H_L^1(0,\beta)) \ \text{ and } \ H_L^1(0,\beta) := \left\{\widetilde{\eta} \in H^1(0,\beta) \mid \widetilde{\eta}(0) = 0\right\}.$$

The space $\mathcal{W}$ is a Hilbert space of $H_L^1(0,\beta)$-valued functions on $(0,1)$, equipped with the following inner product

$$(\eta^1, \eta^2)_{\mathcal{W}} := \int_0^\beta \int_0^1 \eta_x^1 \overline{\eta_x^2} d\rho dx, \quad \forall\, \eta^1, \eta^2 \in \mathcal{W}.$$

The Hilbert space $\mathcal{H}_1$ is equipped with the following inner product

$$\left(U, U^1\right)_{\mathcal{H}_1} = \int_0^L \left(a u_x \overline{u_x^1} + v\overline{v^1} + y_x \overline{y_x^1} + z\overline{z^1}\right) dx + \tau|\kappa_2| \int_0^\beta \int_0^1 \eta_x \overline{\eta_x^1} d\rho dx,$$

where $U = (u, v, y, z, \eta)^\top$, $U^1 = (u^1, v^1, y^1, z^1, \eta^1)^\top \in \mathcal{H}_1$. The linear unbounded operator $\mathcal{A}_1 : D(\mathcal{A}_1) \subset \mathcal{H}_1 \longmapsto \mathcal{H}_1$ is defined by:

$$D(\mathcal{A}_1) = \left\{ \begin{array}{c} U = (u, v, y, z, \eta)^\top \in \mathcal{H}_1 \mid y \in H^2(0,L) \cap H_0^1(0,L), \ v, z \in H_0^1(0,L) \\ (S_b(u, v, \eta))_x \in L^2(0,L), \quad \eta_\rho \in \mathcal{W}, \quad \eta(\cdot, 0) = v(\cdot) \text{ in } (0,\beta) \end{array} \right\}$$

and

$$\mathcal{A}_1 \begin{pmatrix} u \\ v \\ y \\ z \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ (S_b(u, v, \eta))_x - c(\cdot) z \\ z \\ y_{xx} + c(\cdot) v \\ -\tau^{-1}\eta_\rho \end{pmatrix}, \ \forall U = (u, v, y, z, \eta)^\top \in D(\mathcal{A}_1).$$

In chapter three, we investigate the indirect stability of coupled elastic wave equations with localized past history damping. More precisely, we consider the following system:

$$\begin{cases} u_{tt} - \left(a u_x - b(x) \displaystyle\int_0^\infty g(s) u_x(x, t-s) ds\right)_x + c(x) y_t & \\ = 0, & (x,t) \in (0,L) \times (0,\infty), \\ y_{tt} - y_{xx} - c(x) u_t = 0, & (x,t) \in (0,L) \times (0,\infty), \\ u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, & t > 0, \\ (u(x,-s), u_t(x,0)) = (u_0(x,s), u_1(x)), & (x,s) \in (0,L) \times (0,\infty), \\ (y(x,0), y_t(x,0)) = (y_0(x), y_1(x)), & x \in (0,L), \end{cases} \tag{Sys3}$$

where L and a are positive real numbers. We suppose that there exist a non-zero constant c_0 and positive constants α, β, γ, and b_0 such that $0 < \alpha < \beta < \gamma < L$, and define

$$b(x) = \begin{cases} b_0, & x \in (0,\beta), \\ 0, & x \in (\beta, L), \end{cases} \quad \text{and} \quad c(x) = \begin{cases} c_0, & x \in (\alpha,\gamma), \\ 0, & x \in (0,\alpha) \cup (\gamma, L). \end{cases}$$

The general integral term represents a history term with the relaxation function g that is supposed to satisfy the following hypotheses:

$$\begin{cases} g \in L^1([0,\infty)) \cap C^1([0,\infty)) \text{ is a positive function such that} \\ g(0) := g_0 > 0, \quad \displaystyle\int_0^\infty g(s)ds := \widetilde{g}, \quad \widetilde{b}(x) := a - b(x)\widetilde{g} > 0, \quad \text{and} \\ g'(s) \leq -mg(s), \quad \text{for some } m > 0, \forall s \geq 0. \end{cases}$$

Moreover, from the definition of $b(x)$, we have

$$\widetilde{b}(x) := a - b(x)\widetilde{g} = \begin{cases} \widetilde{b_0} := a - b_0\widetilde{g}, & \text{in } (0,\beta), \\ a, & \text{in } (\beta, L). \end{cases}$$

According to the best of our knowledge, it seems that no result in the literature exists concerning the case of coupled wave equations with localized past history damping, especially in the absence of smoothness of the damping and coupling coefficients. This motivates our interest to study the stabilization of system (Sys3) in this chapter. As in [35], we introduce the following auxiliary change of variable

$$\omega(x,s,t) := u(x,t) - u(x,t-s), \quad (x,s,t) \in (0,\beta) \times (0,\infty) \times (0,\infty).$$

Then, system (Sys3) becomes

$$\begin{cases} u_{tt} - \left(S_{\tilde{b}(\cdot)}(u,\omega)\right)_x + c(\cdot)y_t = 0, & (x,t) \in (0,L) \times (0,\infty), \\ y_{tt} - y_{xx} - c(\cdot)u_t = 0, & (x,t) \in (0,L) \times (0,\infty), \\ \omega_t(x,s,t) + \omega_s(x,s,t) - u_t = 0, & (x,s,t) \in (0,\beta) \times (0,\infty) \times (0,\infty), \\ u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, & t > 0, \\ \omega(x,0,t) = 0, & (x,t) \in (0,\beta) \times (0,\infty), \\ \omega(0,s,t) = 0, & (s,t) \in (0,\infty) \times (0,\infty), \\ (u(x,-s), u_t(x,0)) = (u_0(x,s), u_1(x)), & (x,s) \in (0,L) \times (0,\infty), \\ (y(x,0), y_t(x,0)) = (y_0(x), y_1(x)), & x \in (0,L), \\ \omega_0(x,s) := \omega(x,s,0) = u_0(x,0) - u_0(x,s), & (x,s) \in (0,\beta) \times (0,\infty), \end{cases} \tag{Sys4}$$

where

$$S_{\tilde{b}(\cdot)}(u,\omega) := \begin{cases} S_{\tilde{b_0}}(u,\omega) := \widetilde{b_0}u_x + b_0 \displaystyle\int_0^\infty g(s)\omega_x(x,s,t)ds, & \text{in } (0,\beta), \\ au_x, & \text{in } (\beta, L). \end{cases}$$

The energy of system (Sys4) is given by

$$E_2(t) = E_{2,u}(t) + E_{2,y}(t) + E_{2,\omega}(t),$$

where

$$\begin{cases} E_{2,u}(t) = \dfrac{1}{2}\displaystyle\int_0^L \left(|u_t|^2 + \widetilde{b}(\cdot)|u_x|^2\right) dx, \quad E_{2,y}(t) = \dfrac{1}{2}\displaystyle\int_0^L \left(|y_t|^2 + |y_x|^2\right) dx \quad \text{and} \\ E_{2,\omega}(t) = \dfrac{b_0}{2}\displaystyle\int_0^\beta \int_0^\infty g(s)|\omega_x(\cdot,s,t)|^2 dsdx. \end{cases}$$

According to Lemma 3.2.1, we have

$$\frac{d}{dt}E_2(t) = \frac{b_0}{2}\int_0^\beta \int_0^\infty g'(s)|\omega_x(\cdot,s,t)|^2 ds dx \leq 0.$$

Then, system (Sys4) is dissipative in the sense that its energy is non-increasing with respect to time. Now, we write system (Sys4) as the following first order evolution equation

$$U_t = \mathcal{A}_2 U, \quad U(0) = U_0,$$

where $U_0 = (u_0(\cdot,0), u_1, y_0, y_1, \omega_0(\cdot,s))^\top \in \mathcal{H}_2$. The Hilbert space $\mathcal{H}_2$ is defined by

$$\mathcal{H} := \left(H_0^1(0,L) \times L^2(0,L)\right)^2 \times \mathcal{W}_g,$$

where

$$\mathcal{W}_g := L_g^2((0,\infty); H_L^1(0,\beta)) \quad \text{and} \quad H_L^1(0,\beta) := \left\{\widetilde{\omega} \in H^1(0,\beta) \mid \widetilde{\omega}(0) = 0\right\}.$$

The space $\mathcal{W}_g$ is a Hilbert space of $H_L^1(0,\beta)$-valued functions on $(0,\infty)$, equipped with the following inner product

$$(\omega^1, \omega^2)_{\mathcal{W}_g} := \int_0^\beta \int_0^\infty g(s)\omega_x^1 \overline{\omega_x^2} ds dx, \quad \forall\, \omega^1, \omega^2 \in \mathcal{W}_g.$$

The Hilbert space $\mathcal{H}_2$ is equipped with the following inner product

$$\begin{aligned}\left(U, U^1\right)_{\mathcal{H}_2} = &\int_0^L \left(\tilde{b}(\cdot) u_x \overline{u_x^1} + v\overline{v^1} + y_x \overline{y_x^1} + z\overline{z^1}\right) dx \\ &+ b_0 \int_0^\beta \int_0^\infty g(s)\omega_x \overline{\omega_x^1} ds dx,\end{aligned}$$

where $U = (u,v,y,z,\omega)^\top \in \mathcal{H}_2$ and $U^1 = (u^1, v^1, y^1, z^1, \omega^1)^\top \in \mathcal{H}_2$. Now, we define the linear unbounded operator $\mathcal{A}_2 : D(\mathcal{A}_2) \subset \mathcal{H}_2 \longmapsto \mathcal{H}_2$ by:

$$D(\mathcal{A}_2) = \left\{\begin{array}{c} U = (u,v,y,z,\omega)^\top \in \mathcal{H}_2 \mid y \in H^2(0,L) \cap H_0^1(0,L),\ v, z \in H_0^1(0,L) \\ \left(S_{\tilde{b}(\cdot)}(u,\omega)\right)_x \in L^2(0,L), \quad \omega_s \in \mathcal{W}_g, \quad \omega(\cdot,0) = 0 \ \text{ in } \ (0,\beta) \end{array}\right\}$$

and

$$\mathcal{A}_2 \begin{pmatrix} u \\ v \\ y \\ z \\ \omega \end{pmatrix} = \begin{pmatrix} v \\ \left(S_{\tilde{b}(\cdot)}(u,\omega)\right)_x - c(\cdot) z \\ z \\ y_{xx} + c(\cdot) v \\ -\omega_s + v \end{pmatrix}, \ \forall U = (u,v,y,z,\omega)^\top \in D(\mathcal{A}_2).$$

In chapter four, we investigate the stability of a Bresse system with only one discontinuous local internal Kelvin-Voigt damping on the axial force. More precisely, we consider the following system:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) & \\ -\, ld(x)(w_{tx} - l\varphi_t) = 0, & (x,t) \in (0,L) \times (0,\infty), \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) = 0, & (x,t) \in (0,L) \times (0,\infty), \\ \rho_1 w_{tt} - [k_3(w_x - l\varphi) + d(x)(w_{tx} - l\varphi_t)]_x & \\ +\, lk_1(\varphi_x + \psi + lw) = 0, & (x,t) \in (0,L) \times (0,\infty), \\ \varphi(x,t) = \psi(x,t) = w(x,t) = 0, & (x,t) \in \{0,L\} \times (0,\infty), \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), & x \in (0,L), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), & x \in (0,L), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), & x \in (0,L), \end{cases} \tag{Sys5}$$

where $\rho_1, \rho_2, k_1, k_2, k_3, l$ and L are positive real numbers. We suppose that there exist $0 < \alpha < \beta < L$ and a positive constant d_0, such that

$$d(x) = \begin{cases} d_0 & \text{if} \quad x \in (\alpha,\beta), \\ 0 & \text{if} \quad x \in (0,\alpha) \cup (\beta, L). \end{cases}$$

According to the best of our knowledge, it seems that no result in the literature exists concerning the case of Bresse system with only one discontinuous local internal Kelvin-Voigt damping on the axial force, especially under fully Dirichlet boundary conditions and without any condition on the curvature l. This motivates our interest to study the stabilization of system (Sys5) in this chapter. The energy of system (Sys5) is given by

$$E_3(t) = \frac{1}{2}\int_0^L \left(\rho_1 |\varphi_t|^2 + \rho_2|\psi_t|^2 + \rho_1|w_t|^2 + k_1|\varphi_x + \psi + lw|^2 + k_2|\psi_x|^2 + k_3|w_x - l\varphi|^2\right) dx.$$

A direct computation gives

$$E_3'(t) = -\int_0^L d(x)|w_{tx} - l\varphi_t|^2 dx = -d_0\int_\alpha^\beta |w_{tx} - l\varphi_t|^2 dx \le 0.$$

Thus, system (Sys5) is dissipative in the sense that its energy is non-increasing with respect to time. Now, we write system (Sys5) as the following first order evolution equation

$$U_t = \mathcal{A}_3 U, \quad U(0) = U_0,$$

where $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1)^\top \in \mathcal{H}_3$. The Hilbert space $\mathcal{H}_3$ is given by

$$\mathcal{H}_3 := \left(H_0^1(0,L) \times L^2(0,L)\right)^3.$$

The Hilbert space $\mathcal{H}_3$ is equipped with the following inner product and norm

$$(U, \widetilde{U})_{\mathcal{H}_3} = \int_0^L \left\{ k_1(v_x^1 + v^3 + lv^5)\overline{(\widetilde{v_x^1} + \widetilde{v^3} + l\widetilde{v^5})} + \rho_1 v^2\overline{\widetilde{v^2}} + k_2 v_x^3\overline{\widetilde{v_x^3}} + \rho_2 v^4\overline{\widetilde{v^4}} \right.$$
$$\left. + k_3(v_x^5 - lv^1)\overline{(\widetilde{v_x^5} - l\widetilde{v^1})}dx + \rho_1 v^6\overline{\widetilde{v^6}} \right\} dx,$$

and

$$\|U\|^2_{\mathcal{H}_3} = \int_0^L \left\{k_1|v^1_x + v^3 + lv^5|^2 + \rho_1|v^2|^2 + k_2|v^3_x|^2 + \rho_2|v^4|^2 \right.$$
$$\left. + k_3|v^5_x - lv^1|^2 + \rho_1|v^6|^2\right\} dx,$$

where $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in \mathcal{H}_3$ and $\widetilde{U} = (\widetilde{v^1}, \widetilde{v^1}, \widetilde{v^2}, \widetilde{v^3}, \widetilde{v^4}, \widetilde{v^5}, \widetilde{v^6})^\top \in \mathcal{H}_3$. Now, we define the linear unbounded operator $\mathcal{A}_3 : D(\mathcal{A}_3) \subset \mathcal{H}_3 \longmapsto \mathcal{H}_3$ by:

$$D(\mathcal{A}_3) = \left\{ \begin{array}{c} U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in \mathcal{H}_3 \mid v^1, v^3 \in H^2(0, L) \cap H^1_0(0, L) \\ v^2, v^4, v^6 \in H^1_0(0, L), \quad \left[k_3 v^5_x + d(x)(v^6_x - lv^2)\right]_x \in L^2(0, L) \end{array} \right\}$$

and

$$\mathcal{A}_3 \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \\ v^5 \\ v^6 \end{pmatrix} = \begin{pmatrix} v^2 \\ \dfrac{k_1}{\rho_1}(v^1_x + v^3 + lv^5)_x + \dfrac{lk_3}{\rho_1}(v^5_x - lv^1) + \dfrac{ld(x)}{\rho_1}(v^6_x - lv^2) \\ v^4 \\ \dfrac{k_2}{\rho_2} v^3_{xx} - \dfrac{k_1}{\rho_2}(v^1_x + v^3 + lv^5) \\ v^6 \\ \dfrac{1}{\rho_1}\left[k_3(v^5_x - lv^1) + d(x)(v^6_x - lv^2)\right]_x - \dfrac{lk_1}{\rho_1}(v^1_x + v^3 + lv^5) \end{pmatrix},$$

for all $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A}_3)$.

In Sections 2.2, 3.2, and 4.2, we prove that the operators $\mathcal{A}_j$ are m-dissipative in $\mathcal{H}_j$, $j \in \{1, 2, 3\}$. Thus, according to Lumer-Phillips theorem (see Theorem 1.2.8), we deduce that the operators $\mathcal{A}_j$ generate a C_0-semigroup of contractions $e^{t\mathcal{A}_j}$ in $\mathcal{H}_j$ which give the well-posedness of systems (Sys2), (Sys4), and (Sys5).

In Sections 2.3, 3.3, and 4.3, we use a general criteria of Arendt-Batty (see Theorem 1.3.3) to show the strong stability of the C_0-semigroups $e^{t\mathcal{A}_j}$ associated with systems (Sys2), (Sys4), and (Sys5) in the absence of the compactness of the resolvents of $\mathcal{A}_j$. The tools used in these proofs are:

- In Section 2.3, by using a contradiction argument (see Remark 1.3.5) with the help of some multiplier techniques, we prove that $i\mathbb{R} \subset \rho(\mathcal{A}_1)$, $\rho(\mathcal{A}_1)$ being the resolvent set of $\mathcal{A}_1$.

- In Sections 3.3 and 4.3, by using Holmgren uniqueness theorem (see [75]) and Fredholm alternative (see Theorem 1.1.4), for all $\lambda \in \mathbb{R}$, we prove that
  - $\ker(i\lambda I - \mathcal{A}_j) = \{0\}$, $j \in \{2, 3\}$.
  - $\mathcal{R}(i\lambda I - \mathcal{A}_j) = \mathcal{H}_j$, $j \in \{2, 3\}$.

In Sections 2.4, 3.4, amd 4.4, by combining a frequency domain approach with a multiplier method (see Theorems 1.3.6 and 1.3.7), we prove that the energies of systems (Sys2), (Sys4), and (Sys5) decay exponentially or polynomially with the rates summarized in the following table:

System	Energy decay rate
(Sys2)	t^{-1}
(Sys4)	Exponential if $a=1$ t^{-1} if $a\neq 1$
(Sys5)	t^{-1} if $\frac{k_1}{\rho_1}=\frac{k_2}{\rho_2}$ $t^{-\frac{1}{2}}$ if $\frac{k_1}{\rho_1}\neq\frac{k_2}{\rho_2}$

In other words, for all $U_0 \in D(\mathcal{A}_j)$, $j \in \{1,2,3\}$, there exists a constant $C>0$ independent of U_0 such that for all $t>0$, we have

$$E_1(t) \leq \frac{C}{t}\|U_0\|^2_{D(\mathcal{A}_1)}, \quad E_2(t) \leq \frac{C}{t}\|U_0\|^2_{D(\mathcal{A}_2)} \text{ if } \quad a \neq 1,$$

$$E_3(t) \leq \begin{cases} \dfrac{C}{t}\|U_0\|^2_{D(\mathcal{A}_3)} & \text{if } \quad \dfrac{k_1}{\rho_1} = \dfrac{k_2}{\rho_2}, \\ \dfrac{C}{\sqrt{t}}\|U_0\|^2_{D(\mathcal{A}_3)} & \text{if } \quad \dfrac{k_1}{\rho_1} \neq \dfrac{k_2}{\rho_2}, \end{cases}$$

and for all $U_0 \in \mathcal{H}_2$, there exist constants $M \geq 1$ and $\epsilon > 0$ independent of U_0 such that for all $t>0$ we have

$$\|e^{t\mathcal{A}_2}U_0\|_{\mathcal{H}_2} \leq Me^{-\epsilon t}\|U_0\|_{\mathcal{H}_2} \text{ if } \quad a = 1.$$

In Section 3.5, we use Theorem 1.3.6 to prove the lack of exponential stability of system (Sys3) when $b(x) = c(x) = 1$ in case of different speeds of propagation, i.e., when $a \neq 1$.

In the last chapter, we study the boundary stabilization of the Kirchhoff plate equation with time delay. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with boundary Γ of class C^4 consisting of a clamped part $\Gamma_0 \neq \emptyset$ and a rimmed part $\Gamma_1 \neq \emptyset$ such that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. In the first part of this chapter, we study the stability of the Kirchhoff plate equation with delay terms on the dynamical boundary controls, namely we consider

$$\begin{cases} u_{tt}(x,t) + \Delta^2 u(x,t) = 0 \text{ in } \Omega \times (0,\infty), \\ u(x,t) = \partial_\nu u(x,t) = 0 \text{ on } \Gamma_0 \times (0,\infty), \\ \mathcal{B}_1 u(x,t) + \eta(x,t) = 0 \text{ on } \Gamma_1 \times (0,\infty), \\ \mathcal{B}_2 u(x,t) - \xi(x,t) = 0 \text{ on } \Gamma_1 \times (0,\infty), \\ \eta_t(x,t) - \partial_\nu u_t(x,t) + \beta_1 \eta(x,t) + \beta_2 \eta(x,t-\tau_1) = 0 \text{ on } \Gamma_1 \times (0,\infty), \\ \xi_t(x,t) - u_t(x,t) + \gamma_1 \xi(x,t) + \gamma_2 \xi(x,t-\tau_2) = 0 \text{ on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \text{ in } \Omega, \\ \eta(x,0) = \eta_0(x), \quad \xi(x,0) = \xi_0(x) \text{ on } \Gamma_1, \\ \eta(x,t) = f_0(x,t) \text{ on } \Gamma_1 \times (-\tau_1, 0), \\ \xi(x,t) = g_0(x,t) \text{ on } \Gamma_1 \times (-\tau_2, 0). \end{cases} \tag{Sys6}$$

In the second part of this chapter, we study the stability of the Kirchhoff plate equation with delay terms on the boundary controls, by considering:

$$\begin{cases} u_{tt}(x,t)+\Delta^2 u(x,t)=0 \ \text{ in } \ \Omega\times(0,\infty),\\ u(x,t)=\partial_\nu u(x,t)=0 \ \text{ on } \ \Gamma_0\times(0,\infty),\\ \mathcal{B}_1 u(x,t)=-\beta_1\partial_\nu u_t(x,t)-\beta_2\partial_\nu u_t(x,t-\tau_1) \ \text{ on } \ \Gamma_1\times(0,\infty),\\ \mathcal{B}_2 u(x,t)=\gamma_1 u_t(x,t)+\gamma_2 u_t(x,t-\tau_2) \ \text{ on } \ \Gamma_1\times(0,\infty),\\ u(x,0)=u_0(x), \ \ u_t(x,0)=u_1(x) \ \text{ in } \ \Omega,\\ u_t(x,t)=f_0(x,t) \ \text{ on } \ \Gamma_1\times(-\tau_1,0),\\ \partial_\nu u_t(x,t)=g_0(x,t) \ \text{ on } \ \Gamma_1\times(-\tau_2,0). \end{cases} \tag{Sys7}$$

Here and below, β_1, γ_1, τ_1 and τ_2 are positive real numbers, β_2 and γ_2 are non-zero real numbers, $\nu=(\nu_1,\nu_2)$ is the unit outward normal vector along Γ, and $\tau=(-\nu_2,\nu_1)$ is the unit tangent vector along Γ. The constant $0<\mu<\frac{1}{2}$ is the Poisson coefficient and the boundary operators $\mathcal{B}_1$ and $\mathcal{B}_2$ are defined by

$$\mathcal{B}_1\mathsf{f}=\Delta\mathsf{f}+(1-\mu)\mathcal{C}_1\mathsf{f}$$

and

$$\mathcal{B}_2\mathsf{f}=\partial_\nu\Delta\mathsf{f}+(1-\mu)\partial_\tau\mathcal{C}_2\mathsf{f},$$

where

$$\mathcal{C}_1\mathsf{f}=2\nu_1\nu_2\mathsf{f}_{x_1x_2}-\nu_1^2\mathsf{f}_{x_2x_2}-\nu_2^2\mathsf{f}_{x_1x_1} \ \text{ and } \ \mathcal{C}_2\mathsf{f}=(\nu_1^2-\nu_2^2)\mathsf{f}_{x_1x_2}-\nu_1\nu_2\left(\mathsf{f}_{x_1x_1}-\mathsf{f}_{x_2x_2}\right).$$

In Section 5.2, we study the first system (Sys6). For this aim, as in [88], we introduce the following auxiliary variables

$$\begin{aligned} z^1(x,\rho,t)&:=\eta(x,t-\rho\tau_1), \quad x\in\Gamma_1,\ \rho\in(0,1),\ t>0,\\ z^2(x,\rho,t)&:=\xi(x,t-\rho\tau_2), \quad x\in\Gamma_1,\ \rho\in(0,1),\ t>0. \end{aligned}$$

Then, system (Sys6) becomes

$$\begin{cases} u_{tt}+\Delta^2 u=0 \ \text{ in } \ \Omega\times(0,\infty),\\ u=\partial_\nu u=0 \ \text{ on } \ \Gamma_0\times(0,\infty),\\ \mathcal{B}_1 u+\eta=0 \ \text{ on } \ \Gamma_1\times(0,\infty),\\ \mathcal{B}_2 u-\xi=0 \ \text{ on } \ \Gamma_1\times(0,\infty),\\ \eta_t-\partial_\nu u_t+\beta_1\eta+\beta_2 z^1(\cdot,1,t)=0 \ \text{ on } \ \Gamma_1\times(0,\infty),\\ \xi_t-u_t+\gamma_1\xi+\gamma_2 z^2(\cdot,1,t)=0 \ \text{ on } \ \Gamma_1\times(0,\infty),\\ \tau_1 z_t^1(\cdot,\rho,t)+z_\rho^1(\cdot,\rho,t)=0 \ \text{ on } \ \Gamma_1\times(0,1)\times(0,\infty),\\ \tau_2 z_t^2(\cdot,\rho,t)+z_\rho^2(\cdot,\rho,t)=0 \ \text{ on } \ \Gamma_1\times(0,1)\times(0,\infty),\\ u(\cdot,0)=u_0(\cdot), \ \ u_t(\cdot,0)=u_1(\cdot) \ \text{ in } \ \Omega,\\ \eta(\cdot,0)=\eta_0(\cdot), \ \ \xi(\cdot,0)=\xi_0(\cdot) \ \text{ on } \ \Gamma_1,\\ z^1(\cdot,\rho,0)=f_0(\cdot,-\rho\tau_1) \ \text{ on } \ \Gamma_1\times(0,1),\\ z^2(\cdot,\rho,0)=g_0(\cdot,-\rho\tau_2) \ \text{ on } \ \Gamma_1\times(0,1). \end{cases} \tag{Sys8}$$

The energy of system (Sys8) is given by

$$\begin{aligned} E(t)=\frac{1}{2}\Bigg\{&a(u,u)+\int_\Omega |u_t|^2dx+\int_{\Gamma_1}|\eta|^2d\Gamma+\int_{\Gamma_1}|\xi|^2d\Gamma\\ &+\tau_1|\beta_2|\int_{\Gamma_1}\int_0^1\left|z^1(\cdot,\rho,t)\right|^2d\rho d\Gamma+\tau_2|\gamma_2|\int_{\Gamma_1}\int_0^1\left|z^2(\cdot,\rho,t)\right|^2d\rho d\Gamma\Bigg\}, \end{aligned}$$

where the sesquilinear form $a : H^2(\Omega)\times H^2(\Omega)\longmapsto \mathbb{C}$ is defined by

$$a(\mathsf{f},\mathsf{g}) = \int_\Omega \left[\mathsf{f}_{x_1x_1}\overline{\mathsf{g}}_{x_1x_1} + \mathsf{f}_{x_2x_2}\overline{\mathsf{g}}_{x_2x_2} + \mu\left(\mathsf{f}_{x_1x_1}\overline{\mathsf{g}}_{x_2x_2} + \mathsf{f}_{x_2x_2}\overline{\mathsf{g}}_{x_1x_1}\right)\right.$$
$$\left. + 2(1-\mu)\mathsf{f}_{x_1x_2}\overline{\mathsf{g}}_{x_1x_2}\right] dx.$$

According to Lemma 5.2.1, we have

$$\frac{d}{dt}E(t) \leq -(\beta_1 - |\beta_2|)\int_{\Gamma_1}|\eta|^2 d\Gamma - (\gamma_1 - |\gamma_2|)\int_{\Gamma_1}|\xi|^2 d\Gamma \leq 0.$$

In the sequel, we make the following assumptions

$$\beta_1,\gamma_1 > 0,\quad \beta_2,\gamma_2\in\mathbb{R}^*,\quad |\beta_2| < \beta_1 \ \text{ and } \ |\gamma_2| < \gamma_1.$$

Then, system (Sys8) is dissipative in the sense that its energy is non-increasing with respect to time (i.e. $E'(t)\leq 0$).

In Subsection 5.2.1, we write system (Sys8) as the following first order evolution equation

$$U_t = \mathcal{A}U,\quad U(0) = U_0,$$

where $U_0 = (u_0, u_1, \eta_0, \xi_0, f_0(\cdot,-\rho\tau_1), g_0(\cdot,-\rho\tau_2))^\top \in \mathcal{H}$. The Hilbert space $\mathcal{H}$ is defined by

$$\mathcal{H} = H^2_{\Gamma_0}(\Omega)\times L^2(\Omega)\times \left(L^2(\Gamma_1)\right)^2\times\left(L^2(\Gamma_1\times(0,1))\right)^2,$$

where

$$H^2_{\Gamma_0}(\Omega) = \left\{\mathsf{f}\in H^2(\Omega) \mid \mathsf{f} = \partial_\nu \mathsf{f} = 0 \text{ on } \Gamma_0\right\}.$$

The Hilbert space $\mathcal{H}$ is equipped with the following inner product

$$(U,U_1)_{\mathcal{H}} = a(u,u_1) + \int_\Omega v\overline{v_1}dx + \int_{\Gamma_1}\eta\overline{\eta_1}d\Gamma + \int_{\Gamma_1}\xi\overline{\xi_1}d\Gamma$$
$$+\tau_1|\beta_2|\int_{\Gamma_1}\int_0^1 z^1\overline{z_1^1}d\rho d\Gamma + \tau_2|\gamma_2|\int_{\Gamma_1}\int_0^1 z^2\overline{z_1^2}d\rho d\Gamma,$$

where $U = (u,v,\eta,\xi,z^1,z^2)^\top$, $U_1 = (u_1,v_1,\eta_1,\xi_1,z_1^1,z_1^2)^\top\in\mathcal{H}$. Now, we define the linear unbounded operator $\mathcal{A} : D(\mathcal{A})\subset\mathcal{H}\longmapsto\mathcal{H}$ by:

$$D(\mathcal{A}) = \left\{\begin{array}{l} U = (u,v,\eta,\xi,z^1,z^2)^\top \in \mathsf{D}_{\Gamma_0}(\Delta^2)\times H^2_{\Gamma_0}(\Omega)\times (L^2(\Gamma_1))^2\times (L^2(\Gamma_1;H^1(0,1)))^2 \mid \\ \mathcal{B}_1 u = -\eta,\quad \mathcal{B}_2 u = \xi,\quad z^1(\cdot,0) = \eta,\quad z^2(\cdot,0) = \xi \ \text{ on } \ \Gamma_1 \end{array}\right\}$$

where

$$\mathsf{D}_{\Gamma_0}(\Delta^2) = \left\{\mathsf{f}\in H^2_{\Gamma_0}(\Omega) \mid \Delta^2\mathsf{f}\in L^2(\Omega),\ \mathcal{B}_1\mathsf{f}\in L^2(\Gamma_1),\ \text{and } \mathcal{B}_2\mathsf{f}\in L^2(\Gamma_1)\right\}$$

and

$$\mathcal{A}\begin{pmatrix} u\\ v\\ \eta\\ \xi\\ z^1\\ z^2\end{pmatrix} = \begin{pmatrix} v\\ -\Delta^2 u\\ \partial_\nu v - \beta_1\eta - \beta_2 z^1(\cdot,1)\\ v - \gamma_1\xi - \gamma_2 z^2(\cdot,1)\\ -\dfrac{1}{\tau_1}z^1_\rho\\ -\dfrac{1}{\tau_2}z^2_\rho\end{pmatrix}, \forall U = (u,v,\eta,\xi,z^1,z^2)^\top\in D(\mathcal{A}).$$

Next, we prove that the operator $\mathcal{A}$ is m-dissipative in $\mathcal{H}$. Thus, according to Lumer-Phillips theorem (see Theorem 1.2.8), we deduce that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in $\mathcal{H}$ which gives the well-posedness of (Sys8).

In Subsection 5.2.2, we use a general criteria of Arendt-Batty (see Theorem 1.3.3) to show the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$ associated with system (Sys8) in the absence of the compactness of the resolvent of $\mathcal{A}$.

In Subsection 5.2.3, we use Theorem 1.3.6 to prove the lack of exponential stability of system (Sys8).

In Subsection 5.2.4, we use Theorem 1.3.7 to prove under the multiplier geometric control condition **MGC** (see Definition 1.4.1) that the energy of system (Sys8) decays polynomially with rate t^{-1}. In other words, for all $U_0 \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of U_0 such that

$$E(t) \leq \frac{C}{t}\|U_0\|^2_{D(\mathcal{A})}, \quad t > 0.$$

In Section 5.3, we study the second system (Sys7). We use Theorem 1.3.6 to prove under **MGC** geometric condition that system (Sys7) is exponentially stable if

$$\beta_1, \gamma_1 > 0, \quad \beta_2, \gamma_2 \in \mathbb{R}^*, \quad |\beta_2| < \beta_1 \quad \text{and} \quad |\gamma_2| < \gamma_1.$$

Moreover, we give some instability examples of system (Sys7) in the cases $|\beta_2| \geq \beta_1$ and $|\gamma_2| \geq \gamma_1$.

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Chapter 1

Preliminaries

As our analysis is based on the semigroup and spectral theories, in this chapter we will recall some well-known results about semigroups, including some theorems about strong, exponential, polynomial, and analytic stability of a C_0-semigroup. We also recall the definition of the multiplier geometric control condition denoted by **MGC** . All of the theorems are stated without proofs, but the relevant references are given. The reader may skip this chapter in the first reading, then refer to it as a reference of related. For more details see [31, 68, 71, 24, 30, 94, 77, 27, 67].

1.1 Bounded and Unbounded linear operators

We start this chapter by giving some well known results about bounded and unbounded operators. We are not trying to give a complete development, but rather review the basic definitions and theorems, mostly without proof, see [31, 68, 71].

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces over $\mathbb{C}$, and H will always denote a Hilbert space equipped with the scalar product $\langle\cdot,\cdot\rangle_H$ and the corresponding norm $\|\cdot\|_H$.

A linear operator $T : E \longmapsto F$ is a transformation which maps linearly E in F, that is

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v), \quad \forall\, u, v \in E \text{ and } \alpha, \beta \in \mathbb{C}.$$

Definition 1.1.1. A linear operator $T : E \longmapsto F$ is said to be bounded if there exists $C > 0$ such that

$$\|Tu\|_F \leq C\|u\|_E \quad \forall\, u \in E.$$

The set of all bounded linear operators from E into F is denoted by $\mathcal{L}(E, F)$. Moreover, the set of all bounded linear operators from E into E is denoted by $\mathcal{L}(E)$.

Definition 1.1.2. A bounded operator $T \in \mathcal{L}(E, F)$ is said to be compact if for each sequence $(x_n)_{n\in\mathbb{N}} \subset E$ with $\|x_n\|_E = 1$ for each $n \in \mathbb{N}$, the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a subsequence which converges in F.

The set of all compact operators from E into F is denoted by $\mathcal{K}(E, F)$. For simplicity one writes $\mathcal{K}(E) = \mathcal{K}(E, E)$.

Definition 1.1.3. Let $T \in \mathcal{L}(E, F)$, we define

- Range of T by
$$\mathcal{R}\,(T) = \{Tu : \; u \in E\} \subset F.$$

- Kernel of T by

$$\ker(T) = \{u \in E : \; Tu = 0\} \subset E.$$

Theorem 1.1.4. (**Fredholm alternative**, see Theorem 6.6 in [31]). If $T \in \mathcal{K}(E)$, then

- $\ker(I - T)$ is finite dimensional (I is the identity operator on E).
- $\mathcal{R}(I - T)$ is closed.
- $\ker(I - T) = 0 \Leftrightarrow \mathcal{R}(I - T) = E$.

Definition 1.1.5. An unbounded linear operator T from E into F is a pair $(T, D(T))$, consisting of a subspace $D(T) \subset E$ (called the domain of T) and a linear transformation.

$$T : D(T) \subset E \longmapsto F.$$

If $E = F$, then we say $(T, D(T))$ is an unbounded linear operator on E.

Definition 1.1.6. Let $T : D(T) \subset E \longmapsto F$ be an unbounded linear operator.

- The range of T is defined by

$$\mathcal{R}(T) = \{Tu : \; u \in D(T)\} \subset F.$$

- The kernel of T is defined by

$$\ker(T) = \{u \in D(T) : \; Tu = 0\} \subset E.$$

- The graph of T is defined by

$$\mathrm{G}(T) = \{(u, Tu) : \; u \in D(T)\} \subset E \times F.$$

Definition 1.1.7. A map T is said to be closed if $\mathrm{G}(T)$ is closed in $E \times F$. The closedness of an unbounded linear operator T can be characterized as follows

$$\text{if } u_n \in D(T) \text{ such that } u_n \to u \text{ in } E \text{ and } Tu_n \to v \text{ in } F, \text{ then } u \in D(T) \text{ and } Tu = v.$$

Definition 1.1.8. Let $T : D(T) \subset E \longmapsto F$ be a closed unbounded linear operator.

- The resolvent set of T is defined by

$$\rho(T) = \{\lambda \in \mathbb{C} : \; \lambda I - T \text{ is bijective from } D(T) \text{ onto } F\}.$$

- The resolvent of T is defined by

$$R(\lambda, T) = (\lambda I - T)^{-1}, \quad \forall \lambda \in \rho(T).$$

- The spectrum set of T is the complement of the resolvent set in $\mathbb{C}$, denoted by

$$\sigma(T) = \mathbb{C} \backslash \rho(T).$$

Definition 1.1.9. Let $T : D(T) \subset E \longmapsto F$ be a closed unbounded linear operator. We can split the spectrum $\sigma(T)$ of T into three disjoint sets, given by

- The point spectrum of T is defined by

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \ \ker(\lambda I - T) \neq \{0\}\},$$

in this case λ is called an eigenvalue of T.

- The continuous spectrum of T is defined by

$$\sigma_c(T) = \Big\{\lambda \in \mathbb{C} \ \ker(\lambda I - T) = 0, \ \overline{\mathcal{R}(\lambda I - T)} = F \text{ and } (\lambda I - T)^{-1} \text{ is not bounded}\Big\}.$$

- The residual spectrum of T is defined by

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : \ \ker(\lambda I - T) = 0 \text{ and } \mathcal{R}(\lambda I - T) \text{ is not dense in } F\}.$$

Definition 1.1.10. Let $T : D(T) \subset E \longmapsto F$ be a closed unbounded linear operator and let λ be an eigevalue of A. A non-zero element $e \in E$ is called a generalized eigenvector of T associated with the eigenvalue value λ, if there exists $n \in \mathbb{N}^*$ such that

$$(\lambda I - T)^n e = 0 \quad \text{and} \quad (\lambda I - T)^{n-1} e \neq 0.$$

If $n = 1$, then e is called an eigenvector.

Definition 1.1.11. Let $T : D(T) \subset E \longmapsto F$ be a closed unbounded linear operator. We say that T has a compact resolvent, if there exist $\lambda_0 \in \rho(T)$ such that $(\lambda_0 I - T)^{-1}$ is compact.

Theorem 1.1.12. (see Theorem 6.5.5 in [68]). Let $(T, D(T))$ be a closed unbounded linear operator on H, then the space $\Big(D(T), \|\cdot\|_{D(T)}\Big)$ where $\|u\|_{D(T)} = \|Tu\|_H + \|u\|_H$, $\forall\, u \in D(T)$ is a Banach space .

Theorem 1.1.13. (see Theorem 6.7 in [71]). Let $(T, D(T))$ be a closed unbounded linear operator on H, then $\rho(T)$ is an open set of $\mathbb{C}$.

1.2 Semigroups for Cauchy problems

In this section, we introduce some basic concepts concerning semigroups. The majority of evolution equations can be reduced to the form

$$\begin{cases} U_t = AU, \quad t > 0, \text{ in } H, \\ U(0) = U_0, \end{cases} \tag{1.2.1}$$

where A is the infinitesimal generator of a C_0-semigroup $S(t)$ over a Hilbert space H. Let us start by basic definitions and theorems, see [31, 92].
Let $(X, \|\cdot\|_X)$ be a Banach space, and H be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_H$ and the induced norm $\|\cdot\|_H$.

Definition 1.2.1. A family $(S(t))_{t \geq 0}$ of bounded linear operators in X is called a strongly continous semigroup (in short, a C_0-semigroup) if

- $S(0) = I$ (I is the identity operator on X).
- $S(t+s) = S(t)S(s), \quad \forall\, t, s \geq 0.$
- For each $u \in H$, $S(t)u$ is continuous in t on $[0, +\infty[$.

Sometimes we also denote $S(t)$ by e^{tA}.

Definition 1.2.2. For a semigroup $(S(t))_{t\geq 0}$, we define an linear operator A with domain $D(A)$ consisting of points u such that the limit

$$Au := \lim_{t\to 0^+} \frac{S(t)u - u}{t}$$

exists. Then A is called the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$.

Proposition 1.2.3. (See Theorem 2.2 in [92]). Let $(S(t))_{t\geq 0}$ be a C_0-semigroup in X. Then there exist a constant $M \geq 1$ and $\omega \geq 0$ such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad \forall t \geq 0.$$

If $\omega = 0$ then the corresponding semigroup is uniformly bounded; moreover, if $M = 1$ then $(S(t))_{t\geq 0}$ is said to be a C_0-semigroup of contractions.

Definition 1.2.4. An unbounded linear operator $(A, D(A))$ on H, is said to be dissipative if

$$\Re\langle Au, u\rangle_H \leq 0, \quad \forall\, u \in D(A).$$

Definition 1.2.5. An unbounded linear operator $(A, D(A))$ on X, is said to be m-dissipative if

- A is a dissipative operator.
- $\exists\, \lambda_0 > 0$ such that $\mathcal{R}(\lambda_0 I - A) = X$.

Theorem 1.2.6. (See Theorem 4.5 in [92]). Let A be a m-dissipative operator, then

- $\mathcal{R}(\lambda I - A) = X, \quad \forall\, \lambda > 0.$
- $]0, \infty[\subseteq \rho(A)$.

Theorem 1.2.7. (**Hille-Yosida**, see Theorem 3.1 in [92]). An unbounded linear operator $(A, D(A))$ on X, is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$ if and only if

- A is closed and $\overline{D(A)} = X$.
- The resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains $\mathbb{R}^+$, and for all $\lambda > 0$,

$$\left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \lambda^{-1}.$$

Theorem 1.2.8. (**Lumer-Phillips**, see Theorem 4.3 in [92]). Let $(A, D(A))$ be an unbounded linear operator on X, with dense domain $D(A)$ in X. A is the infinitesimal generator of a C_0-semigroup of contractions if and only if it is a m-dissipative operator.

Theorem 1.2.9. (see Theorem 4.6 in [92]). Let $(A, D(A))$ be an unbounded linear operator on X. If A is dissipative with $\mathcal{R}(I - A) = X$ and X is reflexive, then $\overline{D(A)} = X$.

Corollary 1.2.10. Let $(A, D(A))$ be an unbounded linear operator on H. A is the infinitesimal generator of a C_0-semigroup of contractions if and only if A is a m-dissipative operator.

Theorem 1.2.11. Let A be a linear operator with dense domain $D(A)$ in a Hilbert space H. If A is dissipative and $0 \in \rho(A)$, then A is the infinitesimal generator of a C_0-semigroup of contractions on H.

Theorem 1.2.12. (see Theorem 7.4 in [31]). Let $(A, D(A))$ be an unbounded linear operator on H. Assume that A is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$.

1. For $U_0 \in D(A)$, the problem (1.2.1) admits a unique strong solution

$$U(t) = S(t)U_0 \in C^0(\mathbb{R}_+, D(A)) \cap C^1(\mathbb{R}_+, H).$$

2. For $U_0 \in H$, the problem (1.2.1) admits a unique weak solution

$$U(t) \in C^0(\mathbb{R}_+, H).$$

1.3 Stability of semigroups

In this section, we introduce some definitions about strong, exponential, polynomial and analytic stability of a C_0-semigroup. Then, we give some results about the stability of C_0-semigroup. For more details, see [67, 94, 24, 30, 77, 27].
Let $(X, \|\cdot\|_X)$ be a Banach space, and H be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_H$ and the induced norm $\|\cdot\|_H$.

Definition 1.3.1. Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on X. We say that the C_0-semigroup $(S(t))_{t\geq 0}$ is

- Strongly stable if
$$\lim_{t\to+\infty} \|S(t)u\|_X = 0, \quad \forall\ u \in X.$$
- Uniformly stable if
$$\lim_{t\to+\infty} \|S(t)\|_{\mathcal{L}(X)} = 0.$$
- Exponentially stable if there exist two positive constants M and ϵ such that
$$\|S(t)u\|_X \leq Me^{-\epsilon t}\|u\|_X, \quad \forall\ t > 0,\ \forall\ u \in X.$$
- Polynomially stable if there exist two positive constants C and α such that
$$\|S(t)u\|_X \leq Ct^{-\alpha}\|u\|_{D(A)}, \quad \forall\ t > 0,\ \forall\ u \in D(A).$$

Proposition 1.3.2. Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on X. The following statements are equivalent

- $(S(t))_{t\geq 0}$ is uniformly stable.

- $(S(t))_{t\geq 0}$ is exponentially stable.

To show the strong stability of a C_0-semigroup we rely on the following result due to Arendt-Batty [24].

Theorem 1.3.3. (**Arendt and Batty**). Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on a reflexive Banach space X. If

- A has no pure imaginary eigenvalues.
- $\sigma(A) \cap i\mathbb{R}$ is countable.

Then $S(t)$ is strongly stable.

Remark 1.3.4. If the resolvent $(I-T)^{-1}$ of T is compact, then $\sigma(T) = \sigma_p(T)$. Thus, the statement of Theorem 1.3.3 lessens to $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.

An alternative method based on Arendt and Batty theorem and a contradiction argument, see [82, page 25] is presented in the following Remark.

Remark 1.3.5. Assume that the unbounded linear operator $A : D(A) \subset H \longmapsto H$ is the generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$ on a Hilbert space H and suppose that $0 \in \rho(A)$. According to [82, page 25], in order to prove that

$$i\mathbb{R} \equiv \{i\lambda \mid \ \lambda \in \mathbb{R}\} \subseteq \rho(A), \tag{1.3.1}$$

we need the following steps:

(i) It follows from the fact that $0 \in \rho(A)$ and the contraction mapping theorem that for any real number λ with $|\lambda| < \|A^{-1}\|^{-1}$, the operator $i\lambda I - A = A(i\lambda A^{-1} - I)$ is invertible. Furthermore, $\|(i\lambda I - A)^{-1}\|$ is a continuous function of λ in the interval $(-\|A^{-1}\|^{-1}, \|A^{-1}\|^{-1})$.

(ii) If $\sup\{\|(i\lambda I - A)^{-1}\| \mid |\lambda| < \|A^{-1}\|^{-1}\} = M < \infty$, then by the contraction mapping theorem, the operator $i\lambda I - A = (i\lambda_0 I - A)(I + i(\lambda-\lambda_0)(i\lambda_0 I - A)^{-1})$ with $|\lambda_0| < \|A^{-1}\|^{-1}$ is invertible for $|\lambda - \lambda_0| < M^{-1}$. It turns out that by choosing $|\lambda_0|$ as close to $\|A^{-1}\|^{-1}$ as we can, we conclude that $\{\lambda \mid |\lambda| < \|A^{-1}\|^{-1} + M^{-1}\} \subset \rho(A)$ and $\|(i\lambda I - A)^{-1}\|$ is a continuous function of λ in the interval $(-\|A^{-1}\|^{-1} - M^{-1}, \|A^{-1}\|^{-1} + M^{-1})$.

(iii) Thus it follows from the argument in (ii) that if (1.3.1) is false, then there is $\omega \in \mathbb{R}$ with $\|A^{-1}\|^{-1} \leq |\omega| < \infty$ such that $\{i\lambda \mid |\lambda| < |\omega|\} \subset \rho(A)$ and $\sup\{\|(i\lambda - A)^{-1}\| \mid |\lambda| < |\omega|\} = \infty$. It turns out that there exists a sequence $\{(\lambda_n, U_n)\}_{n\geq 1} \subset \mathbb{R} \times D(A)$, with $\lambda_n \to \omega$ as $n \to \infty$, $|\lambda_n| < |\omega|$ and $\|U_n\|_H = 1$, such that

$$(i\lambda_n I - A)U_n = F_n \to 0 \text{ in } H, \qquad \text{as } n \to \infty.$$

Then, we will prove (1.3.1) by showing that $\|U_n\|_H \to 0$ (up to a subsequence) which contradicts $\|U_n\|_H = 1$. □

Next, when the C_0-semigroup is strongly stable, we look for the necessary and sufficient conditions of exponential stability of a C_0-semigroup. In case when the C_0-semigroup is not exponentially stable, we may look for a polynomial one. In fact, exponential and polynomial stability results are obtained using different methods like: multipliers method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them. In this thesis we will review only two methods. The following result is a frequency domain approach method which was obtained by Huang [67] and Prüss [94].

Theorem 1.3.6. Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on H. $S(t)$ is uniformly stable if and only if

- $i\mathbb{R} \subset \rho(A)$.
- $\limsup_{\lambda\in\mathbb{R},|\lambda|\to\infty} \| (i\lambda I - A)^{-1} \|_{\mathcal{L}(H)} < \infty.$

Moreover, the following result is a frequency domain approach method which was obtained by Borichev and Tomilov [30] (see also [27] and [77]) .

Theorem 1.3.7. Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on H. If $i\mathbb{R} \subset \rho(A)$, then for a fixed $\ell > 0$ the following conditions are equivalent

$$\limsup_{\lambda\in\mathbb{R},|\lambda|\to\infty} \frac{1}{|\lambda|^{\ell}} \| (i\lambda I - A)^{-1} \|_{\mathcal{L}(H)} < \infty, \tag{1.3.2}$$

$$\|S(t)\,U_0\|_H \leq \frac{C}{t^{\frac{1}{\ell}}}\, \|U_0\|_{D(A)} \quad \forall\, t>0,\ U_0 \in D(A), \text{ for some } C>0. \tag{1.3.3}$$

Also, the analytic property of a C_0-semigroup of contraction $(S(t))_{t\geq 0}$ is characterized in the following theorem due to [23].

Theorem 1.3.8. Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on H. Assume that

$$i\mathbb{R} \subset \rho(A).$$

Then, $(S(t))_{t\geq 0}$ is analytic if and only if

$$\limsup_{\lambda\in\mathbb{R},|\lambda|\to\infty} |\lambda| \|(i\lambda I - A)^{-1}\|_{\mathcal{L}(H)} < \infty.$$

1.4 The multiplier geometric control condition

In this section, we recall the definition of the multiplier geometric control condition denoted by **MGC**.

Definition 1.4.1. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded open set with the boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. We say that the partition (Γ_0, Γ_1) of the boundary Γ satisfies the multiplier geometric control condition **MGC** (see Fig. 1.1 for an illustration) if there exists a point $x_0 \in \mathbb{R}^n$ and a positive constant δ such that

$$h \cdot \nu \geq \delta^{-1} \ \text{ on } \ \Gamma_1 \quad \text{and} \quad h \cdot \nu \leq 0 \ \text{ on } \ \Gamma_0, \tag{1.4.1}$$

where $h(x) = x - x_0$. □

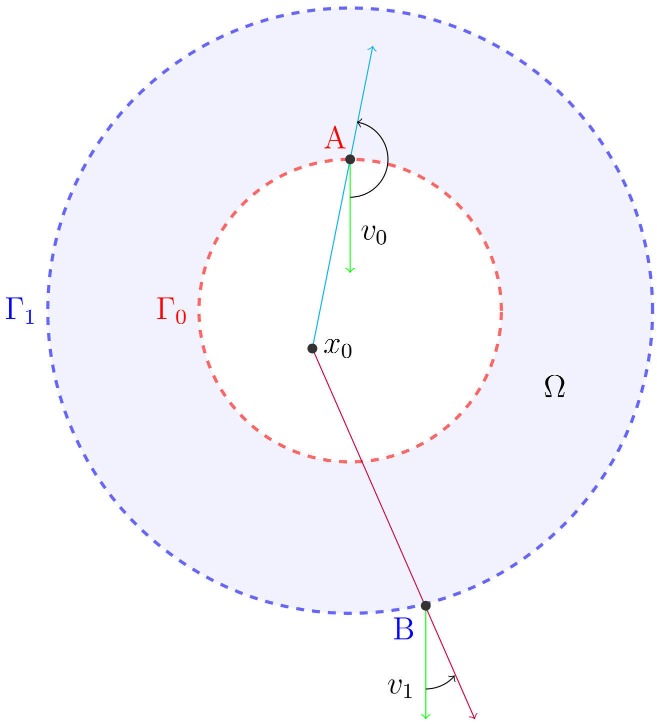

Figure 1.1: An example where the **MGC** boundary condition holds.

Chapter 2

Stability results of a singular local interaction elastic/viscoelastic coupled wave equations with time delay

The purpose of this chapter is to investigate the stabilization of locally coupled wave equations with non-smooth localized viscoelastic damping of Kelvin-Voigt type and localized time delay. Using a general criteria of Arendt-Batty, we show the strong stability of our system in the absence of the compactness of the resolvent. Finally, using frequency domain approach combined with the multiplier method, we prove a polynomial energy decay rate of order t^{-1}. This chapter is published in [7].

2.1 Introduction

2.1.1 Description of the chapter

In this chapter, we investigate the stability of local coupled wave equations with singular localized viscoelastic damping of Kelvin-Voigt type and localized time delay. More precisely, we consider the following system:

$$\begin{cases} u_{tt} - \left[au_x + b(x)(\kappa_1 u_{tx} + \kappa_2 u_{tx}(x, t-\tau))\right]_x + c(x)y_t & \\ = 0, & (x,t) \in (0,L) \times (0,\infty), \\ y_{tt} - y_{xx} - c(x)u_t = 0, & (x,t) \in (0,L) \times (0,\infty), \\ u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, & t > 0, \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in (0,L), \\ (y(x,0), y_t(x,0)) = (y_0(x), y_1(x)), & x \in (0,L), \\ u_t(x,t) = f_0(x,t), & (x,t) \in (0,L) \times (-\tau, 0), \end{cases} \tag{2.1.1}$$

where L, τ, a and κ_1 are positive real numbers, κ_2 is a non-zero real number and $(u_0, u_1, y_0, y_1, f_0)$ belongs to a suitable space. We suppose that there exist $0 < \alpha < \beta < \gamma < L$ and a non-zero constant c_0, such that

$$b(x) = \begin{cases} 1, & x \in (0,\beta), \\ 0, & x \in (\beta, L), \end{cases} \quad \text{and} \quad c(x) = \begin{cases} c_0, & x \in (\alpha,\gamma), \\ 0, & x \in (0,\alpha) \cup (\gamma, L). \end{cases}$$

Figure 2.1 describes system (2.1.1).

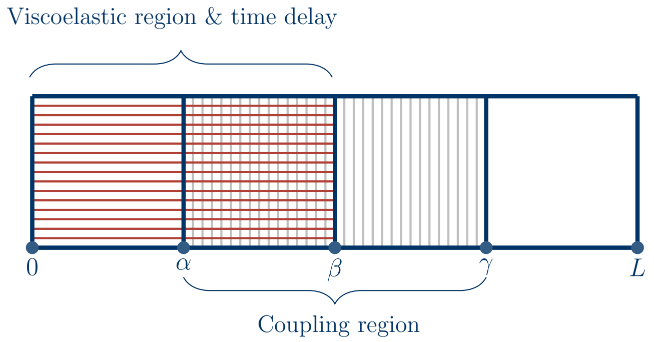

Figure 2.1: Local Kelvin-Voigt damping and local time delay feedback.

System (2.1.1) consists in two wave equations with only one singular viscoelastic damping acting on the first equation, the second one is indirectly damped via a singular coupling between the two equations. The notion of indirect damping mechanisms has been introduced by Russell in [100] and since then, it has attracted the attention of many authors (see for instance [9], [10], [11], [19], [34], [2], [78] and [109]). The study of such systems is also motivated by several physical considerations like Timoshenko and Bresse systems (see for instance [1], [8], [84] and [86]). In fact, there are few results concerning the stability of coupled wave equations with local Kelvin-Voigt damping without time delay, especially in the absence of smoothness of the damping and coupling coefficients (see Subsection 2.1.2). This motivates our interest to study the stabilization of system (2.1.1) in this chapter.

2.1.2 Previous Literature

The wave is created when a vibrating source disturbs the medium. In order to restrain those vibrations, several damping can be added such as Kelvin-Voigt damping which is originated from the extension or compression of the vibrating particles. This damping is a viscoelastic structure having properties of both elasticity and viscosity. In the recent years, many researchers showed interest in problems involving this kind of damping where different types of stability, depend on the smoothness of the damping coefficients, has been showed (see [17], [18], [62], [63], [66], [76], [80], [91] and [98]). However, time delays appear in several applications such as in physics, chemistry, biology, thermal phenomena not only depend on the present state but also on some past occurrences (see [44] and [72]) . In the last years, the control of partial differential equations with time delays have become popular among scientists, since in many cases time delays induce some instabilities see [36, 38, 39, 42].

However, let us recall briefly some systems of wave equations with Kelvin-Voigt damping and time delay represented in previous literature.

Coupled wave equations with Kelvin-Voigt damping and without time delay

In 2020, Hayek *et al.* in [65] studied the stabilization of a multi-dimensional system of weakly coupled wave equations with one or two locally Kelvin-Voigt damping and non-smooth coefficient at the interface. They established different stability results. In 2021, Hassine and Souayeh in [64] studied the behavior of a system with coupled wave equations with a partial Kelvin-Voigt damping, by considering the following system

$$\begin{cases} u_{tt} - (u_x + b_2(x)u_{tx})_x + v_t = 0, & (x,t) \in (-1,1) \times (0,\infty), \\ v_{tt} - cv_{xx} - u_t = 0, & (x,t) \in (-1,1) \times (0,\infty), \\ u(0,t) = v(0,t) = 0, u(1,t) = v(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in (-1,1), \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), & x \in (-1,1), \end{cases} \tag{2.1.2}$$

where $c > 0$ and $b_2 \in L^\infty(-1,1)$ is a non-negative function. They assumed that the damping coefficient is piecewise function in particular they supposed that $b_2(x) = d\mathbb{1}_{[0,1]}(x)$, where d is a strictly positive constant. So, they took the damping coefficient to be near the boundary with a global coupling coefficient. They showed the lack of exponential stability and that the semigroup loses speed and it decays polynomially with a rate as $t^{-\frac{1}{12}}$. In 2021, Akil, Issa and Wehbe in [103] studied the localized coupled wave equations, by considering the following system:

$$\begin{cases} u_{tt} - (au_x + b(x)u_{tx})_x + c(x)y_t = 0, & (x,t) \in (0,L) \times (0,\infty), \\ y_{tt} - y_{xx} - c(x)u_t = 0, & (x,t) \in (0,L) \times (0,\infty), \\ u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, & t > 0, \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in (0,L), \\ (y(x,0), y_t(x,0)) = (y_0(x), y_1(x)), & x \in (0,L), \end{cases}$$

where

$$b(x) = \begin{cases} b_0, & x \in (\alpha_1, \alpha_3), \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad c(x) = \begin{cases} c_0, & x \in (\alpha_2, \alpha_4), \\ 0, & \text{otherwise} \end{cases}$$

where $a > 0$, $b_0 > 0$, $c_0 > 0$ and $0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < L$. They generalized the results of Hassine and Souayeh in [64] by establishing a polynomial decay rate of type t^{-1}.

Wave equations with time delay and without Kelvin-Voigt damping

The delay equations of hyperbolic type is given by

$$u_{tt} - \Delta u(x, t-\tau) = 0. \tag{2.1.3}$$

with a delay parameter $\tau > 0$. This system is not well posed since there exists a sequence of solutions tending to infinity for any fixed $t > 0$ while the norm of the initial data remains bounded (see Theorem 1.1 in [42]). In 2006, Nicaise and Pignotti in [88] studied the multidimensional wave equation considering two cases. The first case concerns a wave equation with boundary feedback and a delay term at the boundary

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \Gamma_D \times (0,\infty), \\ \frac{\partial u}{\partial \nu}(x,t) = -\mu_1 u_t(x,t) - \mu_2 u_t(x,t-\tau), & (x,t) \in \Gamma_N \times (0,\infty), \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega, \\ u_t(x,t) = f_0(x,t), & (x,t) \in \Gamma_N \times (-\tau, 0). \end{cases} \tag{2.1.4}$$

The second case concerns a wave equation with an internal feedback and a delayed velocity term (i.e. an internal delay) and a mixed Dirichlet-Neumann boundary condition

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \Gamma_D \times (0,\infty), \\ \frac{\partial u}{\partial \nu}(x,t) = 0, & (x,t) \in \Gamma_N \times (0,\infty), \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega, \\ u_t(x,t) = f_0(x,t), & (x,t) \in \Omega \times (-\tau, 0), \end{cases} \tag{2.1.5}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ with a boundary Γ of class C^2 and $\Gamma = \Gamma_D \cup \Gamma_N$, such that $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$. Under the assumption $\mu_2 < \mu_1$, an exponential decay is achieved for the both systems (2.1.4)-(2.1.5). If this assumption does not hold, they found a sequences of delays $\{\tau_k\}_k$, $\tau_k \to 0$, for which the corresponding solutions have increasing energy. Furthermore, we refer to [29] for system (2.1.5) in more general abstract setting. In 2010, Ammari *et al.* in [21] studied the wave equation with interior delay damping and dissipative undelayed boundary condition in an open domain Ω of $\mathbb{R}^N$, $N \geq 2$. The system is described by:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \Gamma_0 \times (0,\infty), \\ \frac{\partial u}{\partial \nu}(x,t) = -\kappa u_t(x,t), & (x,t) \in \Gamma_1 \times (0,\infty), \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega, \\ u_t(x,t) = f_0(x,t), & (x,t) \in \Omega \times (-\tau, 0), \end{cases} \tag{2.1.6}$$

where $\tau > 0$, $a > 0$ and $\kappa > 0$. Under the condition that Γ_1 satisfies the Γ-condition introduced in [75], they proved that system (2.1.6) is uniformly asymptotically stable whenever the delay coefficient is sufficiently small. In 2012, Pignotti in [93] considered the wave equation with internal distributed time delay and local damping in a bounded and smooth domain $\Omega \subset \mathbb{R}^N, N \geq 1$. The considered system is given by the following:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a\chi_\omega u_t(x,t) + \kappa u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \Gamma \times (0,\infty), \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega, \\ u_t(x,t) = f(x,t), & (x,t) \in \Omega \times (-\tau, 0), \end{cases} \tag{2.1.7}$$

where $\kappa \in \mathbb{R}$, $\tau > 0$, $a > 0$ and ω is the intersection between an open neighborhood of the set $\Gamma_0 = \{x \in \Gamma; \quad (x - x_0) \cdot \nu(x) > 0\}$ and Ω. Moreover, χ_ω is the characteristic function of ω. We remark that the damping is localized and it acts on a neighborhood of a part of Ω. She showed an exponential stability results if the coefficients of the delay terms satisfy the following assumption $|\kappa| < \kappa_0 < a$.
Several researches were done on wave equation with time delay acting on the boundary see ([39],[37], [108], [59], [58], [102], [107]) and different type of stability has been proved.

Wave equations with Kelvin-Voigt damping and time delay

In 2016, Messaoudi *et al.* in [85] considered the stabilization of the following wave equation with strong time delay

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) - \mu_1 \Delta u_t(x,t) - \mu_2 \Delta u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \Gamma \times (0,\infty), \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega, \\ u_t(x,t) = f_0(x,t), & (x,t) \in \Omega \times (-\tau, 0), \end{cases}$$

where $\mu_1 > 0$ and μ_2 is a non zero real number. Under the assumption that $|\mu_2| < \mu_1$, they obtained an exponential stability result. In 2016, Nicaise *et al.* in [89] studied the multidimensional wave equation with localized Kelvin-Voigt damping and mixed boundary condition with time delay

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) - \operatorname{div}(a(x)\nabla u_t) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \Gamma_0 \times (0,\infty), \\ \frac{\partial u}{\partial \nu}(x,t) = -a(x)\frac{\partial u_t}{\partial \nu}(x,t) - \kappa u_t(x,t-\tau), & (x,t) \in \Gamma_1 \times (0,\infty), \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega, \\ u_t(x,t) = f_0(x,t), & (x,t) \in \Gamma_1 \times (-\tau, 0), \end{cases} \tag{2.1.8}$$

where $\tau > 0$, $\kappa \in \mathbb{R}$, $a(x) \in L^\infty(\Omega)$ and $a(x) \geq a_0 > 0$ on ω such that $\omega \subset \Omega$ is an open neighborhood of Γ_1. Under an appropriate geometric condition on Γ_1 and assuming that $a \in C^{1,1}(\overline{\Omega})$, $\Delta a \in L^\infty(\Omega)$, they proved an exponential decay of the energy of system (2.1.8). In 2019, Anikushyn *et al.* in [41] considered an initial boundary value problem for a viscoelastic wave equation subjected to a strong time localized delay in a Kelvin-Voigt type. The system is given by the following:

$$\begin{cases} u_{tt} - c_1 \Delta u - c_2 \Delta u(x,t-\tau) - d_1 \Delta u_t - d_2 \Delta u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \Gamma_0 \times (0,\infty), \\ \frac{\partial u}{\partial \nu}(x,t) = 0, & (x,t) \in \Gamma_1 \times (0,\infty), \\ (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega, \\ u(x,t) = f_0(x,t), & (x,t) \in \Omega \times (-\tau, 0). \end{cases}$$

Under appropriate conditions on the coefficients, a global exponential decay rate is obtained. In 2015, Ammari *et al.* in [22] considered the stabilization problem for an abstract equation with delay and a Kelvin-Voigt damping. The system is given by the following:

$$\begin{cases} u_{tt}(t) + a\mathcal{B}\mathcal{B}^* u_t(t) + \mathcal{B}\mathcal{B}^* u(t-\tau) = 0, & t \in (0,\infty), \\ (u(0), u_t(0)) = (u_0, u_1), & \\ \mathcal{B}^* u(t) = f_0(t), & t \in (-\tau, 0), \end{cases}$$

for an appropriate class of operator $\mathcal{B}$ and $a > 0$. Using the frequency domain approach, they obtained an exponential stability result.

Thus, to the best of our knowledge, it seems to us that there is no result in the existing literature concerning the case of coupled wave equations with localized Kelvin-Voigt damping and localized time delay, especially in the absence of smoothness of the damping and coupling coefficients. The goal of the present chapter is to fill this gap by studying the stability of system (2.1.1).

This chapter is organized as follows: In Section 2.2, we prove the well-posedness of our system by using semigroup approach. In Section 2.3, by using a general criteria of Arendt-Batty, we show the strong stability of our system in the absence of the compactness of the resolvent. Next, in Section 2.4, by using frequency domain approach combining with a specific multiplier method, we prove a polynomial energy decay rate of order t^{-1}.

2.2 Well-posedness of the system

In this section, we will establish the well-posedness of system (2.1.1) by using semigroup approach. To this aim, as in [88], we introduce the following auxiliary change of variable

$$\eta(x,\rho,t) := u_t(x,t-\rho\tau), \quad x \in (0,\beta),\ \rho \in (0,1),\ t>0. \tag{2.2.1}$$

Then, system (2.1.1) becomes

$$u_{tt} - (S_b(u,u_t,\eta))_x + c(x)y_t = 0, \qquad (x,t) \in (0,L)\times(0,\infty), \tag{2.2.2}$$

$$y_{tt} - y_{xx} - c(x)u_t = 0, \qquad (x,t) \in (0,L)\times(0,\infty), \tag{2.2.3}$$

$$\tau\eta_t(x,\rho,t) + \eta_\rho(x,\rho,t) = 0, \quad (x,\rho,t) \in (0,\beta)\times(0,1)\times(0,\infty), \tag{2.2.4}$$

where $S_b(u,u_t,\eta) := au_x + b(x)(\kappa_1 u_{tx} + \kappa_2 u_{tx}(x,t-\tau))$. Moreover, from the definition of $b(\cdot)$, we have

$$S_b(u,u_t,\eta) = \begin{cases} S_1(u,u_t,\eta) := au_x + \kappa_1 u_{tx} + \kappa_2 \eta_x(\cdot,1,t), & \text{in } (0,\beta), \\ au_x, & \text{in } (\beta,L). \end{cases} \tag{2.2.5}$$

With the following boundary conditions

$$\begin{cases} u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, \quad t>0, \\ \eta(0,\rho,t) = 0, \quad (\rho,t) \in (0,1)\times(0,\infty), \end{cases} \tag{2.2.6}$$

and the following initial conditions

$$\begin{cases} u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in (0,L), \\ y(x,0) = y_0(x), \qquad y_t(x,0) = y_1(x), & x \in (0,L), \\ \eta(x,\rho,0) = f_0(x,-\rho\tau), & (x,\rho) \in (0,\beta)\times(0,1). \end{cases} \tag{2.2.7}$$

The energy of system (2.2.2)-(2.2.7) is given by

$$E(t) = E_1(t) + E_2(t) + E_3(t), \tag{2.2.8}$$

where

$$\begin{cases} E_1(t) = \dfrac{1}{2}\displaystyle\int_0^L \left(|u_t|^2 + a|u_x|^2\right)dx, \quad E_2(t) = \dfrac{1}{2}\displaystyle\int_0^L \left(|y_t|^2 + |y_x|^2\right)dx \ \text{ and} \\ E_3(t) = \dfrac{\tau|\kappa_2|}{2}\displaystyle\int_0^\beta \int_0^1 |\eta_x(\cdot,\rho,t)|^2 d\rho dx. \end{cases}$$

Lemma 2.2.1. Let $U = (u, u_t, y, y_t, \eta)$ be a regular solution of system (2.2.2)-(2.2.7). Then, the energy $E(t)$ satisfies the following estimation

$$\frac{d}{dt}E(t) \leq -(\kappa_1 - |\kappa_2|)\int_0^\beta |u_{tx}|^2 dx. \tag{2.2.9}$$

Proof. First, multiplying (2.2.2) by $\overline{u_t}$, integrating over $(0, L)$, using integration by parts with (2.2.6), then taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L |u_t|^2 dx + \Re\left\{\int_0^L S_b(u, u_t, \eta)\overline{u_{tx}} dx\right\} + \Re\left\{\int_0^L c(\cdot) y_t \overline{u_t} dx\right\} = 0.$$

From the above equation and the definition of $S_b(u, u_t, \eta)$ and $c(\cdot)$, we deduce that

$$\frac{d}{dt}E_1(t) = -\kappa_1 \int_0^\beta |u_{tx}|^2 dx - \Re\left\{\kappa_2 \int_0^\beta \eta_x(\cdot, 1, t)\overline{u_{tx}} dx\right\} - \Re\left\{c_0 \int_\alpha^\gamma y_t \overline{u_t} dx\right\}. \tag{2.2.10}$$

Using Young's inequality in (2.2.10), we get

$$\frac{d}{dt}E_1(t) \leq -\left(\kappa_1 - \frac{|\kappa_2|}{2}\right)\int_0^\beta |u_{tx}|^2 dx + \frac{|\kappa_2|}{2}\int_0^\beta |\eta_x(\cdot, 1, t)|^2 dx - \Re\left\{c_0 \int_\alpha^\gamma y_t \overline{u_t} dx\right\}. \tag{2.2.11}$$

Now, multiplying (2.2.3) by $\overline{y_t}$, integrating over $(0, L)$, using the definition of $c(\cdot)$, then taking the real part, we get

$$\frac{d}{dt}E_2(t) = \Re\left\{c_0 \int_\alpha^\gamma u_t \overline{y_t} dx\right\}. \tag{2.2.12}$$

Deriving (2.2.4) with respect to x, we obtain

$$\tau \eta_{xt}(\cdot, \rho, t) + \eta_{x\rho}(\cdot, \rho, t) = 0. \tag{2.2.13}$$

Multiplying (2.2.13) by $|\kappa_2|\overline{\eta_x}(\cdot, \rho, t)$, integrating over $(0, \beta) \times (0, 1)$, using the fact that $\eta_x(\cdot, 0, t) = u_{tx}$, then taking the real part, we get

$$\begin{aligned} \frac{d}{dt}E_3(t) &= -\frac{|\kappa_2|}{2}\int_0^\beta \left(|\eta_x(\cdot, 1, t)|^2 - |\eta_x(\cdot, 0, t)|^2\right) dx \\ &= -\frac{|\kappa_2|}{2}\int_0^\beta \left(|\eta_x(\cdot, 1, t)|^2 - |u_{tx}|^2\right) dx. \end{aligned} \tag{2.2.14}$$

Finally, adding (2.2.11), (2.2.12) and (2.2.14), we obtain (2.2.9). The proof is thus complete. □

In the sequel, we make the following assumptions

$$\kappa_1 > 0, \quad \kappa_2 \in \mathbb{R}^* \quad \text{and} \quad |\kappa_2| < \kappa_1. \tag{H}$$

Under the hypothesis (H) and from Lemma 2.2.1, the system (2.2.2)-(2.2.7) is dissipative in the sense that its energy is non-increasing with respect to time (i.e. $E'(t) \leq 0$). Let us define the Hilbert space $\mathcal{H}$ by

$$\mathcal{H} := \left(H_0^1(0, L) \times L^2(0, L)\right)^2 \times \mathcal{W},$$

where

$$\mathcal{W} := L^2((0, 1); H_L^1(0, \beta)) \;\; \text{and} \;\; H_L^1(0, \beta) := \left\{\widetilde{\eta} \in H^1(0, \beta) \mid \widetilde{\eta}(0) = 0\right\}.$$

The space $\mathcal{W}$ is a Hilbert space of $H_L^1(0,\beta)$-valued functions on $(0,1)$, equipped with the following inner product

$$(\eta^1,\eta^2)_{\mathcal{W}} := \int_0^\beta \int_0^1 \eta_x^1 \overline{\eta_x^2} d\rho dx, \quad \forall\, \eta^1,\eta^2 \in \mathcal{W}.$$

The Hilbert space $\mathcal{H}$ is equipped with the following inner product

$$\left(U,U^1\right)_{\mathcal{H}} = \int_0^L \left(au_x\overline{u_x^1} + v\overline{v^1} + y_x\overline{y_x^1} + z\overline{z^1}\right) dx + \tau|\kappa_2| \int_0^\beta \int_0^1 \eta_x(\cdot,\rho)\overline{\eta_x^1}(\cdot,\rho) d\rho dx, \quad (2.2.15)$$

where $U = (u,v,y,z,\eta(\cdot,\rho))^\top$, $U^1 = (u^1,v^1,y^1,z^1,\eta^1(\cdot,\rho))^\top \in \mathcal{H}$. Now, we define the linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longmapsto \mathcal{H}$ by:

$$D(\mathcal{A}) = \left\{ \begin{array}{c} U = (u,v,y,z,\eta(\cdot,\rho))^\top \in \mathcal{H} \mid y \in H^2(0,L) \cap H_0^1(0,L),\ v,z \in H_0^1(0,L) \\ (S_b(u,v,\eta))_x \in L^2(0,L), \quad \eta_\rho(\cdot,\rho) \in \mathcal{W}, \quad \eta(\cdot,0) = v(\cdot) \text{ in } (0,\beta) \end{array} \right\}$$

and

$$\mathcal{A}\begin{pmatrix} u \\ v \\ y \\ z \\ \eta(\cdot,\rho) \end{pmatrix} = \begin{pmatrix} v \\ (S_b(u,v,\eta))_x - c(\cdot)z \\ z \\ y_{xx} + c(\cdot)v \\ -\tau^{-1}\eta_\rho(\cdot,\rho) \end{pmatrix}, \quad (2.2.16)$$

for all $U = (u,v,y,z,\eta(\cdot,\rho))^\top \in D(\mathcal{A})$.

Now, if $U = (u,u_t,y,y_t,\eta(\cdot,\rho))^\top$, then system (2.2.2)-(2.2.7) can be written as the following first order evolution equation

$$U_t = \mathcal{A}U, \quad U(0) = U_0, \quad (2.2.17)$$

where $U_0 = (u_0,u_1,y_0,y_1,f_0(\cdot,-\rho\tau))^\top \in \mathcal{H}$.

Remark 2.2.1. The linear unbounded operator $\mathcal{A}$ is injective (i.e. $\ker(\mathcal{A}) = \{0\}$). Indeed, if $U \in D(\mathcal{A})$ such that $\mathcal{A}U = 0$, then $v, z, \eta_\rho(\cdot,\rho) = 0$ and since $\eta(\cdot,0) = v(\cdot)$, we get $\eta(\cdot,\rho) = 0$. Consequently, $(S_b(u,v,\eta))_x = au_{xx} = 0$ and $y_{xx} = 0$. Now, since $u(0) = u(L) = y(0) = y(L) = 0$, then $u = y = 0$. Thus, $U = (u,v,y,z,\eta(\cdot,\rho))^\top = 0$. □

Proposition 2.2.1. Under the hypothesis (H), the unbounded linear operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$.

Proof. For all $U = (u,v,y,z,\eta(\cdot,\rho))^\top \in D(\mathcal{A})$, from (2.2.15) and (2.2.16), we have

$$\begin{aligned} \Re(\mathcal{A}U,U)_{\mathcal{H}} = {} & \Re\left\{\int_0^L av_x\overline{u_x}dx\right\} + \Re\left\{\int_0^L (S_b(u,v,\eta))_x\,\overline{v}dx\right\} + \Re\left\{\int_0^L z_x\overline{y_x}dx\right\} \\ & + \Re\left\{\int_0^L y_{xx}\overline{z}dx\right\} - \Re\left\{|\kappa_2|\int_0^\beta\int_0^1 \eta_{x\rho}(\cdot,\rho)\overline{\eta_x}(\cdot,\rho)d\rho dx\right\}. \end{aligned}$$

Using integration by parts to the second and fourth terms in the above equation, then using the definition of $S_b(u,v,\eta)$ and the fact that $U \in D(\mathcal{A})$, we get

$$\Re(\mathcal{A}U,U)_{\mathcal{H}} = -\kappa_1\int_0^\beta |v_x|^2dx - \Re\left\{\kappa_2\int_0^\beta \eta_x(\cdot,1)\overline{v_x}dx\right\} - \frac{|\kappa_2|}{2}\int_0^\beta\int_0^1 \frac{d}{d\rho}|\eta_x(\cdot,\rho)|^2 d\rho dx,$$

the fact that $\eta(\cdot,0)=v(\cdot)$ in $(0,\beta)$, implies that

$$\Re\left(\mathcal{A}U,U\right)_{\mathcal{H}}=-\left(\kappa_1-\frac{|\kappa_2|}{2}\right)\int_0^\beta |v_x|^2dx-\frac{|\kappa_2|}{2}\int_0^\beta |\eta_x(\cdot,1)|^2dx-\Re\left\{\kappa_2\int_0^\beta \eta_x(\cdot,1)\overline{v_x}dx\right\}.$$

Using Young's inequality in the above equation and the hypothesis (H), we obtain

$$\Re\left(\mathcal{A}U,U\right)_{\mathcal{H}}\leq-\left(\kappa_1-|\kappa_2|\right)\int_0^\beta |v_x|^2dx\leq 0, \tag{2.2.18}$$

which implies that $\mathcal{A}$ is dissipative. Now, let us prove that $\mathcal{A}$ is maximal. To this aim, let $F=(f^1,f^2,f^3,f^4,f^5(\cdot,\rho))^\top\in\mathcal{H}$, we look for $U=(u,v,y,z,\eta(\cdot,\rho))^\top\in D(\mathcal{A})$ unique solution of

$$-\mathcal{A}U=F. \tag{2.2.19}$$

Equivalently, we have the following system

$$\begin{aligned}
-v &= f^1, &(2.2.20)\\
-(S_b(u,v,\eta))_x+c(\cdot)z &= f^2, &(2.2.21)\\
-z &= f^3, &(2.2.22)\\
-y_{xx}-c(\cdot)v &= f^4, &(2.2.23)\\
\tau^{-1}\eta_\rho(\cdot,\rho) &= f^5(\cdot,\rho), &(2.2.24)
\end{aligned}$$

with the following boundary conditions

$$u(0)=u(L)=y(0)=y(L)=0,\quad \eta(0,\rho)=0\quad\text{and}\quad \eta(\cdot,0)=v(\cdot)\text{ in }(0,\beta). \tag{2.2.25}$$

From (2.2.20), (2.2.24) and (2.2.25), we get

$$\eta(x,\rho)=\tau\int_0^\rho f^5(x,s)ds-f^1,\quad (x,\rho)\in(0,\beta)\times(0,1). \tag{2.2.26}$$

Since, $f^1\in H_0^1(0,L)$ and $f^5(\cdot,\rho)\in\mathcal{W}$. Then, from (2.2.24) and (2.2.26), we get $\eta_\rho(\cdot,\rho),\eta(\cdot,\rho)\in\mathcal{W}$. Now, see the definition of $S_b(u,v,\eta)$, substituting (2.2.20), (2.2.22) and (2.2.26) in (2.2.21) and (2.2.23), we get the following system

$$\begin{aligned}
\left[S_b\left(u,f^1,\tau\int_0^1 f^5(x,s)ds-f^1\right)\right]_x+c(\cdot)f^3 &= -f^2, &(2.2.27)\\
y_{xx}-c(\cdot)f^1 &= -f^4, &(2.2.28)\\
u(0)=u(L)=y(0)=y(L) &= 0, &(2.2.29)
\end{aligned}$$

where

$$S_b\left(u,-f^1,\tau\int_0^1 f^5(x,s)ds-f^1\right)=\begin{cases}au_x-\kappa_1 f_x^1+\tau\kappa_2\displaystyle\int_0^1 f_x^5(\cdot,s)ds-k_2f_x^1, & \text{in }(0,\beta),\\ au_x, & \text{in }(\beta,L).\end{cases}$$

Let $(\phi,\psi)\in H_0^1(0,L)\times H_0^1(0,L)$. Multiplying (2.2.27) and (2.2.28) by $\overline{\phi}$ and $\overline{\psi}$ respectively, integrating over $(0,L)$, then using formal integrations by parts, we obtain

$$\begin{aligned}
a\int_0^L u_x\overline{\phi_x}dx=&\int_0^L f^2\overline{\phi}dx+c_0\int_\alpha^\gamma f^3\overline{\phi}dx+(\kappa_1+\kappa_2)\int_0^\beta f_x^1\overline{\phi_x}dx\\
&-\tau\kappa_2\int_0^\beta\left(\int_0^1 f_x^5(\cdot,s)ds\right)\overline{\phi_x}dx
\end{aligned} \tag{2.2.30}$$

and
$$\int_0^L y_x\overline{\psi_x}dx = \int_0^L f^4\overline{\psi}dx - c_0\int_\alpha^\gamma f^1\overline{\psi}dx. \tag{2.2.31}$$
Adding (2.2.30) and (2.2.31), we obtain
$$\mathcal{B}((u,y),(\phi,\psi)) = \mathcal{L}(\phi,\psi), \quad \forall(\phi,\psi)\in H_0^1(0,L)\times H_0^1(0,L), \tag{2.2.32}$$
where
$$\mathcal{B}((u,y),(\phi,\psi)) = a\int_0^L u_x\overline{\phi_x}dx + \int_0^L y_x\overline{\psi_x}dx$$
and
$$\mathcal{L}(\phi,\psi) = \int_0^L \left(f^2\overline{\phi}+f^4\overline{\psi}\right)dx + c_0\int_\alpha^\gamma \left(f^3\overline{\phi}-f^1\overline{\psi}\right)dx - \tau\kappa_2\int_0^\beta \left(\int_0^1 f_x^5(\cdot,s)ds\right)\overline{\phi_x}dx$$
$$+ (\kappa_1+\kappa_2)\int_0^\beta f_x^1\overline{\phi_x}dx.$$

It is easy to see that, $\mathcal{B}$ is a sesquilinear, continuous and coercive form on $\left(H_0^1(0,L)\times H_0^1(0,L)\right)^2$ and $\mathcal{L}$ is an antilinear and continuous form on $H_0^1(0,L)\times H_0^1(0,L)$. Then, it follows by Lax-Milgram theorem that (2.2.32) admits a unique solution $(u,y)\in H_0^1(0,L)\times H_0^1(0,L)$. By using the classical elliptic regularity, we deduce that system (2.2.27)-(2.2.29) admits a unique solution $(u,y)\in H_0^1(0,L)\times(H^2(0,L)\cap H_0^1(0,L))$ such that $(S_b(u,v,\eta))_x\in L^2(0,L)$ and since $\ker(\mathcal{A})=\{0\}$ (see Remark 2.2.1), we get $U=\left(u,-f^1,y,-f^3,\tau\int_0^\rho f^5(\cdot,s)ds - f^1\right)^\top \in D(\mathcal{A})$ is a unique solution of (2.2.19). Then, $\mathcal{A}$ is an isomorphism and since $\rho(\mathcal{A})$ is open set of $\mathbb{C}$ (see Theorem 1.1.13), we easily get $\mathcal{R}(\lambda I-\mathcal{A})=\mathcal{H}$ for a sufficiently small $\lambda>0$. This, together with the dissipativeness of $\mathcal{A}$, imply that $D(\mathcal{A})$ is dense in $\mathcal{H}$ and that $\mathcal{A}$ is m-dissipative in $\mathcal{H}$ (see Theorems 1.2.6, 1.2.9). The proof is thus complete. $\square$

According to Lumer-Phillips theorem (see Theorem 1.2.8), Proposition 2.2.1 implies that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in $\mathcal{H}$ which gives the well-posedness of (2.2.17). Then, we have the following result:

Theorem 2.2.1. Under the hypothesis (H), for all $U_0\in\mathcal{H}$, system (2.2.17) admits a unique weak solution
$$U(x,\rho,t)=e^{t\mathcal{A}}U_0(x,\rho)\in C^0(\mathbb{R}_+,\mathcal{H}).$$
Moreover, if $U_0\in D(\mathcal{A})$, then system (2.2.17) admits a unique strong solution
$$U(x,\rho,t)=e^{t\mathcal{A}}U_0(x,\rho)\in C^0(\mathbb{R}_+,D(\mathcal{A}))\cap C^1(\mathbb{R}_+,\mathcal{H}).$$

2.3 Strong Stability

In this section, we will prove the strong stability of system (2.2.2)-(2.2.7). The main result of this section is the following theorem.

Theorem 2.3.1. Assume that (H) is true. Then, the C_0-semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq0}$ is strongly stable in $\mathcal{H}$; i.e., for all $U_0\in\mathcal{H}$, the solution of (2.2.17) satisfies
$$\lim_{t\to+\infty}\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}}=0.$$

According to Theorem 1.3.3, to prove Theorem 2.3.1, we need to prove that the operator $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable. The proof of Theorem 2.3.1 will be achieved from the following proposition.

Proposition 2.3.1. Under the hypothesis (H), we have

$$i\mathbb{R} \subset \rho(\mathcal{A}). \tag{2.3.1}$$

We will prove Proposition 2.3.1 by a contradiction argument. Remark that, it has been proved in Proposition 2.2.1 that $0 \in \rho(\mathcal{A})$. Now, suppose that (2.3.1) is false, then there exists $\omega \in \mathbb{R}^*$ such that $i\omega \notin \rho(\mathcal{A})$. According to Remark 1.3.5, let $\left\{(\lambda^n, U^n := (u^n, v^n, y^n, z^n, \eta^n(\cdot,\rho))^\top)\right\}_{n\geq 1} \subset \mathbb{R}^* \times D(\mathcal{A})$, with

$$\lambda^n \to \omega \text{ as } n \to \infty \quad \text{and} \quad |\lambda^n| < |\omega| \tag{2.3.2}$$

and

$$\|U^n\|_{\mathcal{H}} = \left\|(u^n, v^n, y^n, z^n, \eta^n(\cdot,\rho))^\top\right\|_{\mathcal{H}} = 1, \forall n \geq 1, \tag{2.3.3}$$

such that

$$(i\lambda^n I - \mathcal{A})U^n = F^n := (f^{1,n}, f^{2,n}, f^{3,n}, f^{4,n}, f^{5,n}(\cdot,\rho))^\top \to 0 \quad \text{in} \quad \mathcal{H}, \quad \text{as} \quad n \to \infty. \tag{2.3.4}$$

Equivalently, we have

$$\begin{aligned}
i\lambda^n u^n - v^n &= f^{1,n} \to 0 && \text{in} \quad H_0^1(0,L), && (2.3.5)\\
i\lambda^n v^n - (S_b(u^n, v^n, \eta^n))_x + c(\cdot)z^n &= f^{2,n} \to 0 && \text{in} \quad L^2(0,L), && (2.3.6)\\
i\lambda^n y^n - z^n &= f^{3,n} \to 0 && \text{in} \quad H_0^1(0,L), && (2.3.7)\\
i\lambda^n z^n - y^n_{xx} - c(\cdot)v^n &= f^{4,n} \to 0 && \text{in} \quad L^2(0,L), && (2.3.8)\\
i\lambda^n \eta^n(.,\rho) + \tau^{-1}\eta^n_\rho(\cdot,\rho) &= f^{5,n}(\cdot,\rho) \to 0 && \text{in} \quad \mathcal{W}. && (2.3.9)
\end{aligned}$$

Then, we will prove condition (2.3.1) by finding a contradiction with (2.3.3) such as $\|U^n\|_{\mathcal{H}} \to 0$. The proof of proposition 2.3.1 has been divided into several Lemmas.

Lemma 2.3.1. Under the hypothesis (H), the solution $U^n = (u^n, v^n, y^n, z^n, \eta^n(\cdot,\rho))^\top \in D(\mathcal{A})$ of system (2.3.5)-(2.3.9) satisfies the following limits

$$\lim_{n\to\infty} \int_0^\beta |v^n_x|^2 dx = 0, \tag{2.3.10}$$

$$\lim_{n\to\infty} \int_0^\beta |v^n|^2 dx = 0, \tag{2.3.11}$$

$$\lim_{n\to\infty} \int_0^\beta |u^n_x|^2 dx = 0, \tag{2.3.12}$$

$$\lim_{n\to\infty} \int_0^\beta \int_0^1 |\eta^n_x(\cdot,\rho)|^2 d\rho dx = 0, \tag{2.3.13}$$

$$\lim_{n\to\infty} \int_0^\beta |\eta^n_x(\cdot,1)|^2 dx = 0, \tag{2.3.14}$$

$$\lim_{n\to\infty} \int_0^\beta |S_1(u^n, v^n, \eta^n)|^2 dx = 0. \tag{2.3.15}$$

Proof. First, taking the inner product of (2.3.4) with U^n in $\mathcal{H}$ and using (2.2.18) with the help of hypothesis (H), we obtain

$$\begin{aligned}\int_0^\beta |v_x^n|^2dx &\leq -\frac{1}{\kappa_1-|\kappa_2|}\Re(\mathcal{A}U^n,U^n)_{\mathcal{H}} = \frac{1}{\kappa_1-|\kappa_2|}\Re(F^n,U^n)_{\mathcal{H}}\\ &\leq \frac{1}{\kappa_1-|\kappa_2|}\|F^n\|_{\mathcal{H}}\|U^n\|_{\mathcal{H}}.\end{aligned}\tag{2.3.16}$$

Passing to the limit in (2.3.16), then using the fact that $\|U^n\|_{\mathcal{H}}=1$ and $\|F^n\|_{\mathcal{H}}\to 0$, we obtain (2.3.10). Now, since $v^n\in H_0^1(0,L)$, then it follows from Poincaré inequality that there exists a constant $C_p>0$ such that

$$\|v^n\|_{L^2(0,\beta)}\leq C_p\|v_x^n\|_{L^2(0,\beta)}.\tag{2.3.17}$$

Thus, from (2.3.10) and (2.3.17), we obtain (2.3.11). Next, from (2.3.5) and the fact that $\int_0^\beta |f_x^{1,n}|^2dx\leq \int_0^L |f_x^{1,n}|^2dx\leq a^{-1}\|F^n\|^2_{\mathcal{H}}$, we deduce that

$$\begin{aligned}\int_0^\beta |u_x^n|^2dx &\leq \frac{2}{(\lambda^n)^2}\int_0^\beta |v_x^n|^2dx+\frac{2}{(\lambda^n)^2}\int_0^\beta |f_x^{1,n}|^2dx\\ &\leq \frac{2}{(\lambda^n)^2}\int_0^\beta |v_x^n|^2dx+\frac{2}{a(\lambda^n)^2}\|F^n\|^2_{\mathcal{H}}.\end{aligned}\tag{2.3.18}$$

Passing to the limit in (2.3.18), then using (2.3.2), (2.3.10) and the fact that $\|F^n\|_{\mathcal{H}}\to 0$, we obtain (2.3.12). Moreover, from (2.3.9) and the fact that $\eta^n(\cdot,0)=v^n(\cdot)$ in $(0,\beta)$, we deduce that

$$\eta^n(x,\rho)=v^ne^{-i\lambda^n\tau\rho}+\tau\int_0^\rho e^{i\lambda^n\tau(s-\rho)}f^{5,n}(x,s)ds,\qquad (x,\rho)\in(0,\beta)\times(0,1).\tag{2.3.19}$$

From (2.3.19), and the fact that $\rho\in(0,1)$ and $\int_0^\beta\int_0^1 |f_x^{5,n}(\cdot,s)|^2dsdx\leq \tau^{-1}|\kappa_2|^{-1}\|F^n\|^2_{\mathcal{H}}$, we obtain

$$\begin{aligned}\int_0^\beta\int_0^1|\eta_x^n(\cdot,\rho)|^2d\rho dx &\leq 2\int_0^\beta |v_x^n|^2dx+2\tau^2\int_0^\beta\int_0^1\int_0^\rho \rho|f_x^{5,n}(\cdot,s)|^2dsd\rho dx\\ &\leq 2\int_0^\beta |v_x^n|^2dx+2\tau^2\int_0^\beta\int_0^1\int_0^1 \rho|f_x^{5,n}(\cdot,s)|^2dsd\rho dx\\ &= 2\int_0^\beta |v_x^n|^2dx+2\tau^2\left(\int_0^1\rho d\rho\right)\int_0^\beta\int_0^1 |f_x^{5,n}(\cdot,s)|^2dsdx\\ &= 2\int_0^\beta |v_x^n|^2dx+\tau^2\int_0^\beta\int_0^1 |f_x^{5,n}(\cdot,s)|^2dsdx\\ &\leq 2\int_0^\beta |v_x^n|^2dx+\tau|\kappa_2|^{-1}\|F^n\|^2_{\mathcal{H}}.\end{aligned}\tag{2.3.20}$$

Passing to the limit in (2.3.20), then using (2.3.10) and the fact that $\|F^n\|_{\mathcal{H}}\to 0$, we obtain (2.3.13). On the other hand, from (2.3.19), we have

$$\eta_x^n(\cdot,1)=v_x^ne^{-i\lambda^n\tau}+\tau\int_0^1 e^{i\lambda^n\tau(s-1)}f_x^{5,n}(\cdot,s)ds,$$

consequently, by using the same argument as proof of (2.3.13), we obtain (2.3.14). Next, it is clear to see that

$$\begin{aligned}\int_0^\beta |S_1(u^n,v^n,\eta^n)|^2dx &= \int_0^\beta |au_x^n+\kappa_1 v_x^n+\kappa_2\eta_x^n(\cdot,1)|^2dx\\ &\leq 3a^2\int_0^\beta |u_x^n|^2dx+3\kappa_1^2\int_0^\beta |v_x^n|^2dx+3\kappa_2^2\int_0^\beta |\eta_x^n(\cdot,1)|^2dx.\end{aligned}$$

Finally, passing to the limit in the above estimation, then using (2.3.10), (2.3.12) and (2.3.14), we obtain (2.3.15). The proof is thus complete.

□

Now, we fix a function $g\in C^1([\alpha,\beta])$ such that

$$g(\alpha)=-g(\beta)=1 \quad \text{and set} \quad \max_{x\in[\alpha,\beta]}|g(x)|=M_g \ \text{and} \ \max_{x\in[\alpha,\beta]}|g'(x)|=M_{g'}. \tag{2.3.21}$$

Remark 2.3.1. To prove the existence of a function g, we need to find an example. For this aim, we can take
$g(x)=1+\dfrac{2(\alpha-x)}{\beta-\alpha}$, then $g\in C^1([\alpha,\beta])$, $g(\alpha)=-g(\beta)=1$, $M_g=1$ and $M_{g'}=\dfrac{2}{\beta-\alpha}$. Also, we can take $g(x)=\cos\left(\dfrac{(\alpha-x)\pi}{\alpha-\beta}\right)$. □

Lemma 2.3.2. Under the hypothesis (H), the solution $U^n=(u^n,v^n,y^n,z^n,\eta^n(\cdot,\rho))^\top\in D(\mathcal{A})$ of system (2.3.5)-(2.3.9) satisfies the following inequalities

$$|z^n(\beta)|^2+|z^n(\alpha)|^2\leq M_{g'}\int_\alpha^\beta |z^n|^2dx+2|\lambda^n|M_g\left(\int_\alpha^\beta |z^n|^2dx\right)^{\frac12}+2M_g\|F^n\|_{\mathcal{H}}, \tag{2.3.22}$$

$$|y_x^n(\beta)|^2+|y_x^n(\alpha)|^2\leq M_{g'}\int_\alpha^\beta |y_x^n|^2dx+2(|\lambda^n|+c_0)M_g\left(\int_\alpha^\beta |y_x^n|^2dx\right)^{\frac12}+2M_g\|F^n\|_{\mathcal{H}} \tag{2.3.23}$$

and the following limits

$$\lim_{n\to\infty}|v^n(\alpha)|=0 \ \text{and} \ \lim_{n\to+\infty}|v^n(\beta)|=0, \tag{2.3.24}$$

$$\lim_{n\to\infty}|(S_1(u^n,v^n,\eta^n))(\alpha)|=0 \ \text{and} \ \lim_{n\to\infty}\left|(S_1(u^n,v^n,\eta^n))(\beta^-)\right|=0. \tag{2.3.25}$$

Proof. First, from (2.3.7), we deduce that

$$i\lambda^n y_x^n-z_x^n=f_x^{3,n}. \tag{2.3.26}$$

Multiplying (2.3.26) and (2.3.8) by $2g\overline{z^n}$ and $2g\overline{y_x^n}$ respectively, integrating over (α,β), using the definition of $c(\cdot)$, then taking the real part, we get

$$\Re\left\{2i\lambda^n\int_\alpha^\beta g y_x^n\overline{z^n}dx\right\}-\int_\alpha^\beta g\left(|z^n|^2\right)_x dx=\Re\left\{2\int_\alpha^\beta g f_x^{3,n}\overline{z^n}dx\right\} \tag{2.3.27}$$

and

$$\begin{aligned}&\Re\left\{2i\lambda^n\int_\alpha^\beta gz^n\overline{y_x^n}dx\right\}-\int_\alpha^\beta g\left(|y_x^n|^2\right)_x dx-\Re\left\{2c_0\int_\alpha^\beta gv^n\overline{y_x^n}dx\right\}\\&=\Re\left\{2\int_\alpha^\beta gf^{4,n}\overline{y_x^n}dx\right\}.\end{aligned}\tag{2.3.28}$$

Using integration by parts in (2.3.27) and (2.3.28), we obtain

$$\left[-g\left|z^n\right|^2\right]_\alpha^\beta=-\int_\alpha^\beta g'|z^n|^2dx-\Re\left\{2i\lambda^n\int_\alpha^\beta gy_x^n\overline{z^n}dx\right\}+\Re\left\{2\int_\alpha^\beta gf_x^{3,n}\overline{z^n}dx\right\}$$

and

$$\begin{aligned}\left[-g\left|y_x^n\right|^2\right]_\alpha^\beta=&-\int_\alpha^\beta g'|y_x^n|^2dx-\Re\left\{2i\lambda^n\int_\alpha^\beta gz^n\overline{y_x^n}dx\right\}+\Re\left\{2c_0\int_\alpha^\beta gv^n\overline{y_x^n}\right\}\\&+\Re\left\{2\int_\alpha^\beta gf^{4,n}\overline{y_x^n}dx\right\}.\end{aligned}$$

Using the definition of g and Cauchy-Schwarz inequality in the above equations, we obtain

$$\begin{aligned}|z^n(\beta)|^2+|z^n(\alpha)|^2\leq M_{g'}\int_\alpha^\beta|z^n|^2dx&+2|\lambda^n|M_g\left(\int_\alpha^\beta|y_x^n|^2dx\right)^{\frac12}\left(\int_\alpha^\beta|z^n|^2dx\right)^{\frac12}\\&+2M_g\left(\int_\alpha^\beta|f_x^{3,n}|^2dx\right)^{\frac12}\left(\int_\alpha^\beta|z^n|^2dx\right)^{\frac12}\end{aligned}\tag{2.3.29}$$

and

$$\begin{aligned}|y_x^n(\beta)|^2+|y_x^n(\alpha)|^2\leq M_{g'}\int_\alpha^\beta|y_x^n|^2dx&+2|\lambda^n|M_g\left(\int_\alpha^\beta|y_x^n|^2dx\right)^{\frac12}\left(\int_\alpha^\beta|z^n|^2dx\right)^{\frac12}\\&+2|c_0|M_g\left(\int_\alpha^\beta|y_x^n|^2dx\right)^{\frac12}\left(\int_\alpha^\beta|v^n|^2dx\right)^{\frac12}\\&+2M_g\left(\int_\alpha^\beta|f^{4,n}|^2dx\right)^{\frac12}\left(\int_\alpha^\beta|y_x^n|^2dx\right)^{\frac12}.\end{aligned}\tag{2.3.30}$$

Therefore, from (2.3.29), (2.3.30) and the fact that $\int_\alpha^\beta|\xi_1^n|^2dx\leq\int_0^L|\xi_1^n|^2dx\leq\|U^n\|_{\mathcal{H}}^2=1$ with $\xi_1^n\in\{v^n,y_x^n,z^n\}$ and $\int_\alpha^\beta|\xi_2^n|^2dx\leq\int_0^L|\xi_2^n|^2dx\leq\|F^n\|_{\mathcal{H}}^2$ with $\xi_2^n\in\{f_x^{3,n},f^{4,n}\}$, we obtain (2.3.22) and (2.3.23). On the other hand, from (2.3.5), we deduce that

$$i\lambda^n u_x^n-v_x^n=f_x^{1,n}.\tag{2.3.31}$$

Multiplying (2.3.31) and (2.3.6) by $2g\overline{v^n}$ and $2g\overline{S_1}(u^n,v^n,\eta^n)$ respectively, integrating over (α,β), using the definition of $c(\cdot)$ and $S_b(u^n,v^n,\eta^n)$, then taking the real part, we get

$$\Re\left\{2i\lambda^n\int_\alpha^\beta gu_x^n\overline{v^n}dx\right\}-\int_\alpha^\beta g(|v^n|^2)_xdx=\Re\left\{2\int_\alpha^\beta gf_x^{1,n}\overline{v^n}dx\right\}\tag{2.3.32}$$

and

$$\begin{aligned}&\Re\left\{2i\lambda^n\int_\alpha^\beta gv^n\overline{S_1}(u^n,v^n,\eta^n)dx\right\}-\int_\alpha^\beta g\left(|S_1(u^n,v^n,\eta^n)|^2\right)_x dx\\&+\Re\left\{2c_0\int_\alpha^\beta gz^n\overline{S_1}(u^n,v^n,\eta^n)dx\right\}=\Re\left\{2\int_\alpha^\beta gf^{2,n}\overline{S_1}(u^n,v^n,\eta^n)dx\right\}.\end{aligned}\tag{2.3.33}$$

Using integration by parts in (2.3.32) and (2.3.33), we get

$$\left[-g\left|v^n\right|^2\right]_\alpha^\beta=-\int_\alpha^\beta g'|v^n|^2dx-\Re\left\{2i\lambda^n\int_\alpha^\beta gu_x^n\overline{v^n}dx\right\}+\Re\left\{2\int_\alpha^\beta gf_x^{1,n}\overline{v^n}dx\right\}$$

and

$$\begin{aligned}\left[-g\left|S_1(u^n,v^n,\eta^n)\right|^2\right]_\alpha^\beta=&-\int_\alpha^\beta g'\left|S_1(u^n,v^n,\eta^n)\right|^2dx-\Re\left\{2i\lambda^n\int_\alpha^\beta gv^n\overline{S_1}(u^n,v^n,\eta^n)dx\right\}\\&-\Re\left\{2c_0\int_\alpha^\beta gz^n\overline{S_1}(u^n,v^n,\eta^n)dx\right\}+\Re\left\{2\int_\alpha^\beta gf^{2,n}\overline{S_1}(u^n,v^n,\eta^n)dx\right\}.\end{aligned}$$

Using the definition of g and Cauchy-Schwarz inequality in the above equations, then using the fact that

$$\begin{cases}\displaystyle\int_\alpha^\beta|z^n|^2dx\le\int_0^L|z^n|^2dx\le\|U^n\|_{\mathcal{H}}^2=1,\quad\int_\alpha^\beta|f_x^{1,n}|^2dx\le\int_0^L|f_x^{1,n}|^2dx\le a^{-1}\|F^n\|_{\mathcal{H}}^2\\ \text{and}\quad\displaystyle\int_\alpha^\beta|f^{2,n}|^2dx\le\int_0^L|f^{2,n}|^2dx\le\|F^n\|_{\mathcal{H}}^2,\end{cases}$$

we obtain

$$\begin{aligned}|v^n(\beta)|^2+|v^n(\alpha)|^2\le{}&M_{g'}\int_\alpha^\beta|v^n|^2dx+2|\lambda^n|M_g\left(\int_\alpha^\beta|u_x^n|^2dx\right)^{\frac12}\left(\int_\alpha^\beta|v^n|^2dx\right)^{\frac12}\\&+\frac{2}{\sqrt{a}}M_g\left(\int_\alpha^\beta|v^n|^2dx\right)^{\frac12}\|F^n\|_{\mathcal{H}}\end{aligned}\tag{2.3.34}$$

and

$$\begin{aligned}&\left|\left(S_1(u^n,v^n,\eta^n)\right)(\beta^-)\right|^2+\left|\left(S_1(u^n,v^n,\eta^n)\right)(\alpha)\right|^2\le M_{g'}\int_\alpha^\beta|S_1(u^n,v^n,\eta^n)|^2dx\\&+2|\lambda^n|M_g\left(\int_\alpha^\beta\left|S_1(u^n,v^n,\eta^n)\right|^2dx\right)^{\frac12}\left(\int_\alpha^\beta|v^n|^2dx\right)^{\frac12}\\&+2|c_0|M_g\left(\int_\alpha^\beta|S_1(u^n,v^n,\eta^n)|^2dx\right)^{\frac12}+2M_g\left(\int_\alpha^\beta|S_1(u^n,v^n,\eta^n)|^2dx\right)^{\frac12}\|F^n\|_{\mathcal{H}}.\end{aligned}\tag{2.3.35}$$

Finally, passing to limit in (2.3.34) and (2.3.35), then using (2.3.2), Lemma 2.3.1 and the fact that $\|F^n\|_{\mathcal{H}}\to0$, we obtain (2.3.24) and (2.3.25). The proof is thus complete. □

Remark 2.3.2. From (2.3.2), (2.3.22), (2.3.23), and the fact that $\|U^n\|_{\mathcal{H}}=1$ and $\|F^n\|_{\mathcal{H}}\to0$, we obtain

$$|z^n(\alpha)|,\ |z^n(\beta)|,\ |y_x^n(\alpha)|,\ |y_x^n(\beta)|\ \text{ are bounded.}\tag{2.3.36}$$

□

Lemma 2.3.3. Under the hypothesis (H), the solution $U^n = (u^n, v^n, y^n, z^n, \eta^n(\cdot,\rho))^\top \in D(\mathcal{A})$ of system (2.3.5)-(2.3.8) satisfies the following limits

$$\lim_{n\to\infty} \int_\alpha^\beta |z^n|^2 dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_\alpha^\beta |y_x^n|^2 dx = 0. \tag{2.3.37}$$

Proof. First, multiplying (2.3.6) by $\overline{z^n}$, integrating over (α,β), using the definition of $c(\cdot)$ and $S_b(u^n, v^n, \eta^n)$, then taking the real part, we get

$$\begin{aligned}
&\Re\left\{i\lambda^n \int_\alpha^\beta v^n \overline{z^n} dx\right\} - \Re\left\{\int_\alpha^\beta (S_1(u^n, v^n, \eta^n))_x \overline{z^n} dx\right\} + c_0 \int_\alpha^\beta |z^n|^2 dx \\
&= \Re\left\{\int_\alpha^\beta f^{2,n} \overline{z^n} dx\right\}.
\end{aligned} \tag{2.3.38}$$

From (2.3.7), we deduce that

$$\overline{z_x^n} = -i\lambda^n \overline{y_x^n} - \overline{f_x^{3,n}}. \tag{2.3.39}$$

Using integration by parts to the second term in (2.3.38), then using (2.3.39), we get

$$\begin{aligned}
c_0 \int_\alpha^\beta |z^n|^2 dx = &\Re\left\{i\lambda^n \int_\alpha^\beta S_1(u^n, v^n, \eta^n) \overline{y_x^n} dx\right\} + \Re\left\{\int_\alpha^\beta S_1(u^n, v^n, \eta^n) \overline{f_x^{3,n}} dx\right\} \\
&+ \Re\left\{\left[S_1(u^n, v^n, \eta^n) \overline{z^n}\right]_\alpha^\beta\right\} + \Re\left\{\int_\alpha^\beta f^{2,n} \overline{z^n} dx\right\} - \Re\left\{i\lambda^n \int_\alpha^\beta v^n \overline{z^n} dx\right\}.
\end{aligned}$$

Using Cauchy-Schwarz inequality in the above equation and the fact that $\int_\alpha^\beta |\xi_1^n|^2 dx \leq \int_0^L |\xi_1^n|^2 dx \leq \|U^n\|_{\mathcal{H}}^2 = 1$ with $\xi_1^n \in \{y_x^n, z^n\}$ and $\int_\alpha^\beta |\xi_2^n|^2 dx \leq \int_0^L |\xi_2^n|^2 dx \leq \|F^n\|_{\mathcal{H}}^2$ with $\xi_2^n \in \{f^{2,n}, f_x^{3,n}\}$, we obtain

$$\begin{aligned}
\left|c_0 \int_\alpha^\beta |z^n|^2 dx\right| \leq &(|\lambda^n| + \|F^n\|_{\mathcal{H}}) \left(\int_\alpha^\beta |S_1(u^n, v^n, \eta^n)|^2 dx\right)^{\frac{1}{2}} + |\lambda^n| \left(\int_\alpha^\beta |v^n|^2 dx\right)^{\frac{1}{2}} + \|F^n\|_{\mathcal{H}} \\
&+ \left|(S_1(u^n, v^n, \eta^n))(\beta^-)\right| |z^n(\beta)| + |(S_1(u^n, v^n, \eta^n))(\alpha)| |z^n(\alpha)|.
\end{aligned}$$

Passing to the limit in the above inequality, then using (2.3.2), (2.3.36), (2.3.25), Lemma 2.3.1 and the fact that $\|F^n\|_{\mathcal{H}} \to 0$, we obtain the first limit in (2.3.37). On the other hand, multiplying (2.3.8) by $-\overline{z^n}(\lambda^n)^{-1}$, integrating over (α,β), using the definition of $c(\cdot)$, then taking the imaginary part, we get

$$\begin{aligned}
&-\int_\alpha^\beta |z^n|^2 dx + \Im\left\{(\lambda^n)^{-1} \int_\alpha^\beta y_{xx}^n \overline{z^n} dx\right\} + \Im\left\{c_0 (\lambda^n)^{-1} \int_\alpha^\beta v^n \overline{z^n} dx\right\} \\
&= -\Im\left\{(\lambda^n)^{-1} \int_\alpha^\beta f^{4,n} \overline{z^n} dx\right\}.
\end{aligned}$$

Using integration by parts to the second term in the above equation, then using (2.3.39), we obtain

$$\begin{aligned}
\int_\alpha^\beta |y_x^n|^2 dx = &\int_\alpha^\beta |z^n|^2 dx - \Im\left\{(\lambda^n)^{-1} \int_\alpha^\beta \overline{f_x^{3,n}} y_x^n dx\right\} - \Im\left\{(\lambda^n)^{-1} \left[y_x^n \overline{z^n}\right]_\alpha^\beta\right\} \\
&- \Im\left\{c_0 (\lambda^n)^{-1} \int_\alpha^\beta v^n \overline{z^n} dx\right\} - \Im\left\{(\lambda^n)^{-1} \int_\alpha^\beta f^{4,n} \overline{z^n} dx\right\}.
\end{aligned}$$

Using Cauchy-Schwarz inequality in the above equation and the fact that $\|U^n\|_{\mathcal{H}} = 1$, we get

$$\begin{aligned}\int_\alpha^\beta |y_x^n|^2 dx \leq \int_\alpha^\beta |z^n|^2 dx + c_0|\lambda^n|^{-1}\left(\int_\alpha^\beta |v^n|^2 dx\right)^{\frac{1}{2}} + 2|\lambda^n|^{-1}\|F^n\|_{\mathcal{H}} \\ + |\lambda^n|^{-1}|y_x^n(\beta)||z^n(\beta)| + |\lambda^n|^{-1}|y_x^n(\alpha)||z^n(\alpha)|.\end{aligned} \tag{2.3.40}$$

Now, passing to the limit in (2.3.22), then using (2.3.2), the first limit in (2.3.37) and the fact that $\|F^n\|_{\mathcal{H}} \to 0$, we get

$$\lim_{n\to\infty} |z^n(\alpha)| = 0 \quad \text{and} \quad \lim_{n\to+\infty} |z^n(\beta)| = 0. \tag{2.3.41}$$

Finally, passing to the limit in (2.3.40), then using (2.3.2), (2.3.11), (2.3.36), the first limit in (2.3.37), (2.3.41), and the fact that $\|F^n\|_{\mathcal{H}} \to 0$, we obtain the second limit in (2.3.37). The proof is thus complete. □

Lemma 2.3.4. Under the hypothesis (H), the solution $U^n = (u^n, v^n, y^n, z^n, \eta^n(\cdot,\rho))^\top \in D(\mathcal{A})$ of system (2.3.5)-(2.3.9) satisfies the following estimations

$$\lim_{n\to\infty} |u^n(\beta)|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} |y^n(\beta)|^2 = 0, \tag{2.3.42}$$

$$\lim_{n\to\infty} |u_x^n(\beta^+)|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} |y_x^n(\beta)|^2 = 0, \tag{2.3.43}$$

$$\lim_{n\to\infty}\left(\int_\beta^\gamma |u^n|^2 dx + \int_\beta^\gamma |u_x^n|^2 dx + \int_\beta^\gamma |y^n|^2 dx + \int_\beta^\gamma |y_x^n|^2 dx\right) = 0, \tag{2.3.44}$$

$$\lim_{n\to\infty} \int_\beta^\gamma |v^n|^2 dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_\beta^\gamma |z^n|^2 dx = 0. \tag{2.3.45}$$

Proof. First, from (2.3.5) and (2.3.7), we get

$$|u^n(\beta)|^2 \leq 2(\lambda^n)^{-2}|v^n(\beta)|^2 + 2(\lambda^n)^{-2}|f^{1,n}(\beta)|^2$$

and

$$|y^n(\beta)|^2 \leq 2(\lambda^n)^{-2}|z^n(\beta)|^2 + 2(\lambda^n)^{-2}|f^{3,n}(\beta)|^2.$$

Using the fact that $|f^{1,n}(\beta)|^2 \leq \beta\int_0^\beta |f_x^{1,n}|^2 dx \leq \beta a^{-1}\|F^n\|^2_{\mathcal{H}}$ and $|f^{3,n}(\beta)|^2 \leq \beta\int_0^\beta |f_x^{3,n}|^2 dx \leq \beta\|F^n\|^2_{\mathcal{H}}$ in the above inequalities, we obtain

$$|u^n(\beta)|^2 \leq 2(\lambda^n)^{-2}|v^n(\beta)|^2 + 2\beta a^{-1}(\lambda^n)^{-2}\|F^n\|^2_{\mathcal{H}}$$

and

$$|y^n(\beta)|^2 \leq 2(\lambda^n)^{-2}|z^n(\beta)|^2 + 2\beta(\lambda^n)^{-2}\|F^n\|^2_{\mathcal{H}}.$$

Passing to the limit in the above inequalities, then using (2.3.2), (2.3.24), (2.3.41) and the fact that $\|F^n\|_{\mathcal{H}} \to 0$, we obtain (2.3.42). Secondly, since $S_b(u^n, v^n, \eta^n) \in H^1(0,L) \subset C([0,L])$, then we deduce that

$$\left|\left(S_1(u^n, v^n, \eta^n)\right)(\beta^-)\right|^2 = |a u_x^n(\beta^+)|^2. \tag{2.3.46}$$

Thus, from (2.3.25) and (2.3.46), we obtain the first limit in (2.3.43). Moreover, passing to the limit in inequality (2.3.23), then using (2.3.2), the second limit in (2.3.37) and the fact that

$\|F^n\|_{\mathcal{H}} \to 0$, we obtain the second limit in (2.3.43). On the other hand, (2.3.5)-(2.3.8) can be written in (β,γ) as the following form

$$\begin{aligned} (\lambda^n)^2 u^n + a u^n_{xx} - i\lambda^n c_0 y^n &= G^{1,n} \quad \text{in} \quad (\beta,\gamma), && (2.3.47)\\ (\lambda^n)^2 y^n + y^n_{xx} + i\lambda^n c_0 u^n &= G^{2,n} \quad \text{in} \quad (\beta,\gamma), && (2.3.48) \end{aligned}$$

where

$$G^{1,n} = -f^{2,n} - i\lambda^n f^{1,n} - c_0 f^{3,n} \quad \text{and} \quad G^{2,n} = -f^{4,n} - i\lambda^n f^{3,n} + c_0 f^{1,n}. \tag{2.3.49}$$

Let $V^n = (u^n, u^n_x, y^n, y^n_x)^\top$, then (2.3.47)-(2.3.48) can be written as the following

$$V^n_x = B^n V^n + G^n, \tag{2.3.50}$$

where

$$B^n = \begin{pmatrix} 0 & 1 & 0 & 0\\ -a^{-1}(\lambda^n)^2 & 0 & a^{-1} i\lambda^n c_0 & 0\\ 0 & 0 & 0 & 1\\ -i\lambda^n c_0 & 0 & -(\lambda^n)^2 & 0 \end{pmatrix} = (b_{ij})_{1\le i,j\le 4} \quad \text{and} \quad G^n = \begin{pmatrix} 0\\ a^{-1}G^{1,n}\\ 0\\ G^{2,n} \end{pmatrix}.$$

The solution of the differential equation (2.3.50) is given by

$$V^n(x) = e^{B^n(x-\beta)} V^n(\beta^+) + \int_\beta^x e^{B^n(s-x)} G^n(s)\,ds, \tag{2.3.51}$$

where $e^{B^n(x-\beta)} = (c_{ij})_{1\le i,j\le 4}$ and $e^{B^n(s-x)} = (d_{ij})_{1\le i,j\le 4}$ are denoted by the exponential of the matrices $B^n(x-\beta)$ and $B^n(s-x)$ respectively. Now, from (2.3.2), the entries b_{ij} are bounded for all $1 \le i,j \le 4$ and consequently, the entries $b_{ij}\ (x-\beta)$ and $b_{ij}\ (s-x)$ are bounded. In addition, from the definition of the exponential of a square matrix, we obtain

$$e^{B^n\zeta} = \sum_{k=0}^{\infty} \frac{(B^n\zeta)^k}{k!} \quad \text{for} \quad \zeta \in \{x-\beta, s-x\}.$$

Therefore, the entries c_{ij} and d_{ij} are also bounded for all $1 \le i,j \le 4$ and consequently, $e^{B^n(x-\beta)}$ and $e^{B^n(s-x)}$ are two bounded matrices. From (2.3.42) and (2.3.43), we directly obtain

$$V^n(\beta^+) \to 0 \quad \text{in} \quad (L^2(\beta,\gamma))^4, \quad \text{as} \quad n \to \infty. \tag{2.3.52}$$

Moreover, from (2.3.49), we deduce that

$$\int_\beta^\gamma |G^{1,n}|^2 dx \le 3\int_0^L |f^{2,n}|^2 dx + 3(\lambda^n)^2 \int_0^L |f^{1,n}|^2 dx + 3c_0^2 \int_0^L |f^{3,n}|^2 dx \tag{2.3.53}$$

and

$$\int_\beta^\gamma |G^{2,n}|^2 dx \le 3\int_0^L |f^{4,n}|^2 dx + 3(\lambda^n)^2 \int_0^L |f^{3,n}|^2 dx + 3c_0^2 \int_0^L |f^{1,n}|^2 dx. \tag{2.3.54}$$

Now, since $f^{1,n}, f^{3,n} \in H_0^1(0,L)$, then it follows from Poincaré inequality that there exist two constants $C_1 > 0$ and $C_2 > 0$ such that

$$\|f^{1,n}\|_{L^2(0,L)} \le C_1 \|f^{1,n}_x\|_{L^2(0,L)} \quad \text{and} \quad \|f^{3,n}\|_{L^2(0,L)} \le C_2 \|f^{3,n}_x\|_{L^2(0,L)}. \tag{2.3.55}$$

Consequently, from (2.3.53), (2.3.54) and (2.3.55), we get

$$\int_\beta^\gamma |G^{1,n}|^2 dx \leq 3\left(1 + a^{-1}(\lambda^n C_1)^2 + (c_0 C_2)^2\right) \|F^n\|^2_{\mathcal{H}}, \tag{2.3.56}$$

and

$$\int_\beta^\gamma |G^{2,n}|^2 dx \leq 3\left(1 + (\lambda^n C_1)^2 + a^{-1}(c_0 C_2)^2\right) \|F^n\|^2_{\mathcal{H}}. \tag{2.3.57}$$

Hence, from (2.3.2), (2.3.56), (2.3.57) and the fact that $\|F^n\|_{\mathcal{H}} \to 0$, we obtain

$$G^n \to 0 \quad \text{in} \quad (L^2(\beta,\gamma))^4, \quad \text{as} \quad n \to \infty. \tag{2.3.58}$$

Therefore, from (2.3.51), (2.3.52), (2.3.58) and as $e^{B^n(x-\beta)}$, $e^{B^n(s-x)}$ are two bounded matrices, we get $V^n \to 0$ in $(L^2(\beta,\gamma))^4$ and consequently, we obtain (2.3.44). Next, from (2.3.5) , (2.3.7) and (2.3.55), we deduce that

$$\int_\beta^\gamma |v^n|^2 dx \leq 2(\lambda^n)^2 \int_\beta^\gamma |u^n|^2 dx + 2\int_\beta^\gamma |f^{1,n}|^2 dx \leq 2(\lambda^n)^2 \int_\beta^\gamma |u^n|^2 dx + 2C_1 a^{-1}\|F^n\|^2_{\mathcal{H}},$$
$$\int_\beta^\gamma |z^n|^2 dx \leq 2(\lambda^n)^2 \int_\beta^\gamma |y^n|^2 dx + 2\int_\beta^\gamma |f^{3,n}|^2 dx \leq 2(\lambda^n)^2 \int_\beta^\gamma |y^n|^2 dx + 2C_2\|F^n\|^2_{\mathcal{H}}.$$

Finally, passing to the limit in the above inequalities, then using (2.3.2), (2.3.44) and the fact that $\|F^n\|_{\mathcal{H}} \to 0$, we obtain (2.3.45). The proof is thus complete. □

Lemma 2.3.5. Let $h \in C^1([0,L])$ be a function. Under the hypothesis (H), the solution $U^n = (u^n, v^n, y^n, z^n, \eta^n(\cdot,\rho))^\top \in D(\mathcal{A})$ of system (2.3.5)-(2.3.9) satisfies the following equality

$$\begin{aligned}
&\int_0^L h'\left(a^{-1}|S_b(u^n,v^n,\eta^n)|^2 + |v^n|^2 + |z^n|^2 + |y^n_x|^2\right) dx \\
&\quad - \left[h\left(a^{-1}|S_b(u^n,v^n,\eta^n)|^2 + |y^n_x|^2\right)\right]_0^L - \Re\left\{2\int_0^L c(\cdot) h v^n \overline{y^n_x} dx\right\} \\
&\quad + \Re\left\{\frac{2}{a}\int_0^L c(\cdot) h z^n \overline{S_b}(u^n,v^n,\eta^n) dx\right\} + \Re\left\{\frac{2i\lambda^n}{a}\int_0^\beta h v^n(\kappa_1 \overline{v^n_x} + \kappa_2 \overline{\eta^n_x}(\cdot,1)) dx\right\} \\
&= \Re\left\{2\int_0^L h\overline{f^{1,n}_x} v^n dx\right\} + \Re\left\{\frac{2}{a}\int_0^L h f^{2,n}\overline{S_b}(u^n,v^n,\eta^n) dx\right\} \\
&\quad + \Re\left\{2\int_0^L h\overline{f^{3,n}_x} z^n dx\right\} + \Re\left\{2\int_0^L h f^{4,n}\overline{y^n_x} dx\right\}.
\end{aligned}$$

Proof. First, multiplying (2.3.6) and (2.3.8) by $2a^{-1}h\overline{S_b}(u^n,v^n,\eta^n)$ and $2h\overline{y^n_x}$ respectively, integrating over $(0,L)$, then taking the real part, we get

$$\begin{aligned}
&\Re\left\{\frac{2i\lambda^n}{a}\int_0^L h v^n \overline{S_b}(u^n,v^n,\eta^n) dx\right\} - a^{-1}\int_0^L h\left(|S_b(u^n,v^n,\eta^n)|^2\right)_x dx \\
&+ \Re\left\{\frac{2}{a}\int_0^L c(\cdot) h z^n \overline{S_b}(u^n,v^n,\eta^n) dx\right\} = \Re\left\{\frac{2}{a}\int_0^L h f^{2,n}\overline{S_b}(u^n,v^n,\eta^n) dx\right\}
\end{aligned} \tag{2.3.59}$$

and

$$\begin{aligned}
&\Re\left\{2i\lambda^n\int_0^L h z^n \overline{y^n_x} dx\right\} - \int_0^L h\left(|y^n_x|^2\right)_x dx - \Re\left\{2\int_0^L c(\cdot) h v^n \overline{y^n_x} dx\right\} \\
&= \Re\left\{2\int_0^L h f^{4,n}\overline{y^n_x} dx\right\}.
\end{aligned} \tag{2.3.60}$$

From (2.3.5) and (2.3.7), we deduce that

$$i\lambda^n \overline{u_x^n} = -\overline{v_x^n} - \overline{f_x^{1,n}}, \tag{2.3.61}$$
$$i\lambda^n \overline{y_x^n} = -\overline{z_x^n} - \overline{f_x^{3,n}}. \tag{2.3.62}$$

Consequently, from (2.3.61) and the definition $S_b(u^n, v^n, \eta^n)$, we have

$$i\lambda^n \overline{S_b}(u^n, v^n, \eta^n) = \begin{cases} -a\left(\overline{v_x^n} + \overline{f_x^{1,n}}\right) + i\lambda^n\left(\kappa_1 \overline{v_x^n} + \kappa_2 \overline{\eta_x^n}(\cdot,1)\right), & \text{in } (0,\beta), \\ -a\left(\overline{v_x^n} + \overline{f_x^{1,n}}\right), & \text{in } (\beta, L). \end{cases} \tag{2.3.63}$$

Substituting (2.3.63) and (2.3.62) in (2.3.59) and (2.3.60) respectively, we obtain

$$\begin{aligned} &-\int_0^L h\left(|v^n|^2 + a^{-1}|S_b(u^n,v^n,\eta^n)|^2\right)_x dx + \Re\left\{\frac{2i\lambda^n}{a}\int_0^\beta hv^n(\kappa_1\overline{v_x^n} + \kappa_2\overline{\eta_x^n}(\cdot,1))dx\right\} \\ &+\Re\left\{\frac{2}{a}\int_0^L c(\cdot)hz^n\overline{S_b}(u^n,v^n,\eta^n)dx\right\} \\ &= \Re\left\{2\int_0^L h\overline{f_x^{1,n}}v^n dx\right\} + \Re\left\{\frac{2}{a}\int_0^L hf^{2,n}\overline{S_b}(u^n,v^n,\eta^n)dx\right\} \end{aligned}$$

and

$$\begin{aligned} &-\int_0^L h\left(|z^n|^2 + |y_x^n|^2\right)_x dx - \Re\left\{2\int_0^L c(\cdot)hv^n\overline{y_x^n}dx\right\} \\ &= \Re\left\{2\int_0^L hf^{4,n}\overline{y_x^n}dx\right\} + \Re\left\{2\int_0^L h\overline{f_x^{3,n}}z^n dx\right\}. \end{aligned}$$

Finally, adding the above equations, then using integration by parts and the fact that $v^n(0) = v^n(L) = 0$ and $z^n(0) = z^n(L) = 0$, we obtain the desired result. The proof is thus complete. $\square$

Now, we fix the cut-off functions $\chi_1, \chi_2 \in C^1([0,L])$ (see Figure 2.2) such that $0 \le \chi_1(x) \le 1$, $0 \le \chi_2(x) \le 1$, for all $x \in [0,L]$ and

$$\chi_1(x) = \begin{cases} 1 & \text{if} \quad x \in [0,\alpha], \\ 0 & \text{if} \quad x \in [\beta, L], \end{cases} \quad \text{and} \quad \chi_2(x) = \begin{cases} 0 & \text{if} \quad x \in [0,\beta], \\ 1 & \text{if} \quad x \in [\gamma, L], \end{cases}$$

and set $\max_{x\in[0,L]} |\chi_1'(x)| = M_{\chi_1'}$ and $\max_{x\in[0,L]} |\chi_2'(x)| = M_{\chi_2'}$,

Lemma 2.3.6. Under the hypothesis (H), the solution $U^n = (u^n, v^n, y^n, z^n, \eta^n(\cdot,\rho))^\top \in D(\mathcal{A})$ of system (2.3.5)-(2.3.9) satisfies the following limits

$$\lim_{n\to\infty}\left(\int_0^\alpha |y_x^n|^2 dx + \int_0^\alpha |z^n|^2 dx\right) = 0, \tag{2.3.64}$$
$$\lim_{n\to\infty}\left(a\int_\gamma^L |u_x^n|^2 dx + \int_\gamma^L |v^n|^2 dx + \int_\gamma^L |y_x^n|^2 dx + \int_\gamma^L |z^n|^2 dx\right) = 0. \tag{2.3.65}$$

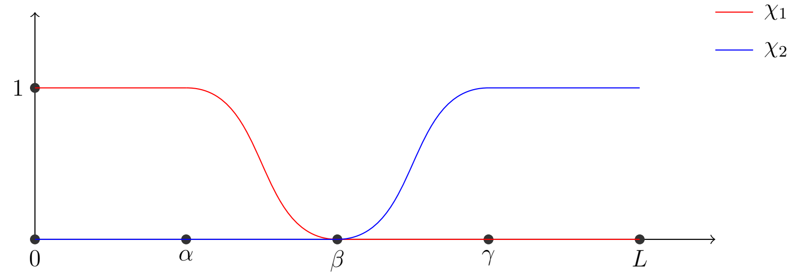

Figure 2.2: Geometric description of the functions χ_1 and χ_2.

Proof. First, using the result of Lemma 2.3.5 with $h = x\chi_1$, then using the definition of $c(\cdot)$, $S_b(u^n, v^n, \eta^n)$ and χ_1, we get

$$\begin{aligned}
&\int_0^\alpha |y_x^n|^2dx + \int_0^\alpha |z^n|^2dx = -\int_0^\alpha |v^n|^2dx - a^{-1}\int_0^\alpha |S_1(u^n,v^n,\eta^n)|^2dx\\
&-\int_\alpha^\beta (\chi_1 + x\chi_1')\left(a^{-1}|S_1(u^n,v^n,\eta^n)|^2 + |v^n|^2 + |y_x^n|^2 + |z^n|^2\right)dx\\
&-\Re\left\{\frac{2c_0}{a}\int_\alpha^\beta x\chi_1 z^n\overline{S_1}(u^n,v^n,\eta^n)dx\right\} + \Re\left\{2c_0\int_\alpha^\beta x\chi_1 v^n\overline{y_x^n}dx\right\}\\
&-\Re\left\{\frac{2i\lambda^n}{a}\int_0^\beta x\chi_1 v^n\left(\kappa_1\overline{v_x^n} + \kappa_2\overline{\eta_x^n}(\cdot,1)\right)dx\right\} + \Re\left\{\frac{2}{a}\int_0^\beta x\chi_1 f^{2,n}\overline{S_1}(u^n,v^n,\eta^n)dx\right\}\\
&+\Re\left\{2\int_0^L x\chi_1\left(\overline{f_x^{1,n}}v^n + \overline{f_x^{3,n}}z^n + f^{4,n}\overline{y_x^n}\right)dx\right\}.
\end{aligned}$$

Using Cauchy-Schwarz inequality in the above equation and the fact that $\|U^n\|_{\mathcal{H}} = 1$, we obtain

$$\begin{aligned}
&\int_0^\alpha |y_x^n|^2dx + \int_0^\alpha |z^n|^2dx \leq \int_0^\alpha |v^n|^2dx + a^{-1}\int_0^\alpha |S_1(u^n,v^n,\eta^n)|^2dx\\
&+\left(1+\beta M_{\chi_1'}\right)\int_\alpha^\beta \left(a^{-1}|S_1(u^n,v^n,\eta^n)|^2 + |v^n|^2 + |z^n|^2 + |y_x^n|^2\right)dx\\
&+\frac{2|c_0|\beta}{a}\left(\int_\alpha^\beta |S_1(u^n,v^n,\eta^n)|^2dx\right)^{\frac12} + 2|c_0|\beta\left(\int_\alpha^\beta |v^n|^2dx\right)^{\frac12}\\
&+\frac{2\beta|\lambda^n|}{a}\left[\kappa_1\left(\int_0^\beta |v_x^n|^2dx\right)^{\frac12} + |\kappa_2|\left(\int_0^\beta |\eta_x^n(\cdot,1)|^2dx\right)^{\frac12}\right]\\
&+\frac{2\beta}{a}\left(\int_0^\beta |S_1(u^n,v^n,\eta^n)|^2dx\right)^{\frac12}\|F^n\|_{\mathcal{H}} + 2L\left(\frac{1}{\sqrt{a}}+2\right)\|F^n\|_{\mathcal{H}}.
\end{aligned}$$

Passing to the limit in the above inequality, then using (2.3.2), Lemmas 2.3.1, 2.3.3 and the fact that $\|F^n\|_{\mathcal{H}} \to 0$, we obtain (2.3.64). On the other hand, using the result of Lemma 2.3.5 with $h = (x-L)\chi_2$, then using Cauchy-Schwarz inequality and the fact that $\|U^n\|_{\mathcal{H}} = 1$, we get

$$
\begin{aligned}
&a\int_\gamma^L |u_x^n|^2 + \int_\gamma^L |v^n|^2 dx + \int_\gamma^L |y_x^n|^2 dx + \int_\gamma^L |z^n|^2 dx \\
&\leq \left(1+(L-\beta)M_{\chi_2'}\right)\int_\beta^\gamma \left(a|u_x^n|^2+|v^n|^2+|y_x^n|^2+|z^n|^2\right)dx \\
&+2|c_0|(L-\beta)\left(\int_\beta^\gamma |v^n|^2 dx\right)^{\frac{1}{2}}\left(\int_\beta^\gamma |y_x^n|^2 dx\right)^{\frac{1}{2}} \\
&+2|c_0|(L-\beta)\left(\int_\beta^\gamma |z^n|^2 dx\right)^{\frac{1}{2}}\left(\int_\beta^\gamma |u_x^n|^2 dx\right)^{\frac{1}{2}} + 4L\left(\frac{1}{\sqrt{a}}+1\right)\|F^n\|_{\mathcal{H}}.
\end{aligned}
$$

Finally, passing to the limit in the above inequality, then using Lemma 2.3.4 and the fact that $\|F^n\|_{\mathcal{H}} \to 0$, we obtain (2.3.65). The proof is thus complete. □

Proof of Proposition 2.3.1. From Lemmas 2.3.1-2.3.6, we obtain $\|U^n\|_{\mathcal{H}} \to 0$ as $n\to\infty$, which contradicts $\|U^n\|_{\mathcal{H}} = 1$. Thus, (2.3.1) holds. The proof is thus complete. □

Proof of Theorem 2.3.1. From proposition 2.3.1, we have $i\mathbb{R}\subset\rho(\mathcal{A})$ and consequently $\sigma(\mathcal{A})\cap i\mathbb{R}=\emptyset$. Therefore, according to Theorem 1.3.3, we get that the C_0-semigroup of contraction $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable. The proof is thus complete. □

2.4 Polynomial Stability

In this section, we will prove the polynomial stability of system (2.2.2)-(2.2.7). The main result of this section is the following theorem.

Theorem 2.4.1. Under the hypothesis (H), for all $U_0\in D(\mathcal{A})$, there exists a constant $C>0$ independent of U_0 such that the energy of system (2.2.2)-(2.2.7) satisfies the following estimation

$$E(t)\leq \frac{C}{t}\|U_0\|^2_{D(\mathcal{A})},\quad \forall\, t>0.$$

According to Theorem 1.3.7, to prove Theorem 2.4.1, we still need to prove the following two conditions

$$i\mathbb{R}\subset\rho(\mathcal{A}) \tag{2.4.1}$$

and

$$\limsup_{\lambda\in\mathbb{R},\ |\lambda|\to\infty}\frac{1}{|\lambda|^2}\left\|(i\lambda I-\mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})}<\infty. \tag{2.4.2}$$

From Proposition 2.3.1, we obtain condition (2.4.1). Next, we will prove condition (2.4.2) by a contradiction argument. For this purpose, suppose that (2.4.2) is false, then there exists $\left\{(\lambda^n, U^n := (u^n,v^n,y^n,z^n,\eta^n(\cdot,\rho))^\top)\right\}_{n\geq 1}\subset\mathbb{R}^*\times D(\mathcal{A})$ with

$$|\lambda^n|\to\infty \text{ as } n\to\infty \quad\text{and}\quad \|U^n\|_{\mathcal{H}}=\left\|(u^n,v^n,y^n,z^n,\eta^n(\cdot,\rho))^\top\right\|_{\mathcal{H}}=1, \forall n\geq 1, \tag{2.4.3}$$

such that

$$(\lambda^n)^2(i\lambda^n I-\mathcal{A})U^n = F^n := (f^{1,n},f^{2,n},f^{3,n},f^{4,n},f^{5,n}(\cdot,\rho))^\top\to 0 \quad\text{in}\quad \mathcal{H},\ \text{ as } n\to\infty. \tag{2.4.4}$$

For simplicity, we drop the index n. Equivalently, from (2.4.4), we have

$$\begin{aligned}
i\lambda u - v &= \lambda^{-2} f^1, \quad f^1 \to 0 \quad \text{in} \quad H_0^1(0,L), && (2.4.5)\\
i\lambda v - (S_b(u,v,\eta))_x + c(\cdot)z &= \lambda^{-2} f^2, \quad f^2 \to 0 \quad \text{in} \quad L^2(0,L), && (2.4.6)\\
i\lambda y - z &= \lambda^{-2} f^3, \quad f^3 \to 0 \quad \text{in} \quad H_0^1(0,L), && (2.4.7)\\
i\lambda z - y_{xx} - c(\cdot)v &= \lambda^{-2} f^4, \quad f^4 \to 0 \quad \text{in} \quad L^2(0,L), && (2.4.8)\\
i\lambda\eta(\cdot,\rho) + \tau^{-1}\eta_\rho(\cdot,\rho) &= \lambda^{-2} f^5(\cdot,\rho), \quad f^5(\cdot,\rho) \to 0 \quad \text{in} \quad \mathcal{W}. && (2.4.9)
\end{aligned}$$

Here we will check the condition (2.4.2) by finding a contradiction with (2.4.3) such as $\|U\|_{\mathcal{H}} = o(1)$. For clarity, we divide the proof into several Lemmas.

Lemma 2.4.1. Under the hypothesis (H), the solution $U = (u, v, y, z, \eta(\cdot,\rho))^\top \in D(\mathcal{A})$ of system (2.4.5)-(2.4.9) satisfies the following estimations

$$\int_0^\beta |v_x|^2 dx = o(\lambda^{-2}), \tag{2.4.10}$$

$$\int_0^\beta |u_x|^2 dx = o(\lambda^{-4}), \tag{2.4.11}$$

$$\int_0^\beta \int_0^1 |\eta_x(\cdot,\rho)|^2 d\rho dx = o(\lambda^{-2}), \tag{2.4.12}$$

$$\int_0^\beta |\eta_x(\cdot,1)|^2 dx = o(\lambda^{-2}), \tag{2.4.13}$$

$$\int_0^\beta |S_1(u,v,\eta)|^2 dx = o(\lambda^{-2}). \tag{2.4.14}$$

Proof. First, taking the inner product of (2.4.4) with U in $\mathcal{H}$ and using (2.2.18) with the help of hypothesis (H), we obtain

$$\int_0^\beta |v_x|^2 dx \leq -\frac{1}{\kappa_1 - |\kappa_2|} \Re(\mathcal{A}U, U)_{\mathcal{H}} = \frac{\lambda^{-2}}{\kappa_1 - |\kappa_2|} \Re(F, U)_{\mathcal{H}} \leq \frac{\lambda^{-2}}{\kappa_1 - |\kappa_2|} \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}. \tag{2.4.15}$$

Thus, from (2.4.15) and the fact that $\|F\|_{\mathcal{H}} = o(1)$ and $\|U\|_{\mathcal{H}} = 1$, we obtain (2.4.10). Now, from (2.4.5), we deduce that

$$\begin{aligned}
\int_0^\beta |u_x|^2 dx &\leq 2\lambda^{-2} \int_0^\beta |v_x|^2 dx + 2\lambda^{-4} \int_0^\beta |f_x^1|^2 dx \\
&\leq 2\lambda^{-2} \int_0^\beta |v_x|^2 dx + 2\lambda^{-4} \int_0^L |f_x^1|^2 dx
\end{aligned}. \tag{2.4.16}$$

Therefore, from (2.4.10), (2.4.16) and the fact that $\|f_x^1\|_{L^2(0,L)} = o(1)$, we obtain (2.4.11). Next, from (2.4.9) and the fact that $\eta(\cdot,0) = v(\cdot)$, we get

$$\eta(x,\rho) = v e^{-i\lambda\tau\rho} + \tau\lambda^{-2} \int_0^\rho e^{i\lambda\tau(s-\rho)} f^5(x,s) ds, \quad (x,\rho) \in (0,\beta) \times (0,1). \tag{2.4.17}$$

From (2.4.17), we deduce that

$$\int_0^\beta \int_0^1 |\eta_x(\cdot,\rho)|^2 d\rho dx \leq 2 \int_0^\beta |v_x|^2 dx + \tau^2 \lambda^{-4} \int_0^\beta \int_0^1 |f_x^5(\cdot,s)|^2 ds dx. \tag{2.4.18}$$

Thus, from (2.4.10), (2.4.18) and the fact that $f^5(\cdot,\rho)\to 0$ in $\mathcal{W}$, we obtain (2.4.12). On the other hand, from (2.4.17), we have

$$\eta_x(\cdot,1)=v_x e^{-i\lambda\tau}+\tau\lambda^{-2}\int_0^1 e^{i\lambda\tau(s-1)}f_x^5(\cdot,s)ds,$$

consequently, similar to the previous proof, we obtain (2.4.13). Next, it is clear to see that

$$\begin{aligned}\int_0^\beta |S_1(u,v,\eta)|^2dx &= \int_0^\beta |au_x+\kappa_1 v_x+\kappa_2\eta_x(\cdot,1)|^2dx\\ &\leq 3a^2\int_0^\beta |u_x|^2dx+3\kappa_1^2\int_0^\beta |v_x|^2dx+3\kappa_2^2\int_0^\beta |\eta_x(\cdot,1)|^2dx.\end{aligned}$$

Finally, from (2.4.10), (2.4.11), (2.4.13) and the above estimation, we obtain (2.4.14). The proof is thus complete. $\square$

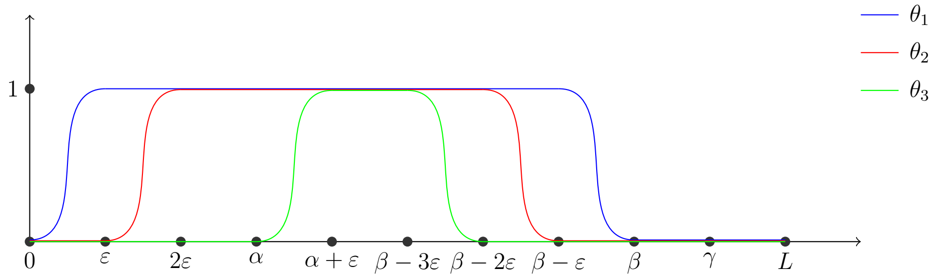

Figure 2.3: Geometric description of the functions θ_1, θ_2 and θ_3.

Lemma 2.4.2. Let $0<\varepsilon<\min\left(\frac{\alpha}{2},\frac{\beta-\alpha}{4}\right)$. Under the hypothesis (H), the solution $U=(u,v,y,z,\eta(\cdot,\rho))^\top\in D(\mathcal{A})$ of system (2.4.5)-(2.4.9) satisfies the following estimation

$$\int_\varepsilon^{\beta-\varepsilon}|v|^2dx=o(1). \tag{2.4.19}$$

Proof. First, we fix a cut-off function $\theta_1\in C^1([0,L])$ (see Figure 2.3) such that $0\leq\theta_1(x)\leq 1$, for all $x\in[0,L]$ and

$$\theta_1(x)=\begin{cases}1 & \text{if} \quad x\in[\varepsilon,\beta-\varepsilon],\\ 0 & \text{if} \quad x\in\{0\}\cup[\beta,L],\end{cases}$$

and set

$$\max_{x\in[0,L]}|\theta_1'(x)|=M_{\theta_1'}.$$

Multiplying (2.4.6) by $\lambda^{-1}\theta_1\overline{v}$, integrating over $(0,L)$, then taking the imaginary part, we obtain

$$\begin{aligned}&\int_0^L\theta_1|v|^2dx-\Im\left\{\lambda^{-1}\int_0^L\theta_1\left(S_b\left(u,v,\eta\right)\right)_x\overline{v}dx\right\}+\Im\left\{\lambda^{-1}\int_0^L c(\cdot)\theta_1 z\overline{v}dx\right\}\\ &=\Im\left\{\lambda^{-3}\int_0^L\theta_1 f^2\overline{v}dx\right\}.\end{aligned}$$

Using integration by parts in the above equation and the fact that $v(0) = v(L) = 0$, we get

$$\begin{aligned} \int_0^L \theta_1 |v|^2 dx = &-\Im\left\{\frac{1}{\lambda}\int_0^L (\theta_1'\overline{v} + \theta_1\overline{v_x}) S_b(u,v,\eta)dx\right\} - \Im\left\{\frac{1}{\lambda}\int_0^L c(\cdot)\theta_1 z\overline{v}dx\right\} \\ &+\Im\left\{\frac{1}{\lambda^3}\int_0^L \theta_1 f^2\overline{v}dx\right\}. \end{aligned} \tag{2.4.20}$$

Using the definition of $c(\cdot)$, $S_b(u,v,\eta)$ and θ_1, then using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &\left|\Im\left\{\lambda^{-1}\int_0^L (\theta_1'\overline{v} + \theta_1\overline{v_x}) S_b(u,v,\eta)dx\right\}\right| = \left|\Im\left\{\lambda^{-1}\int_0^\beta (\theta_1'\overline{v} + \theta_1\overline{v_x}) S_1(u,v,\eta)dx\right\}\right| \\ &\leq |\lambda|^{-1}\left[M_{\theta_1'}\left(\int_0^\beta |v|^2 dx\right)^{\frac{1}{2}} + \left(\int_0^\beta |v_x|^2 dx\right)^{\frac{1}{2}}\right]\left(\int_0^\beta |S_1(u,v,\eta)|^2 dx\right)^{\frac{1}{2}} \end{aligned}$$

and

$$\left|\Im\left\{\lambda^{-1}\int_0^L c(\cdot)\theta_1 z\overline{v}dx\right\}\right| = \left|\Im\left\{c_0\lambda^{-1}\int_\alpha^\beta \theta_1 z\overline{v}dx\right\}\right| \leq |c_0||\lambda|^{-1}\left(\int_\alpha^\beta |z|^2 dx\right)^{\frac{1}{2}}\left(\int_\alpha^\beta |v|^2 dx\right)^{\frac{1}{2}}.$$

From the above inequalities, Lemma 2.4.1 and the fact that v and z are uniformly bounded in $L^2(0,L)$, we obtain

$$\left\{\begin{aligned} &-\Im\left\{\lambda^{-1}\int_0^L (\theta_1'\overline{v} + \theta_1\overline{v_x}) S_b(u,v,\eta)dx\right\} = o(\lambda^{-2}), \\ &-\Im\left\{\lambda^{-1}\int_0^L c(\cdot)\theta_1 z\overline{v}dx\right\} = O(|\lambda|^{-1}) = o(1). \end{aligned}\right. \tag{2.4.21}$$

Inserting (2.4.21) in (2.4.20), then using the fact that v is uniformly bounded in $L^2(0,L)$ and $\|f^2\|_{L^2(0,L)} = o(1)$, we obtain

$$\int_0^L \theta_1 |v|^2 dx = o(1).$$

Finally, from the above estimation and the definition of θ_1, we obtain (2.4.19). The proof is thus complete. □

Lemma 2.4.3. Let $0 < \varepsilon < \min\left(\frac{\alpha}{2}, \frac{\beta-\alpha}{4}\right)$. Under the hypothesis (H), the solution $U = (u,v,y,z,\eta(\cdot,\rho))^\top \in D(\mathcal{A})$ of system (2.4.5)-(2.4.9) satisfies the following estimations

$$\int_\alpha^{\beta-2\varepsilon} |z|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha+\varepsilon}^{\beta-3\varepsilon} |y_x|^2 dx = o(1). \tag{2.4.22}$$

Proof. First, we fix a cut-off function $\theta_2 \in C^1([0,L])$ (see figure 2.3) such that $0 \leq \theta_2(x) \leq 1$, for all $x \in [0,L]$ and

$$\theta_2(x) = \left\{\begin{array}{lll} 0 & \text{if} & x \in [0,\varepsilon] \cup [\beta-\varepsilon, L], \\ 1 & \text{if} & x \in [2\varepsilon, \beta-2\varepsilon], \end{array}\right.$$

and set

$$\max_{x\in[0,L]} |\theta_2'(x)| = M_{\theta_2'}.$$

Multiplying (2.4.6) and (2.4.8) by $\theta_2\overline{z}$ and $\theta_2\overline{v}$ respectively, integrating over $(0,L)$, then taking the real part, we obtain

$$\begin{aligned}&\Re\left\{i\lambda\int_0^L \theta_2 v\overline{z}dx\right\}-\Re\left\{\int_0^L \theta_2(S_b(u,v,\eta))_x\overline{z}dx\right\}+\int_0^L c(\cdot)\theta_2|z|^2dx\\&=\Re\left\{\lambda^{-2}\int_0^L \theta_2 f^2\overline{z}dx\right\}\end{aligned}\tag{2.4.23}$$

and

$$\begin{aligned}&\Re\left\{i\lambda\int_0^L \theta_2 z\overline{v}dx\right\}-\Re\left\{\int_0^L \theta_2 y_{xx}\overline{v}dx\right\}-\int_0^L c(\cdot)\theta_2|v|^2dx\\&=\Re\left\{\lambda^{-2}\int_0^L \theta_2 f^4\overline{v}dx\right\}.\end{aligned}\tag{2.4.24}$$

Adding (2.4.23) and (2.4.24), then using integration by parts and the fact that $v(0)=v(L)=0$ and $z(0)=z(L)=0$, we get

$$\begin{aligned}&\int_0^L c(\cdot)\theta_2|z|^2dx=\int_0^L c(\cdot)\theta_2|v|^2dx-\Re\left\{\int_0^L(\theta_2'\overline{z}+\theta_2\overline{z_x})S_b(u,v,\eta)dx\right\}\\&-\Re\left\{\int_0^L(\theta_2'\overline{v}+\theta_2\overline{v_x})y_xdx\right\}+\Re\left\{\lambda^{-2}\int_0^L\theta_2 f^2\overline{z}dx\right\}+\Re\left\{\lambda^{-2}\int_0^L\theta_2 f^4\overline{v}dx\right\}.\end{aligned}\tag{2.4.25}$$

From (2.4.7), we deduce that

$$\overline{z_x}=-i\lambda\overline{y_x}-\lambda^{-2}\overline{f_x^3}.\tag{2.4.26}$$

Using (2.4.26) and the definition of $S_b(u,v,\eta)$ and θ_2, then using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\left|\Re\left\{\int_0^L(\theta_2'\overline{z}+\theta_2\overline{z_x})S_b(u,v,\eta)dx\right\}\right|=\left|\Re\left\{\int_\varepsilon^{\beta-\varepsilon}\left[\theta_2'\overline{z}+\theta_2(-i\lambda\overline{y_x}-\lambda^{-2}\overline{f_x^3})\right]S_1(u,v,\eta)dx\right\}\right|\\&\leq\left[M_{\theta_2'}\left(\int_\varepsilon^{\beta-\varepsilon}|z|^2dx\right)^{\frac12}+|\lambda|\left(\int_\varepsilon^{\beta-\varepsilon}|y_x|^2dx\right)^{\frac12}\right.\\&\qquad\left.+\lambda^{-2}\left(\int_\varepsilon^{\beta-\varepsilon}|f_x^3|^2dx\right)^{\frac12}\right]\left(\int_\varepsilon^{\beta-\varepsilon}|S_1(u,v,\eta)|^2dx\right)^{\frac12}\end{aligned}$$

and

$$\begin{aligned}&\left|\Re\left\{\int_0^L(\theta_2'\overline{v}+\theta_2\overline{v_x})y_xdx\right\}\right|=\left|\Re\left\{\int_\varepsilon^{\beta-\varepsilon}(\theta_2'\overline{v}+\theta_2\overline{v_x})y_xdx\right\}\right|\\&\qquad\leq\left[M_{\theta_2'}\left(\int_\varepsilon^{\beta-\varepsilon}|v|^2dx\right)^{\frac12}+\left(\int_\varepsilon^{\beta-\varepsilon}|v_x|^2dx\right)^{\frac12}\right]\left(\int_\varepsilon^{\beta-\varepsilon}|y_x|^2dx\right)^{\frac12}.\end{aligned}$$

From the above inequalities, Lemmas 2.4.1, 2.4.2 and the fact that y_x, z are uniformly bounded in $L^2(0,L)$ and $\|f_x^3\|_{L^2(0,L)}=o(1)$, we obtain

$$-\Re\left\{\int_0^L(\theta_2'\overline{z}+\theta_2\overline{z_x})S_b(u,v,\eta)dx\right\}=o(1)\ \text{ and }\ -\Re\left\{\int_0^L(\theta_2'\overline{v}+\theta_2\overline{v_x})y_xdx\right\}=o(1).\tag{2.4.27}$$

Inserting (2.4.27) in (2.4.25), then using the fact that v, z are uniformly bounded in $L^2(0,L)$ and $\|f^2\|_{L^2(0,L)} = o(1)$, $\|f^4\|_{L^2(0,L)} = o(1)$, we obtain

$$\int_0^L c(\cdot)\theta_2|z|^2dx = \int_0^L c(\cdot)\theta_2|v|^2dx + o(1).$$

Therefore, from the above estimation, Lemma 2.4.2 and the definition of $c(\cdot)$ and θ_2, we obtain the first estimation in (2.4.22). On the other hand, let us fix a cut-off function $\theta_3 \in C^1([0,L])$ (see Figure 2.3) such that $0 \le \theta_3(x) \le 1$, for all $x \in [0,L]$ and

$$\theta_3(x) = \begin{cases} 0 & \text{if} \quad x \in [0,\alpha] \cup [\beta - 2\varepsilon, L], \\ 1 & \text{if} \quad x \in [\alpha+\varepsilon, \beta - 3\varepsilon], \end{cases}$$

Now, multiplying (2.4.8) by $-\lambda^{-1}\theta_3\overline{z}$, integrating over $(0,L)$, then taking the imaginary part, we obtain

$$-\int_0^L \theta_3|z|^2dx + \Im\left\{\lambda^{-1}\int_0^L \theta_3 y_{xx}\overline{z}dx\right\} + \Im\left\{\lambda^{-1}\int_0^L c(\cdot)\theta_3 v\overline{z}dx\right\} = -\Im\left\{\lambda^{-3}\int_0^L \theta_3 f^4\overline{z}dx\right\}.$$

Using integration by parts in the above equation and the fact that $z(0) = z(L) = 0$, then using (2.4.26), we get

$$\begin{aligned}\int_0^L \theta_3|y_x|^2dx = \int_0^L \theta_3|z|^2dx + \Im\left\{\lambda^{-1}\int_0^L \theta_3' y_x\overline{z}dx\right\} - \Im\left\{\lambda^{-1}\int_0^L c(\cdot)\theta_3 v\overline{z}dx\right\} \\ - \Im\left\{\lambda^{-3}\int_0^L \theta_3\overline{f_x^3}y_xdx\right\} - \Im\left\{\lambda^{-3}\int_0^L \theta_3 f^4\overline{z}dx\right\}.\end{aligned} \tag{2.4.28}$$

From the definition of $c(\cdot)$ and θ_3, the first estimation of (2.4.22) and the fact that v and y_x are uniformly bounded in $L^2(0,L)$, we obtain

$$\begin{cases} \Im\left\{\lambda^{-1}\int_0^L \theta_3' y_x\overline{z}dx\right\} = \Im\left\{\lambda^{-1}\int_\alpha^{\beta-2\varepsilon} \theta_3' y_x\overline{z}dx\right\} = o\left(|\lambda|^{-1}\right), \\ -\Im\left\{\lambda^{-1}\int_0^L c(\cdot)\theta_3 v\overline{z}dx\right\} = -\Im\left\{c_0\lambda^{-1}\int_\alpha^{\beta-2\varepsilon} \theta_3 v\overline{z}dx\right\} = o\left(|\lambda|^{-1}\right). \end{cases} \tag{2.4.29}$$

Inserting (2.4.29) in (2.4.28), then using the fact that y_x, z are uniformly bounded in $L^2(0,L)$ and $\|f_x^3\|_{L^2(0,L)} = o(1)$, $\|f^4\|_{L^2(0,L)} = o(1)$, we get

$$\int_0^L \theta_3|y_x|^2dx = \int_0^L \theta_3|z|^2dx + o(|\lambda|^{-1}).$$

Finally, from the above estimation, the first estimation of (2.4.22) and the definition of θ_3, we obtain the second estimation in (2.4.22). The proof is thus complete. □

Lemma 2.4.4. $0 < \varepsilon < \min\left(\frac{\alpha}{2}, \frac{\beta-\alpha}{4}\right)$. Under the hypothesis (H), the solution $U = (u, v, y, z, \eta(\cdot,\rho))^\top \in D(\mathcal{A})$ of system (2.4.5)-(2.4.9) satisfies the following estimations

$$|v(\gamma)|^2 + |v(\beta-3\varepsilon)|^2 + a|u_x(\gamma)|^2 + a^{-1}\left|\left(S_1(u,v,\eta)\right)(\beta-3\varepsilon)\right|^2 = O(1), \tag{2.4.30}$$

$$|z(\gamma)|^2 + |z(\beta-3\varepsilon)|^2 + |y_x(\gamma)|^2 + |y_x(\beta-3\varepsilon)|^2 = O(1). \tag{2.4.31}$$

Proof. First, we fix a function $g_2 \in C^1([\beta-3\varepsilon,\gamma])$ such that

$$g_2(\beta-3\varepsilon) = -g_2(\gamma) = 1 \quad \text{and set} \quad \max_{x\in[\beta-3\varepsilon,\gamma]} |g_2(x)| = M_{g_2} \quad \text{and} \quad \max_{x\in[\beta-3\varepsilon,\gamma]} |g_2'(x)| = M_{g_2'}.$$

From (2.4.5), we deduce that

$$i\lambda u_x - v_x = \lambda^{-2} f_x^1. \tag{2.4.32}$$

Multiplying (2.4.32) and (2.4.6) by $2g_2\overline{v}$ and $2a^{-1}g_2\overline{S_b}(u,v,\eta)$ respectively, integrating over $(\beta-3\varepsilon,\gamma)$, using the definition of $c(\cdot)$ and $S_b(u,v,\eta)$, then taking the real part, we obtain

$$\Re\left\{2i\lambda\int_{\beta-3\varepsilon}^{\gamma} g_2 u_x\overline{v}dx\right\} - \int_{\beta-3\varepsilon}^{\gamma} g_2\left(|v|^2\right)_x dx = \Re\left\{2\lambda^{-2}\int_{\beta-3\varepsilon}^{\gamma} g_2 f_x^1\overline{v}dx\right\}$$

and

$$\begin{aligned}
&\Re\left\{2i\lambda\int_{\beta-3\varepsilon}^{\gamma} g_2 v\overline{u_x}dx\right\} + \Re\left\{\frac{2i\lambda}{a}\int_{\beta-3\varepsilon}^{\beta} g_2 v\left(\kappa_1\overline{v_x}+\kappa_2\overline{\eta_x}(\cdot,1)\right)dx\right\}\\
&-a^{-1}\int_{\beta-3\varepsilon}^{\beta} g_2\left(|S_1(u,v,\eta)|^2\right)_x dx - a\int_{\beta}^{\gamma} g_2\left(|u_x|^2\right)_x dx\\
&+\Re\left\{\frac{2c_0}{a}\int_{\beta-3\varepsilon}^{\beta} g_2 z\overline{S_1}(u,v,\eta)dx\right\} + \Re\left\{2c_0\int_{\beta}^{\gamma} g_2 z\overline{u_x}dx\right\}\\
&= \Re\left\{\frac{2}{a\lambda^2}\int_{\beta-3\varepsilon}^{\beta} g_2 f^2\overline{S_1}(u,v,\eta)dx\right\} + \Re\left\{\frac{2}{\lambda^2}\int_{\beta}^{\gamma} g_2 f^2\overline{u_x}dx\right\}.
\end{aligned}$$

Adding the above equations, then using integration by parts, we get

$$\begin{aligned}
&\left[-g_2|v|^2\right]_{\beta-3\varepsilon}^{\gamma} + \left[-a^{-1}g_2|S_1(u,v,\eta)|^2\right]_{\beta-3\varepsilon}^{\beta} + \left[-ag_2|u_x|^2\right]_{\beta}^{\gamma}\\
&= -\int_{\beta-3\varepsilon}^{\gamma} g_2'|v|^2dx - a^{-1}\int_{\beta-3\varepsilon}^{\beta} g_2'|S_1(u,v,\eta)|^2dx - a\int_{\beta}^{\gamma} g_2'|u_x|^2dx\\
&-\Re\left\{\frac{2i\lambda}{a}\int_{\beta-3\varepsilon}^{\beta} g_2 v\left(\kappa_1\overline{v_x}+\kappa_2\overline{\eta_x}(\cdot,1)\right)dx\right\} - \Re\left\{\frac{2c_0}{a}\int_{\beta-3\varepsilon}^{\beta} g_2 z\overline{S_1}(u,v,\eta)dx\right\}\\
&-\Re\left\{2c_0\int_{\beta}^{\gamma} g_2 z\overline{u_x}dx\right\} + \Re\left\{\frac{2}{\lambda^2}\int_{\beta-3\varepsilon}^{\gamma} g_2 f_x^1\overline{v}dx\right\} + \Re\left\{\frac{2}{a\lambda^2}\int_{\beta-3\varepsilon}^{\beta} g_2 f^2\overline{S_1}(u,v,\eta)dx\right\}\\
&+\Re\left\{\frac{2}{\lambda^2}\int_{\beta}^{\gamma} g_2 f^2\overline{u_x}dx\right\}.
\end{aligned}$$

Using the definition of g_2 and Cauchy-Schwarz inequality in the above equation, we obtain

$$
\begin{aligned}
&|v(\gamma)|^2 + |v(\beta-3\varepsilon)|^2 + a|u_x(\gamma)|^2 + a^{-1}\left|\left(S_1(u,v,\eta)\right)(\beta-3\varepsilon)\right|^2 + \mathcal{K}(\beta)\\
&\leq M_{g_2'}\left[\int_{\beta-3\varepsilon}^{\gamma}|v|^2dx + a^{-1}\int_{\beta-3\varepsilon}^{\beta}|S_1(u,v,\eta)|^2dx + a\int_{\beta}^{\gamma}|u_x|^2dx\right]\\
&+\frac{2|\lambda|M_{g_2}}{a}\left[\kappa_1\left(\int_{\beta-3\varepsilon}^{\beta}|v_x|^2dx\right)^{\frac12} + |\kappa_2|\left(\int_{\beta-3\varepsilon}^{\beta}|\eta_x(\cdot,1)|^2dx\right)^{\frac12}\right]\left(\int_{\beta-3\varepsilon}^{\beta}|v|^2dx\right)^{\frac12}\\
&+\frac{2|c_0|M_{g_2}}{a}\left(\int_{\beta-3\varepsilon}^{\beta}|S_1(u,v,\eta)|^2dx\right)^{\frac12}\left(\int_{\beta-3\varepsilon}^{\beta}|z|^2dx\right)^{\frac12} + 2|c_0|M_{g_2}\left(\int_{\beta}^{\gamma}|z|^2dx\right)^{\frac12}\left(\int_{\beta}^{\gamma}|u_x|^2dx\right)^{\frac12}\\
&+\frac{2M_{g_2}}{\lambda^2}\left(\int_{\beta-3\varepsilon}^{\gamma}|f_x^1|^2dx\right)^{\frac12}\left(\int_{\beta-3\varepsilon}^{\gamma}|v|^2dx\right)^{\frac12} + \frac{2M_{g_2}}{a\lambda^2}\left(\int_{\beta-3\varepsilon}^{\beta}|f^2|^2dx\right)^{\frac12}\left(\int_{\beta-3\varepsilon}^{\beta}|S_1(u,v,\eta)|^2dx\right)^{\frac12}\\
&+\frac{2M_{g_2}}{\lambda^2}\left(\int_{\beta}^{\gamma}|f^2|^2dx\right)^{\frac12}\left(\int_{\beta}^{\gamma}|u_x|^2dx\right)^{\frac12}.
\end{aligned}
$$

where $\mathcal{K}(\beta) = g_2(\beta)\left(a|u_x(\beta^+)|^2 - a^{-1}\left|\left(S_1(u,v,\eta)\right)(\beta^-)\right|^2\right)$. Moreover, since $S_b(u,v,\eta) \in H^1(0,L) \subset C([0,L])$, then we obtain

$$
\left|\left(S_1(u,v,\eta)\right)(\beta^-)\right|^2 = |au_x(\beta^+)|^2 \text{ and consequently } \mathcal{K}(\beta) = 0. \tag{2.4.33}
$$

Inserting (2.4.33) in the above inequality, then using Lemma 2.4.1 and the fact that u_x, v, z are uniformly bounded in $L^2(0,L)$ and $\|f_x^1\|_{L^2(0,L)} = o(1)$, $\|f^2\|_{L^2(0,L)} = o(1)$, we obtain (2.4.30). Next, from (2.4.7), we deduce that

$$
i\lambda y_x - z_x = \lambda^{-2}f_x^3. \tag{2.4.34}
$$

Multiplying (2.4.34) and (2.4.8) by $2g_2\overline{z}$ and $2g_2\overline{y_x}$ respectively, integrating over $(\beta-3\varepsilon,\gamma)$, using the definition of $c(\cdot)$, then taking the real part, we obtain

$$
\Re\left\{2i\lambda\int_{\beta-3\varepsilon}^{\gamma}g_2y_x\overline{z}dx\right\} - \int_{\beta-3\varepsilon}^{\gamma}g_2\left(|z|^2\right)_x dx = \Re\left\{2\lambda^{-2}\int_{\beta-3\varepsilon}^{\gamma}g_2f_x^3\overline{z}dx\right\} \tag{2.4.35}
$$

and

$$
\begin{aligned}
&\Re\left\{2i\lambda\int_{\beta-3\varepsilon}^{\gamma}g_2z\overline{y_x}dx\right\} - \int_{\beta-3\varepsilon}^{\gamma}g_2\left(|y_x|^2\right)_x dx - \Re\left\{2c_0\int_{\beta-3\varepsilon}^{\gamma}g_2v\overline{y_x}dx\right\}\\
&= \Re\left\{2\lambda^{-2}\int_{\beta-3\varepsilon}^{\gamma}g_2f^4\overline{y_x}dx\right\}.
\end{aligned} \tag{2.4.36}
$$

Adding (2.4.35) and (2.4.36), then using integration by parts, we obtain

$$
\begin{aligned}
\left[-g_2\left(|z|^2+|y_x|^2\right)\right]_{\beta-3\varepsilon}^{\gamma} = &-\int_{\beta-3\varepsilon}^{\gamma}g_2'(|z|^2+|y_x|^2)dx + \Re\left\{2c_0\int_{\beta-3\varepsilon}^{\gamma}g_2v\overline{y_x}dx\right\}\\
&+\Re\left\{2\lambda^{-2}\int_{\beta-3\varepsilon}^{\gamma}g_2f_x^3\overline{z}dx\right\} + \Re\left\{2\lambda^{-2}\int_{\beta-3\varepsilon}^{\gamma}g_2f^4\overline{y_x}dx\right\}.
\end{aligned}
$$

Using the definition of g_2 and Cauchy-Schwarz inequality in the above equation, we obtain

$$
\begin{aligned}
&|z(\gamma)|^2+|z(\beta-3\varepsilon)|^2+|y_x(\gamma)|^2+|y_x(\beta-3\varepsilon)|^2\\
&\leq M_{g_2'}\int_{\beta-3\varepsilon}^{\gamma}\left(|z|^2+|y_x|^2\right)dx+2|c_0|M_{g_2}\left(\int_{\beta-3\varepsilon}^{\gamma}|v|^2dx\right)^{\frac{1}{2}}\left(\int_{\beta-3\varepsilon}^{\gamma}|y_x|^2dx\right)^{\frac{1}{2}}\\
&+2\lambda^{-2}M_{g_2}\left[\left(\int_{\beta-3\varepsilon}^{\gamma}|f_x^3|^2dx\right)^{\frac{1}{2}}\left(\int_{\beta-3\varepsilon}^{\gamma}|z|^2dx\right)^{\frac{1}{2}}+\left(\int_{\beta-3\varepsilon}^{\gamma}|f^4|^2dx\right)^{\frac{1}{2}}\left(\int_{\beta-3\varepsilon}^{\gamma}|y_x|^2dx\right)^{\frac{1}{2}}\right].
\end{aligned}
$$

Finally, from the above inequality, the fact that v, y_x, z are uniformly bounded in $L^2(0,L)$ and $\|f_x^3\|_{L^2(0,L)}=o(1)$, $\|f^4\|_{L^2(0,L)}=o(1)$, we obtain (2.4.31). The proof is thus complete. □

Lemma 2.4.5. Let $h_2\in C^1([0,L])$ be a function. Under the hypothesis (H), the solution $U=(u,v,y,z,\eta(\cdot,\rho))^\top\in D(\mathcal{A})$ of system (2.4.5)-(2.4.9) satisfies the following equality

$$
\begin{aligned}
&\int_0^L h_2'\left(a^{-1}|S_b(u,v,\eta)|^2+|v|^2+|z|^2+|y_x|^2\right)dx\\
&-\left[h_2\left(a^{-1}|S_b(u,v,\eta)|^2+|y_x|^2\right)\right]_0^L-\Re\left\{2\int_0^L c(\cdot)h_2v\overline{y_x}dx\right\}\\
&+\Re\left\{\frac{2}{a}\int_0^L c(\cdot)h_2z\overline{S_b}(u,v,\eta)dx\right\}+\Re\left\{\frac{2i\lambda}{a}\int_0^\beta h_2v^n(\kappa_1\overline{v_x}+\kappa_2\overline{\eta_x}(\cdot,1))dx\right\}\\
&=\Re\left\{\frac{2}{\lambda^2}\int_0^L h_2\overline{f_x^1}vdx\right\}+\Re\left\{\frac{2}{a\lambda^2}\int_0^L h_2f^2\overline{S_b}(u,v,\eta)dx\right\}\\
&+\Re\left\{\frac{2}{\lambda^2}\int_0^L h_2\overline{f_x^3}zdx\right\}+\Re\left\{\frac{2}{\lambda^2}\int_0^L h_2f^4\overline{y_x}dx\right\}.
\end{aligned}
$$

Proof. See the proof of Lemma 2.3.5. □

Let $0<\varepsilon<\min\left(\frac{\alpha}{2},\frac{\beta-\alpha}{4}\right)$, we fix the cut-off functions $\theta_4,\theta_5\in C^1([0,L])$ (see Figure 2.4) such that $0\leq\theta_4(x)\leq 1$, $0\leq\theta_5(x)\leq 1$, for all $x\in[0,L]$ and

$$
\theta_4(x)=\begin{cases}1 & \text{if} \quad x\in[0,\alpha+\varepsilon],\\ 0 & \text{if} \quad x\in[\beta-3\varepsilon,L],\end{cases}\quad\text{and}\quad \theta_5(x)=\begin{cases}0 & \text{if} \quad x\in[0,\alpha+\varepsilon],\\ 1 & \text{if} \quad x\in[\beta-3\varepsilon,L],\end{cases}
$$

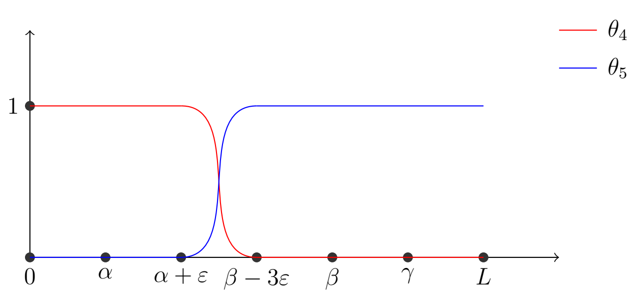

Figure 2.4: Geometric description of the functions θ_4 and θ_5.

Lemma 2.4.6. Let $0 < \varepsilon < \min\left(\frac{\alpha}{2}, \frac{\beta-\alpha}{4}\right)$. Under the hypothesis (H), the solution $U = (u, v, y, z, \eta(\cdot, \rho))^\top \in D(\mathcal{A})$ of system (2.4.5)-(2.4.9) satisfies the following estimations

$$\int_0^{\alpha+\varepsilon} |v|^2 dx + \int_0^{\alpha+\varepsilon} |y_x|^2 dx + \int_0^{\alpha+\varepsilon} |z|^2 dx = o(1), \tag{2.4.37}$$

$$a \int_\beta^L |u_x|^2 dx + \int_{\beta-3\varepsilon}^L |v|^2 dx + \int_{\beta-3\varepsilon}^L |y_x|^2 dx + \int_{\beta-3\varepsilon}^L |z|^2 dx = o(1). \tag{2.4.38}$$

Proof. First, using the result of Lemma 2.4.5 with $h_2 = x\theta_4$, we obtain

$$\begin{aligned}
&\int_0^{\alpha+\varepsilon} |v|^2 dx + \int_0^{\alpha+\varepsilon} |y_x|^2 dx + \int_0^{\alpha+\varepsilon} |z|^2 dx = -\, a^{-1} \int_0^{\alpha+\varepsilon} |S_1(u, v, \eta)|^2 dx \\
&- \int_{\alpha+\varepsilon}^{\beta-3\varepsilon} (\theta_4 + x\theta_4') \left(a^{-1}|S_1(u, v, \eta)|^2 + |v|^2 + |y_x|^2 + |z|^2\right) dx \\
&+ \Re\left\{2 \int_0^L xc(\cdot)\theta_4 v\overline{y_x} dx\right\} - \Re\left\{\frac{2}{a} \int_0^L xc(\cdot)\theta_4 z\overline{S_b}(u, v, \eta) dx\right\} \\
&- \Re\left\{\frac{2i\lambda}{a} \int_0^\beta x\theta_4 v \left(\kappa_1 \overline{v_x} + \kappa_2 \overline{\eta_x}(\cdot, 1)\right) dx\right\} + \Re\left\{\frac{2}{\lambda^2} \int_0^L x\theta_4 \overline{f_x^1} v dx\right\} \\
&+ \Re\left\{\frac{2}{a\lambda^2} \int_0^L x\theta_4 f^2 \overline{S_b}(u, v, \eta) dx\right\} + \Re\left\{\frac{2}{\lambda^2} \int_0^L x\theta_4 \overline{f_x^3} z dx\right\} + \Re\left\{\frac{2}{\lambda^2} \int_0^L x\theta_4 f^4 \overline{y_x} dx\right\}.
\end{aligned}$$

From the above equation, Lemmas 2.4.1-2.4.3 and the fact that v, y_x, z are uniformly bounded in $L^2(0, L)$ and $\|f_x^1\|_{L^2(0,L)} = o(1)$, $\|f_x^3\|_{L^2(0,L)} = o(1)$, $\|f^4\|_{L^2(0,L)} = o(1)$, we obtain

$$\begin{aligned}
&\int_0^{\alpha+\varepsilon} |v|^2 dx + \int_0^{\alpha+\varepsilon} |y_x|^2 dx + \int_0^{\alpha+\varepsilon} |z|^2 dx = \Re\left\{2 \int_0^L xc(\cdot)\theta_4 v\overline{y_x} dx\right\} \\
&- \Re\left\{\frac{2}{a} \int_0^L xc(\cdot)\theta_4 z\overline{S_b}(u, v, \eta) dx\right\} + \Re\left\{\frac{2}{a\lambda^2} \int_0^L x\theta_4 f^2 \overline{S_b}(u, v, \eta) dx\right\} \\
&- \Re\left\{\frac{2i\lambda}{a} \int_0^\beta x\theta_4 v \left(\kappa_1 \overline{v_x} + \kappa_2 \overline{\eta_x}(\cdot, 1)\right) dx\right\} + o(1).
\end{aligned} \tag{2.4.39}$$

Using the definition of $c(\cdot)$, $S_b(u,v,\eta)$ and θ_4, then using Cauchy-Schwarz inequality, we obtain

$$\left\{\begin{array}{l}
\left|\Re\left\{2\int_0^L xc(\cdot)\theta_4 v\overline{y_x}dx\right\}\right| = \left|\Re\left\{2c_0\int_\alpha^{\beta-3\varepsilon} x\theta_4 v\overline{y_x}dx\right\}\right| \\
\leq 2|c_0|(\beta-3\varepsilon)\left(\int_\alpha^{\beta-3\varepsilon}|v|^2dx\right)^{\frac12}\left(\int_\alpha^{\beta-3\varepsilon}|y_x|^2dx\right)^{\frac12}, \\
\\
\left|\Re\left\{\frac{2}{a}\int_0^L xc(\cdot)\theta_4 z\overline{S_b}(u,v,\eta)dx\right\}\right| = \left|\Re\left\{\frac{2c_0}{a}\int_\alpha^{\beta-3\varepsilon} x\theta_4 z\overline{S_1}(u,v,\eta)dx\right\}\right| \\
\leq \frac{2|c_0|}{a}(\beta-3\varepsilon)\left(\int_\alpha^{\beta-3\varepsilon}|z|^2dx\right)^{\frac12}\left(\int_\alpha^{\beta-3\varepsilon}|S_1(u,v,\eta)|^2dx\right)^{\frac12}, \\
\\
\left|\Re\left\{\frac{2}{a\lambda^2}\int_0^L x\theta_4 f^2\overline{S_b}(u,v,\eta)dx\right\}\right| = \left|\Re\left\{\frac{2}{a\lambda^2}\int_0^{\beta-3\varepsilon} x\theta_4 f^2\overline{S_1}(u,v,\eta)dx\right\}\right| \\
\leq \frac{2(\beta-3\varepsilon)}{a\lambda^2}\left(\int_0^{\beta-3\varepsilon}|f^2|^2dx\right)^{\frac12}\left(\int_0^{\beta-3\varepsilon}|S_1(u,v,\eta)|^2dx\right)^{\frac12}, \\
\\
\left|\Re\left\{\frac{2i\lambda}{a}\int_0^\beta x\theta_4 v\left(\kappa_1\overline{v_x}+\kappa_2\overline{\eta_x}(\cdot,1)\right)dx\right\}\right| = \left|\Re\left\{\frac{2i\lambda}{a}\int_0^{\beta-3\varepsilon} x\theta_4 v\left(\kappa_1\overline{v_x}+\kappa_2\overline{\eta_x}(\cdot,1)\right)dx\right\}\right| \\
\leq \frac{2|\lambda|(\beta-3\varepsilon)}{a}\left[\kappa_1\left(\int_0^{\beta-3\varepsilon}|v_x|^2dx\right)^{\frac12}+|\kappa_2|\left(\int_0^{\beta-3\varepsilon}|\eta_x(\cdot,1)|^2dx\right)^{\frac12}\right]\left(\int_0^{\beta-3\varepsilon}|v|^2dx\right)^{\frac12}.
\end{array}\right.$$

From the above inequalities, Lemmas 2.4.1-2.4.3 and the fact that v, y_x are uniformly bounded in $L^2(0,L)$ and $\|f^2\|_{L^2(0,L)} = o(1)$, we obtain

$$\left\{\begin{array}{l}
\Re\left\{2\int_0^L xc(\cdot)\theta_4 v\overline{y_x}dx\right\} = o(1), \\
-\Re\left\{\frac{2}{a}\int_0^L xc(\cdot)\theta_4 z\overline{S_b}(u,v,\eta)dx\right\} = o(|\lambda|^{-1}), \\
\Re\left\{\frac{2}{a\lambda^2}\int_0^L x\theta_4 f^2\overline{S_b}(u,v,\eta)dx\right\} = o(|\lambda|^{-3}), \\
-\Re\left\{\frac{2i\lambda}{a}\int_0^\beta x\theta_4 v\left(\kappa_1\overline{v_x}+\kappa_2\overline{\eta_x}(\cdot,1)\right)dx\right\} = o(1).
\end{array}\right. \tag{2.4.40}$$

Therefore, by inserting (2.4.40) in (2.4.39), we obtain (2.4.37). On the other hand, using the result of Lemma 2.4.5 with $h = (x-L)\theta_5$, then using the definition of $S_b(u,v,\eta)$ and θ_5, Lemmas 2.4.1-2.4.3, and the fact that u_x, v, y_x, z are uniformly bounded in $L^2(0,L)$ and $\|f_x^1\|_{L^2(0,L)} = o(1)$, $\|f^2\|_{L^2(0,L)} = o(1)$, $\|f_x^3\|_{L^2(0,L)} = o(1)$, $\|f^4\|_{L^2(0,L)} = o(1)$, we obtain

$$a\int_\beta^L |u_x|^2dx + \int_{\beta-3\varepsilon}^L |v|^2dx + \int_{\beta-3\varepsilon}^L |y_x|^2dx + \int_{\beta-3\varepsilon}^L |z|^2dx = \mathcal{I} + o(1), \tag{2.4.41}$$

where

$$\mathcal{I} := \Re\left\{2\int_0^L (x-L)c(\cdot)\theta_5 v\overline{y_x}dx\right\} - \Re\left\{2a^{-1}\int_0^L (x-L)c(\cdot)\theta_5 z\overline{S_b}(u,v,\eta)dx\right\}.$$

Moreover, using the definition of $c(\cdot)$, $S_b(u,v,\eta)$ and θ_5, we get

$$\begin{aligned}\mathcal{I} = &\ \Re\left\{2c_0\int_{\alpha+\varepsilon}^{\beta-3\varepsilon}(x-L)\theta_5 v\overline{y_x}dx\right\} + \Re\left\{2c_0\int_{\beta-3\varepsilon}^{\gamma}(x-L)v\overline{y_x}dx\right\}\\ &-\Re\left\{\frac{2c_0}{a}\int_{\alpha+\varepsilon}^{\beta-3\varepsilon}(x-L)\theta_5 z\overline{S_1}(u,v,\eta)dx\right\} - \Re\left\{2c_0\int_{\beta-3\varepsilon}^{\gamma}(x-L)z\overline{u_x}dx\right\}\\ &-\Re\left\{\frac{2c_0}{a}\int_{\beta-3\varepsilon}^{\beta}(x-L)z(\kappa_1\overline{v_x}+\kappa_2\overline{\eta_x}(\cdot,1))dx\right\}.\end{aligned}$$

Using Cauchy-Schwarz inequality, Lemmas 2.4.1-2.4.3 and the fact that z is uniformly bounded in $L^2(0,L)$, we obtain

$$\left\{\begin{aligned}&\Re\left\{2c_0\int_{\alpha+\varepsilon}^{\beta-3\varepsilon}(x-L)\theta_5 v\overline{y_x}dx\right\} = o(1),\\ &-\Re\left\{\frac{2c_0}{a}\int_{\alpha+\varepsilon}^{\beta-3\varepsilon}(x-L)\theta_5 z\overline{S_1}(u,v,\eta)dx\right\} = o(|\lambda|^{-1}),\\ &-\Re\left\{\frac{2c_0}{a}\int_{\beta-3\varepsilon}^{\beta}(x-L)z(\kappa_1\overline{v_x}+\kappa_2\overline{\eta_x}(\cdot,1))dx\right\} = o(|\lambda|^{-1}).\end{aligned}\right. \tag{2.4.42}$$

Inserting (2.4.42) in the above equation, we get

$$\mathcal{I} = \Re\left\{2c_0\int_{\beta-3\varepsilon}^{\gamma}(x-L)v\overline{y_x}dx\right\} - \Re\left\{2c_0\int_{\beta-3\varepsilon}^{\gamma}(x-L)z\overline{u_x}dx\right\} + o(1). \tag{2.4.43}$$

From (2.4.5) and (2.4.7), we deduce that

$$\overline{u_x} = i\lambda^{-1}\overline{v_x} + i\lambda^{-3}\overline{f_x^1} \quad \text{and} \quad \overline{y_x} = i\lambda^{-1}\overline{z_x} + i\lambda^{-3}\overline{f_x^3}. \tag{2.4.44}$$

Substituting (2.4.44) in (2.4.43), then using the fact that v, z are uniformly bounded in $L^2(0,L)$ and $\|f_x^1\|_{L^2(0,L)} = o(1)$, $\|f_x^3\|_{L^2(0,L)} = o(1)$, we obtain

$$\mathcal{I} = \Re\left\{\frac{2c_0 i}{\lambda}\int_{\beta-3\varepsilon}^{\gamma}(x-L)v\overline{z_x}dx\right\} - \Re\left\{\frac{2c_0 i}{\lambda}\int_{\beta-3\varepsilon}^{\gamma}(x-L)z\overline{v_x}dx\right\} + o(1).$$

Using integration by parts to the second term in the above equation, we obtain

$$\mathcal{I} = \Re\left\{\frac{2c_0 i}{\lambda}\int_{\beta-3\varepsilon}^{\gamma} z\overline{v}dx\right\} - \Re\left\{\frac{2c_0 i}{\lambda}\left[(x-L)\,z\overline{v}\right]_{\beta-3\varepsilon}^{\gamma}\right\} + o(1). \tag{2.4.45}$$

Furthermore, by using Cauchy-Schwarz inequality, we get

$$\left|\Re\left\{\frac{2ic_0}{\lambda}\int_{\beta-3\varepsilon}^{\gamma} z\overline{v}dx\right\}\right| \leq 2|c_0||\lambda|^{-1}\left(\int_{\beta-3\varepsilon}^{\gamma}|z|^2dx\right)^{\frac{1}{2}}\left(\int_{\beta-3\varepsilon}^{\gamma}|v|^2dx\right)^{\frac{1}{2}} \tag{2.4.46}$$

and

$$\begin{aligned}
&\left|\Re\left\{\frac{2ic_0}{\lambda}\left[(x-L)\,z\overline{v}\right]_{\beta-3\varepsilon}^{\gamma}\right\}\right| \\
&\leq 2|c_0||\lambda|^{-1}\left[(L-\gamma)\,|z(\gamma)||v(\gamma)|+(L-\beta+3\varepsilon)|z(\beta-3\varepsilon)|\,|v(\beta-3\varepsilon)|\right].
\end{aligned}\tag{2.4.47}$$

From Lemma 2.4.4, we deduce that

$$|v(\beta-3\varepsilon)|=O(1),\quad |v(\gamma)|=O(1),\quad |z(\beta-3\varepsilon)|=O(1)\quad\text{and}\quad |z(\gamma)|=O(1).\tag{2.4.48}$$

Using the fact that v, z are uniformly bounded in $L^2(0,L)$ in (2.4.46) and inserting (2.4.48) in (2.4.47), we obtain

$$\left\{\begin{aligned}
&\Re\left\{\frac{2c_0 i}{\lambda}\int_{\beta-3\varepsilon}^{\gamma} z\overline{v}dx\right\}=O\left(|\lambda|^{-1}\right)=o(1),\\
&-\Re\left\{\frac{2c_0 i}{\lambda}\left[(x-L)z\overline{v}\right]_{\beta-3\varepsilon}^{\gamma}\right\}=O\left(|\lambda|^{-1}\right)=o(1).
\end{aligned}\right.\tag{2.4.49}$$

Inserting (2.4.49) in (2.4.45), we get

$$\mathcal{I}=o(1).\tag{2.4.50}$$

Finally, inserting (2.4.50) in (2.4.41), we obtain (2.4.38). The proof is thus complete. ☐

Proof of Theorem 2.4.1. The proof of Theorem 2.4.1 is divided into three steps.
Step 1. From Lemmas 2.4.1-2.4.3, we obtain

$$\left\{\begin{aligned}
&\int_0^\beta |u_x|^2dx=o(\lambda^{-4}),\ \int_0^\beta\int_0^1|\eta_x(\cdot,\rho)|^2d\rho dx=o(\lambda^{-2}),\ \int_\varepsilon^{\beta-\varepsilon}|v|^2dx=o(1),\\
&\int_\alpha^{\beta-2\varepsilon}|z|^2dx=o(1)\text{ and }\int_{\alpha+\varepsilon}^{\beta-3\varepsilon}|y_x|^2dx=o(1).
\end{aligned}\right.\tag{2.4.51}$$

Step 2. From Lemma 2.4.6 and (2.4.51), we deduce that

$$\left\{\begin{aligned}
&\int_0^\varepsilon|v|^2dx=o(1),\ \int_0^{\alpha+\varepsilon}|y_x|^2dx=o(1),\ \int_0^\alpha|z|^2dx=o(1),\\
&\int_\beta^L|u_x|^2dx=o(1),\ \int_{\beta-\varepsilon}^L|v|^2dx=o(1),\ \int_{\beta-3\varepsilon}^L|y_x|^2dx=o(1)\text{ and }\int_{\beta-2\varepsilon}^L|z|^2dx=o(1).
\end{aligned}\right.$$

Step 3. According to **Step 1** and **Step 2**, we obtain $\|U\|_{\mathcal{H}}=o(1)$, which contradicts (2.4.3). Thus, (2.4.2) holds. Next, since conditions (2.4.1) and (2.4.2) are proved, then according to Theorem 1.3.7, the proof of Theorem 2.4.1 is achieved. The proof is thus complete. ☐

2.5 Conclusion

We have studied the stabilization of a locally coupled wave equations with non smooth localized viscoelastic damping of Kelvin-Voigt type and localized time delay. We proved the strong stability of the system by using Arendt-Batty criteria. Finally, we established a polynomial energy decay rate of order t^{-1}.

Chapter 3

Stability results of coupled wave models with locally memory in a past history framework via non-smooth coefficients on the interface

In this chapter, we investigate the stabilization of locally coupled wave equations with local viscoelastic damping of past history type acting only in one equation via non-smooth coefficients. First, using a general criteria of Arendt-Batty, we prove the strong stability of our system. Second, using a frequency domain approach combined with the multiplier method, we establish the exponential stability of the solution if the two waves have the same speed of propagation. In the case of different propagation speeds, we prove that the energy of our system decays polynomially with rate t^{-1}. Finally, we show the lack of exponential stability if the speeds of wave propagation are different with a global damping and a global coupling. This chapter is published in [6].

3.1 Introduction

In this chapter, we investigate the indirect stability of coupled elastic wave equations with localized past history damping. More precisely, we consider the following system:

$$\begin{cases} u_{tt} - \left(au_x - b(x)\displaystyle\int_0^\infty g(s)u_x(x,t-s)ds\right)_x + c(x)y_t & \\ = 0, & (x,t) \in (0,L)\times(0,\infty), \\ y_{tt} - y_{xx} - c(x)u_t = 0, & (x,t) \in (0,L)\times(0,\infty), \\ u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, & t > 0, \\ (u(x,-s), u_t(x,0)) = (u_0(x,s), u_1(x)), & (x,s) \in (0,L)\times(0,\infty), \\ (y(x,0), y_t(x,0)) = (y_0(x), y_1(x)), & x \in (0,L), \end{cases} \tag{3.1.1}$$

where L and a are positive real numbers. We suppose that there exist a non-zero constant c_0 and positive constants α, β, γ, and b_0 such that $0 < \alpha < \beta < \gamma < L$, and define

$$b(x) = \begin{cases} b_0, & x \in (0,\beta), \\ 0, & x \in (\beta, L), \end{cases} \tag{$b(\cdot)$}$$

$$c(x)=\begin{cases} c_0, & x\in(\alpha,\gamma),\\ 0, & x\in(0,\alpha)\cup(\gamma,L).\end{cases} \tag{$c(\cdot)$}$$

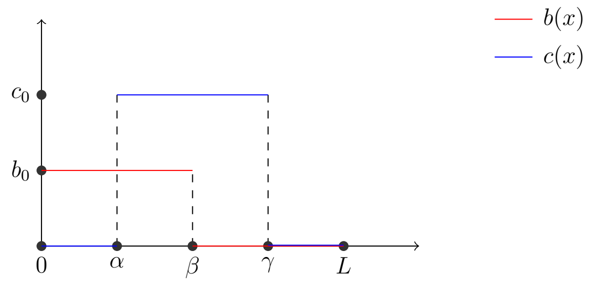

Figure 3.1: Geometric description of the functions $b(x)$ and $c(x)$.

The general integral term represents a history term with the relaxation function g that is supposed to satisfy the following hypotheses:

$$\begin{cases} g\in L^1([0,\infty))\cap C^1([0,\infty)) \text{ is a positive function such that}\\ g(0):=g_0>0,\quad \displaystyle\int_0^\infty g(s)ds:=\widetilde{g},\quad \widetilde{b}(x):=a-b(x)\widetilde{g}>0,\quad\text{and}\\ g'(s)\leq -mg(s),\quad \text{for some } m>0,\forall s\geq 0.\end{cases} \tag{H}$$

Remark that, the last assumption in (H) implies that

$$g(s)\leq g_0e^{-ms},\ \ \forall s\geq 0. \tag{3.1.2}$$

Moreover, from the definition of $b(\cdot)$ (see Figure 3.1), we have

$$\widetilde{b}(x):=a-b(x)\widetilde{g}=\begin{cases} \widetilde{b_0}:=a-b_0\widetilde{g}, & \text{in } (0,\beta),\\ a, & \text{in } (\beta,L).\end{cases} \tag{$\widetilde{b}(\cdot)$}$$

The notion of indirect damping mechanisms has been introduced by Russell in [100] and since this time, it retains the attention of many authors. In particular, the fact that only one equation of the coupled system is damped refers to the so-called class of "indirect" stabilization problems initiated and studied in [10, 11, 12] and further studied by many authors, see for instance [13, 78, 109] and the rich references therein. In 1996, Liu and Zheng in [81] studied the one-dimensional linear thermoviscoelastic system

$$\begin{cases} u_{tt}-\alpha u_{xx}+\displaystyle\int_0^\infty g(s)u_{xx}(\cdot,t-s)ds+\gamma c\theta_x=0, & \text{in } (0,\pi)\times(0,\infty),\\ \theta_t+\gamma u_{xt}-\theta_{xx}=0, & \text{in } (0,\pi)\times(0,\infty),\end{cases} \tag{3.1.3}$$

where $\alpha>0$, $\gamma\geq 0$ and $c>0$; and proved that the system is exponential stable. In 2008, Rivera *et al.* in [97] studied the stability of 1-dimensional Timoshenko system with past history acting only in one equation, they showed that the system is exponential stable if and only if the equations have the same wave speeds of propagation. In case that the wave speeds of the equations are different, they proved that the solution of the system decays polynomially to zero. In 2011, Guesmia in [54] studied the asymptotic stability of the following abstract linear dissipative integrodifferential equation with infinite memory

$$u_{tt}(t) + Au(t) - \int_0^\infty g(s)Bu(t-s)ds = 0, \ \forall\, t > 0. \tag{3.1.4}$$

where $A : D(A) \longmapsto H$ and $B : D(B) \longmapsto H$ are self-adjoint linear positive definite operators with domains $D(A) \subset D(B) \subset H$ such that the embeddings are dense and compact, H is a Hilbert space, and $g : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ is the convolution kernel function. He showed that the stability of the system holds for a relatively large class of convolution kernels and he provided a relation between the decay rate of the solution and the growth of the kernel at infinity. In 2012, Matos *et al.* in [83] studied the stability of the abstract coupled wave equations with past history, by considering:

$$\begin{cases} u_{tt} + \mathbb{A}_1 u - \displaystyle\int_0^\infty g(s)\mathbb{A}_2 u(t-s)ds + \beta v = 0, \\ v_{tt} + \mathbb{B}v + \beta u = 0, \quad \text{in} \quad L^2(\mathbb{R}^+, \mathcal{H}), \\ u(-t) = u_0(t), \quad t \geq 0, \\ v(0) = v_0, \\ u_t(0) = u_1, \quad v_t(0) = v_1, \end{cases} \tag{3.1.5}$$

where $\mathbb{A}_1$, $\mathbb{A}_2$ and $\mathbb{B}$ are self-adjoint positive-definite operators with the domain $D(\mathbb{A}_1) \subseteq D(\mathbb{A}_2) \subset \mathcal{H}$ and $D(\mathbb{B}) \subset \mathcal{H}$ with compact embeddings in $\mathcal{H}$, $g : [0,\infty) \longmapsto [0,\infty)$ is a smooth and summable function and β is a small positive constant. They showed that the abstract setting is not strong enough to produce exponential stability and they proved that the solution decays polynomially to zero. In 2014, Fatori *et al.* in [45] studied a fully hyperbolic thermoelastic Timoshenko system with past history where the thermal effects are given by Cattaneo's law, they established the exponential stability of the solution if and only if the coefficients of their System satisfy the next relation $\chi_0 := \left(\tau - \frac{\rho_1}{\rho_3\kappa}\right)\left(\rho_2 - \frac{b\rho_1}{\kappa}\right) - \frac{\tau\rho_1\delta^2}{\rho_3\kappa} = 0$. In the case $\chi_0 \neq 0$, they established optimal polynomial stability rates. In the same year, Santos *et al.* in [101] studied the stability of 1-dimensional Bresse system with past history acting in the shear angle displacement, they showed the exponential decay of the solution if and only if the wave speeds are the same. Otherwise, they showed that the Bresse system is polynomial stable with optimal decay rate. In 2014, Jin *et al.* in [69] studied the stability of the abstract Cauchy problem for a system of coupled equations with fading memory

$$\begin{cases} u_{tt}(t) + Au(t) - \displaystyle\int_0^t g(t-s)Au(s)ds + \alpha u(t) + \beta Bv(t) = f(u(t)), & t > 0, \\ v_{tt}(t) + Av(t) + \beta Bu(t) = 0, & t > 0, \\ u(0) = u_0, \quad u_t(0) = u_1, \\ v(0) = v_0, \quad v_t(0) = v_1, \end{cases} \tag{3.1.6}$$

where $\alpha \geq 0$, $\beta \geq 0$, A is a positive self-adjoint linear operator in a Hilbert space H, B is a symmetric linear operator in H, $f : D(\sqrt{A}) \longmapsto H$ denotes external forces, and g is the memory kernel. If $\beta > 0$, they established a polynomial decay rate of order t^{-1} of the full energy, while if $\beta = 0$, they proved the same decay rate but only on the energy of u. In 2015, Guesmia in [55] studied the asymptotic behavior for coupled abstract evolution equations with one infinite memory

$$\begin{cases} u_{tt}(t) + Au(t) - \displaystyle\int_0^\infty g(s)Bu(t-s)ds + \widetilde{B}v(t) = 0, & \forall\ t > 0, \\ v_{tt}(t) + \widetilde{A}v(t) + \widetilde{B}u(t) = 0, & \forall\ t > 0, \end{cases} \tag{3.1.7}$$

where $A : D(A) \longmapsto H$, $\widetilde{A} : D(\widetilde{A}) \longmapsto H$, and $B : D(B) \longmapsto H$ are self-adjoint linear positive definite operators with domains $D(A) \subset D(B) \subset H$ and $D(\widetilde{A}) \subset H$ such that the embeddings are dense and compact, $\widetilde{B} : H \longmapsto H$ is a self-adjoint bounded operator, H is a real Hilbert space, and $g : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ is the convolution kernel. He proved under a boundedness condition on the past history data that the stability of the system holds for convolution kernels having much weaker decay rates than the exponential one. In 2017, Alabau-Boussouira *et al.* in [15] studied the energy decay of the coupled wave equations

$$\begin{cases} u_{tt} - \Delta u + \rho(x, u_t) + \alpha(x)v_t = 0, & \text{in } \ \Omega \times (0, \infty), \\ v_{tt} - \Delta v - \alpha(x)u_t = 0, & \text{in } \ \Omega \times (0, \infty), \\ u = v = 0, & \text{on } \ \Gamma \times (0, \infty), \end{cases} \tag{3.1.8}$$

where Ω is a bounded subset of $\mathbb{R}^n$, Γ is a smooth boundary of Ω, $\rho(x, u_t)$ is a nonlinear damping, and $\alpha \in C(\overline{\Omega})$ is positive on a subset of positive measure (but may vanish in some parts of Ω). They proved that the total energy of the whole system (3.1.8) decays as fast as the damped single equation. Also, they gave a one-step general explicit decay formula for arbitrary nonlinearity. In 2018, Abdallah, Ghader and Wehbe in [1] studied the stability of a 1-dimensional Bresse system with infinite memory type control and /or with heat conduction given by Cattaneo's law acting in the shear angle displacement. In the absence of thermal effect, under the same speed propagation, they established the exponential stability of the system. However, in the case of different speed propagation, they established a polynomial energy decay rate. In 2018, Cavalcanti *et al.* in [32] studied the asymptotic stability of the multidimensional damped wave equation, by considering:

$$\rho(x)u_{tt} - \Delta u + \int_0^\infty g(s)\text{div}[a(x)\nabla u(\cdot, t-s)]ds + b(x)u_t = 0, \quad \text{in } \ \Omega \times (0, \infty), \tag{3.1.9}$$

where Ω is an open bounded and connected set of $\mathbb{R}^n$, $n \geq 2$, $\rho(x)$ is constant, $a(x) \geq 0$ is a smooth function, $b(x) \geq 0$ is a bounded function acting effectively in a region A of Ω where $a = 0$. Considering that the well-known geometric control condition (ω, T_0) holds and supposing that the relaxation function g is bounded by a function that decays exponentially to zero, they proved that the solution to the corresponding partial viscoelastic model decays exponentially to zero, even in the absence of the frictional dissipative effect. Moreover, they proved by removing the frictional damping term $b(x)u_t$ and by assuming that ρ is not constant, that localized viscoelastic damping is strong enough to assure that the system is exponentially stable. In 2019, Hao and Wang in [60] studied the stability of the abstract thermoelastic system with infinite memory

$$\begin{cases} u_{tt} + Au + Bu_t - \displaystyle\int_0^\infty g(s)Au(t-s)ds - A^\alpha\theta = 0, & t > 0, \\ \theta_t + kA^\beta\theta + A^\alpha u_t = 0, & t > 0, \\ u(-t) = u_0(t), & t \geq 0, \\ u_t(0) = u_1, \quad \theta(0) = \theta_0, \end{cases} \tag{3.1.10}$$

where $\alpha \in [0,1)$, $\beta \in (0,1]$, $A : D(A) \longmapsto H$ and $B : D(B) \longmapsto H$ are self-adjoint linear positive definite operators with domains $D(A) \subset D(B) \subset H$ such that the embeddings are dense and compact, and H is a real Hilbert space. They obtained the stability result and provided a direct relationship between the decay rate of the energy and the decay rate of kernel function g. In 2019, Hassan and Messaoudi in [61] studied the stability of an abstract class of weakly dissipative second-order system with finite memory

$$\begin{cases} u_{tt} + Au - \displaystyle\int_{-\infty}^{t} g(t-s)A^{\alpha}u(s)ds = 0, \ t > 0, \\ u(-t) = u_0(t), \ t \geq 0, \quad u_t(0) = u_1. \end{cases} \tag{3.1.11}$$

where $A : D(A) \subset H \longmapsto H$ is a positive definite self-adjoint operator on H, H is a real separable Hilbert space, g is the convolution kernel, and $\alpha \in [0,1]$. They established a new general decay rate for the solution of the system under approbritae conditions on the memory kernel g. In 2019, Jin *et al.* in [70] studied the stability of an abstract Cauchy problem for a system of coupled equations with one infinite memory, by considering:

$$\begin{cases} u_{tt}(t) + A_1u(t) - \displaystyle\int_0^{\infty} g(s)Au(t-s)ds + Bv(t) = 0, & t > 0, \\ v_{tt}(t) + A_2v(t) + Bu(t) = 0, & t > 0, \\ u(-t) = u_0(t), \ \forall t \geq 0, \quad u_t(0) = u_1, \\ v(0) = v_0, \quad v_t(0) = v_1, \end{cases} \tag{3.1.12}$$

where A, A_1 and A_2 are positive self-adjoint linear operators in a Hilbert space H, B is a positive self-adjoint bounded linear operator in H, and g is the memory kernel. They established a polynomial energy decay rate of order t^{-1}. In 2011, Almeida *et al.* in [16] studied the stability of coupled wave equations with past history effective only in one equation, by considering the following system:

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^{\infty} g(s)\Delta u(\cdot, t-s)ds + \alpha v = 0, \quad \text{in} \ \ \Omega \times (0,\infty), \\ v_{tt} - \Delta v + \alpha u = 0, \quad \text{in} \ \ \Omega \times (0,\infty), \\ u = v = 0, \quad \text{on} \ \ \Gamma \times (0,\infty) \\ u(x,0), v(x,0)) = (u_0(x), v_0(x)) \ \ \text{in} \ \ \Omega, \\ u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)) \ \ \text{in} \ \ \Omega, \end{cases} \tag{3.1.13}$$

where Ω is an open bounded set of $\mathbb{R}^n$ with smooth boundary Γ and $\alpha > 0$. They showed that the dissipation given by the memory effect is not strong enough to produce exponential decay. They proved that the solution of the system (3.1.13) decays polynomially with rate $t^{-\frac{1}{2}}$. Also, in 2020, Cordeiro *et al.* in [33] established the optimality of the decay rate.

But to the best of our knowledge, it seems that no result in the literature exists concerning the case of coupled wave equations with localized past history damping, especially in the absence of smoothness of the damping and coupling coefficients. The goal of the present chapter is to fill this gap by studying the stability of system (3.1.1).

This chapter is organized as follows: In Section 3.2, we prove the well-posedness of our system by using semigroup approach. In Section 3.3, following a general criteria of Arendt Batty, we show the strong stability of our system in the absence of the compactness of the resolvent. Next, in Section 3.4, by using the frequency domain approach combining with a specific multiplier method, we establish exponential stability of the solution if the two waves have the same speed of propagation (i.e. $a = 1$). In the case $a \neq 1$, we prove that the energy of our system decays polynomially with the rate t^{-1}. Finally, in Section 3.5, we show the lack of exponential stability in case that the speeds of wave propagation are different with a global damping and a global coupling (i.e., when $a \neq 1$ and $b(x) = c(x) = 1$).

3.2 Well-posedness of the system

In this section, we will establish the well-posedness of system (3.1.1) by using semigroup approach. To this aim, as in [35], we introduce the following auxiliary change of variable

$$\omega(x,s,t) := u(x,t) - u(x,t-s), \quad (x,s,t) \in (0,\beta) \times (0,\infty) \times (0,\infty). \tag{3.2.1}$$

Then, system (3.1.1) becomes

$$u_{tt} - \left(S_{\tilde{b}(\cdot)}(u,\omega)\right)_x + c(\cdot)y_t = 0, \qquad (x,t) \in (0,L) \times (0,\infty), \tag{3.2.2}$$

$$y_{tt} - y_{xx} - c(\cdot)u_t = 0, \qquad (x,t) \in (0,L) \times (0,\infty), \tag{3.2.3}$$

$$\omega_t(\cdot,s,t) + \omega_s(\cdot,s,t) - u_t = 0, \quad (x,s,t) \in (0,\beta) \times (0,\infty) \times (0,\infty), \tag{3.2.4}$$

where

$$S_{\tilde{b}(\cdot)}(u,\omega) := \begin{cases} S_{\widetilde{b_0}}(u,\omega) := \widetilde{b_0} u_x + b_0 \displaystyle\int_0^\infty g(s)\omega_x(x,s)ds, & \text{in } (0,\beta), \\ a u_x, & \text{in } (\beta, L). \end{cases} \tag{$S_{\tilde{b}(\cdot)}(u,\omega)$}$$

With the following boundary conditions

$$\begin{cases} u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, \quad t > 0, \\ \omega(\cdot,0,t) = 0, \quad (x,t) \in (0,\beta) \times (0,\infty), \\ \omega(0,s,t) = 0, \quad (s,t) \in (0,\infty) \times (0,\infty), \end{cases} \tag{3.2.5}$$

and the following initial conditions

$$\begin{cases} u(\cdot,-s) = u_0(\cdot,s), \qquad u_t(\cdot,0) = u_1(\cdot), \qquad (x,s) \in (0,L) \times (0,\infty), \\ y(\cdot,0) = y_0(\cdot), \qquad y_t(\cdot,0) = y_1(\cdot), \qquad x \in (0,L), \\ \omega_0(\cdot,s) := \omega(\cdot,s,0) = u_0(\cdot,0) - u_0(\cdot,s), \quad (x,s) \in (0,\beta) \times (0,\infty). \end{cases} \tag{3.2.6}$$

The energy of system (3.2.2)-(3.2.6) is given by

$$E(t) = E_1(t) + E_2(t) + E_3(t), \tag{3.2.7}$$

where

$$\begin{cases} E_1(t) = \dfrac{1}{2}\displaystyle\int_0^L \left(|u_t|^2 + \tilde{b}(\cdot)|u_x|^2\right) dx, \quad E_2(t) = \dfrac{1}{2}\displaystyle\int_0^L \left(|y_t|^2 + |y_x|^2\right) dx \quad \text{and} \\ E_3(t) = \dfrac{b_0}{2}\displaystyle\int_0^\beta \int_0^\infty g(s)|\omega_x(\cdot,s,t)|^2 ds dx. \end{cases}$$

Lemma 3.2.1. Under the hypotheses (H). Let $U = (u, u_t, y, y_t, \omega)$ be a regular solution of system (3.2.2)-(3.2.6). Then, the energy $E(t)$ satisfies the following estimation

$$\frac{d}{dt}E(t) = \frac{b_0}{2}\int_0^\beta\int_0^\infty g'(s)|\omega_x(\cdot,s,t)|^2dsdx. \tag{3.2.8}$$

Proof. First, multiplying (3.2.2) by $\overline{u_t}$, integrating over $(0, L)$, using integration by parts with (3.2.5), using the definition of $S_{\tilde{b}(\cdot)}(u,\omega)$, $\widetilde{b}(\cdot)$ and $c(\cdot)$, then taking the real part, we obtain

$$\frac{d}{dt}E_1(t) = -\Re\left\{b_0\int_0^\beta\int_0^\infty g(s)\omega_x(\cdot,s,t)\overline{u_{tx}}dsdx\right\} - \Re\left\{c_0\int_\alpha^\gamma y_t\overline{u_t}dx\right\}. \tag{3.2.9}$$

Now, multiplying (3.2.3) by $\overline{y_t}$, integrating over $(0, L)$, using the definition of $c(\cdot)$, then taking the real part, we get

$$\frac{d}{dt}E_2(t) = \Re\left\{c_0\int_\alpha^\gamma u_t\overline{y_t}dx\right\}. \tag{3.2.10}$$

Deriving (3.2.4) with respect to x, we obtain

$$\omega_{xt}(\cdot,s,t) + \omega_{xs}(\cdot,s,t) - u_{tx} = 0 \quad \text{in} \quad (0,\beta)\times(0,\infty)\times(0,\infty). \tag{3.2.11}$$

Multiplying (3.2.11) by $b_0 g(s)\overline{\omega_x}(\cdot,s,t)$, integrating over $(0,\beta)\times(0,\infty)$, then taking the real part, we get

$$\frac{d}{dt}E_3(t) = -\frac{b_0}{2}\int_0^\beta\int_0^\infty g(s)\frac{d}{ds}|\omega_x(\cdot,s,t)|^2dsdx + \Re\left\{b_0\int_0^\beta\int_0^\infty g(s)\overline{\omega_x}(\cdot,s,t)u_{tx}dsdx\right\}.$$

Using integration by parts with respect to s in the above equation with the help of (3.2.5) and the hypotheses (H), we obtain

$$\frac{d}{dt}E_3(t) = \frac{b_0}{2}\int_0^\beta\int_0^\infty g'(s)|\omega_x(\cdot,s,t)|^2dsdx + \Re\left\{b_0\int_0^\beta\int_0^\infty g(s)\overline{\omega_x}(\cdot,s,t)u_{tx}dsdx\right\}. \tag{3.2.12}$$

Finally, adding (3.2.9), (3.2.10) and (3.2.12), we obtain (3.2.8). The proof is thus complete. $\square$

Under the hypotheses (H) and from Lemma 3.2.1, system (3.2.2)-(3.2.6) is dissipative in the sense that its energy is non-increasing with respect to time (i.e. $E'(t) \leq 0$). Now, we define the following Hilbert space $\mathcal{H}$ by:

$$\mathcal{H} := \left(H_0^1(0,L)\times L^2(0,L)\right)^2 \times \mathcal{W}_g,$$

where

$$\mathcal{W}_g := L_g^2((0,\infty); H_L^1(0,\beta)) \quad \text{and} \quad H_L^1(0,\beta) := \left\{\widetilde{\omega}\in H^1(0,\beta) \mid \widetilde{\omega}(0) = 0\right\}.$$

The space $\mathcal{W}_g$ is a Hilbert space of $H_L^1(0,\beta)$-valued functions on $(0,\infty)$, equipped with the following inner product

$$(\omega^1,\omega^2)_{\mathcal{W}_g} := \int_0^\beta\int_0^\infty g(s)\omega_x^1\overline{\omega_x^2}dsdx, \quad \forall\,\omega^1,\omega^2\in\mathcal{W}_g.$$

The Hilbert space $\mathcal{H}$ is equipped with the following inner product

$$\begin{aligned}\left(U,U^1\right)_{\mathcal{H}} = \int_0^L \left(\widetilde{b}(\cdot)u_x\overline{u_x^1} + v\overline{v^1} + y_x\overline{y_x^1} + z\overline{z^1}\right)dx \\ + b_0\int_0^\beta\int_0^\infty g(s)\omega_x(\cdot,s)\overline{\omega_x^1}(\cdot,s)dsdx,\end{aligned} \tag{3.2.13}$$

where $U = (u,v,y,z,\omega(\cdot,s))^\top \in \mathcal{H}$ and $U^1 = (u^1,v^1,y^1,z^1,\omega^1(\cdot,s))^\top \in \mathcal{H}$. Now, we define the linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longmapsto \mathcal{H}$ by:

$$D(\mathcal{A}) = \left\{\begin{array}{c} U = (u,v,y,z,\omega(\cdot,s))^\top \in \mathcal{H} \mid y \in H^2(0,L)\cap H_0^1(0,L),\ v,z\in H_0^1(0,L) \\ \left(S_{\tilde{b}(\cdot)}(u,\omega)\right)_x \in L^2(0,L), \quad \omega_s(\cdot,s)\in\mathcal{W}_g, \quad \omega(\cdot,0)=0 \ \text{ in } \ (0,\beta)\end{array}\right\}$$

and

$$\mathcal{A}\begin{pmatrix} u \\ v \\ y \\ z \\ \omega(\cdot,s)\end{pmatrix} = \begin{pmatrix} v \\ \left(S_{\tilde{b}(\cdot)}(u,\omega)\right)_x - c(\cdot)z \\ z \\ y_{xx} + c(\cdot)v \\ -\omega_s(\cdot,s) + v\end{pmatrix}, \tag{3.2.14}$$

for all $U = (u,v,y,z,\omega(\cdot,s))^\top \in D(\mathcal{A})$.

Now, if $U = (u,u_t,y,y_t,\omega(\cdot,s))^\top$, then system (3.2.2)-(3.2.6) can be written as the following first order evolution equation

$$U_t = \mathcal{A}U, \quad U(0) = U_0, \tag{3.2.15}$$

where $U_0 = (u_0(\cdot,0),u_1,y_0,y_1,\omega_0(\cdot,s))^\top \in \mathcal{H}$.

Proposition 3.2.1. Under the hypotheses (H), the unbounded linear operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$.

Proof. For all $U = (u,v,y,z,\omega(\cdot,s))^\top \in D(\mathcal{A})$, from (3.2.13) and (3.2.14), we have

$$\begin{aligned}\Re(\mathcal{A}U,U)_{\mathcal{H}} = \Re\left\{\int_0^L \widetilde{b}(\cdot)v_x\overline{u_x}dx\right\} + \Re\left\{\int_0^L \left(S_{\tilde{b}(\cdot)}(u,\omega)\right)_x\overline{v}dx\right\} + \Re\left\{\int_0^L z_x\overline{y_x}dx\right\} \\ + \Re\left\{\int_0^L y_{xx}\overline{z}dx\right\} + \Re\left\{b_0\int_0^\beta\int_0^\infty g(s)v_x\overline{\omega_x}(\cdot,s)dsdx\right\} \\ - \Re\left\{b_0\int_0^\beta\int_0^\infty g(s)\omega_{xs}(\cdot,s)\overline{\omega_x}(\cdot,s)dsdx\right\}.\end{aligned}$$

Using integration by parts to the second and fourth terms in the above equation, then using the fact that $U \in D(\mathcal{A})$, we obtain

$$\Re(\mathcal{A}U,U)_{\mathcal{H}} = -\Re\left\{b_0\int_0^\beta\int_0^\infty g(s)\omega_{xs}(\cdot,s)\overline{\omega_x}(\cdot,s)dsdx\right\} = -\frac{b_0}{2}\int_0^\beta\int_0^\infty g(s)\frac{d}{ds}|\omega_x(\cdot,s)|^2dsdx.$$

Using integration by parts with respect to s in the above equation and the fact that $\omega(\cdot,0)=0$ in $(0,\beta)$ with the help of hypotheses (H), we get

$$\Re\,(\mathcal{A}U,U)_{\mathcal{H}}=\frac{b_0}{2}\int_0^\beta\int_0^\infty g'(s)|\omega_x(\cdot,s)|^2 ds dx\leq 0, \tag{3.2.16}$$

which implies that $\mathcal{A}$ is dissipative. Now, let us prove that $\mathcal{A}$ is maximal. To this aim, let $F=(f^1,f^2,f^3,f^4,f^5(\cdot,s))^\top\in\mathcal{H}$, we want to find $U=(u,v,y,z,\omega(\cdot,s))^\top\in D(\mathcal{A})$ unique solution of

$$-\mathcal{A}U=F. \tag{3.2.17}$$

Equivalently, we have the following system

$$\begin{aligned} -v &= f^1, && (3.2.18)\\ -\left(S_{\tilde{b}(\cdot)}(u,\omega)\right)_x+c(\cdot)z &= f^2, && (3.2.19)\\ -z &= f^3, && (3.2.20)\\ -y_{xx}-c(\cdot)v &= f^4, && (3.2.21)\\ \omega_s(\cdot,s)-v &= f^5(\cdot,s), && (3.2.22)\end{aligned}$$

with the following boundary conditions

$$u(0)=u(L)=y(0)=y(L)=0,\quad \omega(\cdot,0)=0\ \text{ in }\ (0,\beta)\ \text{ and }\ \omega(0,s)=0\ \text{ in }\ (0,\infty). \tag{3.2.23}$$

From (3.2.18), (3.2.22) and (3.2.23), we get

$$\omega(x,s)=\int_0^s f^5(x,\xi)d\xi-sf^1,\quad (x,s)\in(0,\beta)\times(0,\infty). \tag{3.2.24}$$

Since $v=-f^1\in H_0^1(0,L)$ and $f^5(\cdot,s)\in\mathcal{W}_g$, then from (3.2.22) and (3.2.24) we get $\omega_s(\cdot,s)\in\mathcal{W}_g$ and $\omega(\cdot,s)\in H_L^1(0,\beta)$ a.e. in $(0,\infty)$. Now, to obtain that $\omega(\cdot,s)\in\mathcal{W}_g$, it is sufficient to prove that $\displaystyle\int_0^\infty g(s)\|\omega_x(\cdot,s)\|^2_{L^2_{0,\beta}}ds<\infty$ where $\|\cdot\|_{L^2_{0,\beta}}:=\|\cdot\|_{L^2(0,\beta)}$. For this aim, let $\epsilon_1,\epsilon_2>0$, under the hypotheses (H), we have

$$\int_{\epsilon_1}^{\epsilon_2} g(s)\|\omega_x(\cdot,s)\|^2_{L^2_{0,\beta}}ds\leq-\frac{1}{m}\int_{\epsilon_1}^{\epsilon_2} g'(s)\|\omega_x(\cdot,s)\|^2_{L^2_{0,\beta}}ds. \tag{3.2.25}$$

Using integration by parts in (3.2.25), we obtain

$$\begin{aligned}\int_{\epsilon_1}^{\epsilon_2} g(s)\|\omega_x(\cdot,s)\|^2_{L^2_{0,\beta}}ds\leq\frac{1}{m}\Bigg[\int_{\epsilon_1}^{\epsilon_2} g(s)\frac{d}{ds}\left(\|\omega_x(\cdot,s)\|^2_{L^2_{0,\beta}}\right)ds+g(\epsilon_1)\|\omega_x(\cdot,\epsilon_1)\|^2_{L^2_{0,\beta}}\\ -g\left(\epsilon_2\right)\left\|\omega_x\left(\cdot,\epsilon_2\right)\right\|^2_{L^2_{0,\beta}}\Bigg].\end{aligned}$$

Moreover, from Young's inequality, we have

$$\begin{aligned}\frac{1}{m}\int_{\epsilon_1}^{\epsilon_2} g(s)\frac{d}{ds}\left(\|\omega_x(\cdot,s)\|^2_{L^2_{0,\beta}}\right)ds &= \frac{2}{m}\int_{\epsilon_1}^{\epsilon_2} g(s)\Re\left\{\int_0^\beta \omega_x(\cdot,s)\overline{\omega_{sx}}(\cdot,s)dx\right\}ds\\ &\leq \frac{1}{2}\int_{\epsilon_1}^{\epsilon_2} g(s)\|\omega_x(\cdot,s)\|^2_{L^2_{0,\beta}}ds \qquad (3.2.26)\\ &\quad+\frac{2}{m^2}\int_{\epsilon_1}^{\epsilon_2} g(s)\|\omega_{sx}(\cdot,s)\|^2_{L^2_{0,\beta}}ds.\end{aligned}$$

Inserting (3.2.26) in the above inequality, we get

$$\int_{\epsilon_1}^{\epsilon_2} g(s)\|\omega_x(\cdot,s)\|^2_{L^2_{0,\beta}}ds \leq \frac{4}{m^2}\int_{\epsilon_1}^{\epsilon_2} g(s)\|\omega_{sx}(\cdot,s)\|^2_{L^2_{0,\beta}}ds + \frac{2}{m}g(\epsilon_1)\|\omega_x(\cdot,\epsilon_1)\|^2_{L^2_{0,\beta}} - \frac{2}{m}g\left(\epsilon_2\right)\left\|\omega_x\left(\cdot,\epsilon_2\right)\right\|^2_{L^2_{0,\beta}}.$$

Using the fact that $\omega_s(\cdot,s)\in\mathcal{W}_g$, $\omega(\cdot,0)=0$ in $(0,\beta)$ and the hypotheses (H) in the above inequality, (in particular (3.1.2)) we obtain, as $\epsilon_1\to 0^+$ and $\epsilon_2\to\infty$, that

$$\int_0^\infty g(s)\|\omega_x(\cdot,s)\|^2_{L^2_{0,\beta}}ds < \infty,$$

and consequently, $\omega(\cdot,s)\in\mathcal{W}_g$. Now, see the definition of $S_{\tilde{b}(\cdot)}(u,\omega)$, substituting (3.2.18), (3.2.20) and (3.2.24) in (3.2.19) and (3.2.21), we get the following system

$$\left[S_{\tilde{b}(\cdot)}\left(u,\int_0^s f^5(\cdot,\xi)d\xi - sf^1\right)\right]_x + c(\cdot)f^3 = -f^2, \tag{3.2.27}$$

$$y_{xx} - c(\cdot)f^1 = -f^4, \tag{3.2.28}$$

$$u(0)=u(L)=y(0)=y(L)=0, \tag{3.2.29}$$

where

$$S_{\tilde{b}(\cdot)}\left(u,\int_0^s f^5(\cdot,\xi)d\xi - sf^1\right) = \begin{cases} \widetilde{b}_0 u_x + b_0\displaystyle\int_0^\infty g(s)\left(\int_0^s f^5_x(x,\xi)d\xi - sf^1_x\right), & \text{in } (0,\beta),\\ au_x, & \text{in } (\beta,L).\end{cases}$$

Let $(\phi,\psi)\in H^1_0(0,L)\times H^1_0(0,L)$. Multiplying (3.2.27) and (3.2.28) by $\overline{\phi}$ and $\overline{\psi}$ respectively, integrating over $(0,L)$, using formal integrations by parts, then using the definition of $S_{\tilde{b}(\cdot)}(u,\omega)$, $\widetilde{b}(\cdot)$ and $c(\cdot)$, we obtain

$$\begin{aligned}\int_0^L \widetilde{b}(\cdot)u_x\overline{\phi_x}dx &= \int_0^L f^2\overline{\phi}dx + c_0\int_\alpha^\gamma f^3\overline{\phi}dx\\ &- b_0\int_0^\beta\int_0^\infty g(s)\left(\int_0^s f^5_x(\cdot,\xi)d\xi - sf^1_x\right)\overline{\phi_x}dsdx\end{aligned} \tag{3.2.30}$$

and

$$\int_0^L y_x\overline{\psi_x}dx = \int_0^L f^4\overline{\psi}dx - c_0\int_\alpha^\gamma f^1\overline{\psi}dx. \tag{3.2.31}$$

Adding (3.2.30) and (3.2.31), we obtain

$$\mathcal{B}((u,y),(\phi,\psi)) = \mathcal{L}(\phi,\psi),\quad \forall(\phi,\psi)\in H^1_0(0,L)\times H^1_0(0,L), \tag{3.2.32}$$

where

$$\mathcal{B}((u,y),(\phi,\psi)) = \int_0^L \widetilde{b}(\cdot)u_x\overline{\phi_x}dx + \int_0^L y_x\overline{\psi_x}dx$$

and

$$\begin{aligned}\mathcal{L}(\phi,\psi) = &\int_0^L \left(f^2\overline{\phi} + f^4\overline{\psi}\right)dx + c_0\int_\alpha^\gamma\left(f^3\overline{\phi} - f^1\overline{\psi}\right)dx\\ &- b_0\int_0^\beta\int_0^\infty g(s)\left(\int_0^s f^5_x(\cdot,\xi)d\xi - sf^1_x\right)\overline{\phi_x}dsdx.\end{aligned}$$

It is easy to see that, $\mathcal{B}$ is a sesquilinear, continuous and coercive form on $(H_0^1(0,L)\times H_0^1(0,L))^2$ and $\mathcal{L}$ is an antilinear and continuous form on $H_0^1(0,L)\times H_0^1(0,L)$. Then, it follows by Lax-Milgram theorem that (3.2.32) admits a unique solution $(u,y)\in H_0^1(0,L)\times H_0^1(0,L)$. By taking test-functions $(\phi,\psi)\in(\mathcal{D}(0,L))^2$, we see that (3.2.27)-(3.2.29) hold in the distributional sense, from which we deduce that $y\in H^2(0,L)\cap H_0^1(0,L)$, while $(S_{\tilde{b}(\cdot)}(u,\omega))_x\in L^2(0,L)$. Consequently, $U\in D(\mathcal{A})$ is a unique solution of (3.2.17). Then, $\mathcal{A}$ is an isomorphism and since $\rho(\mathcal{A})$ is open set of $\mathbb{C}$ (see Theorem 1.1.13), we easily get $\mathcal{R}(\lambda I-\mathcal{A})=\mathcal{H}$ for a sufficiently small $\lambda>0$. This, together with the dissipativeness of $\mathcal{A}$, imply that $D(\mathcal{A})$ is dense in $\mathcal{H}$ and that $\mathcal{A}$ is m-dissipative in $\mathcal{H}$ (see Theorems 1.2.6, 1.2.9). The proof is thus complete. $\square$

According to Lumer-Philips theorem (see Theorem 1.2.8), Proposition 3.2.1 implies that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in $\mathcal{H}$ which gives the well-posedness of (3.2.15). Then, we have the following result:

Theorem 3.2.1. Under the hypotheses (H), for all $U_0\in\mathcal{H}$, system (3.2.15) admits a unique weak solution
$$U(x,s,t)=e^{t\mathcal{A}}U_0(x,s)\in C^0(\mathbb{R}_+,\mathcal{H}).$$
Moreover, if $U_0\in D(\mathcal{A})$, then the system (3.2.15) admits a unique strong solution
$$U(x,s,t)=e^{t\mathcal{A}}U_0(x,s)\in C^0(\mathbb{R}_+,D(\mathcal{A}))\cap C^1(\mathbb{R}_+,\mathcal{H}).$$

3.3 Strong Stability

This section is devoted to the proof of the strong stability of the C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$. To obtain the strong stability of the C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$, we use the theorem of Arendt and Batty in [24] (see Theorem 1.3.3).

Theorem 3.3.1. Assume that the hypotheses (H) hold. Then, the C_0-semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is strongly stable in $\mathcal{H}$; i.e., for all $U_0\in\mathcal{H}$, the solution of (3.2.15) satisfies
$$\lim_{t\to+\infty}\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}}=0.$$
According to Theorem 1.3.3, to prove Theorem 3.3.1, we need to prove that the operator $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A})\cap i\mathbb{R}$ is countable. The proof of Theorem 3.3.1 has been divided into the following two Lemmas.

Lemma 3.3.1. Under the hypotheeis (H), we have
$$\ker(i\lambda I-\mathcal{A})=\{0\},\quad\forall\lambda\in\mathbb{R}.$$
Proof. From Proposition 3.2.1, we have $0\in\rho(\mathcal{A})$. We still need to show the result for $\lambda\in\mathbb{R}^\star$. To this aim, suppose that there exists a real number $\lambda\neq 0$ and $U=(u,v,y,z,\omega(\cdot,s))^\top\in D(\mathcal{A})$ such that
$$\mathcal{A}U=i\lambda U. \tag{3.3.1}$$
Equivalently, we have the following system
$$\begin{aligned} v &= i\lambda u, && (3.3.2)\\ \left(S_{\tilde{b}(\cdot)}(u,\omega)\right)_x-c(\cdot)z &= i\lambda v, && (3.3.3)\\ z &= i\lambda y, && (3.3.4)\\ y_{xx}+c(\cdot)v &= i\lambda z, && (3.3.5)\\ -\omega_s(\cdot,s)+v &= i\lambda\omega(\cdot,s). && (3.3.6)\end{aligned}$$

From (3.2.16) and (3.3.1), we obtain

$$0 = \Re\left(i\lambda U, U\right)_{\mathcal{H}} = \Re\left(\mathcal{A}U, U\right)_{\mathcal{H}} = \frac{b_0}{2}\int_0^\beta \int_0^\infty g'(s)|\omega_x(\cdot, s)|^2 ds dx. \tag{3.3.7}$$

Thus, we have

$$\omega_x(\cdot, s) = 0 \ \text{ in } \ (0, \beta) \times (0, \infty). \tag{3.3.8}$$

From (3.3.8), we have

$$\omega(\cdot, s) = k(s) \ \text{ in } \ (0, \beta) \times (0, \infty), \tag{3.3.9}$$

where $k(s)$ is a constant depending on s. Then, from (3.3.9) and the fact that $\omega(\cdot, s) \in \mathcal{W}_g$ (i.e. $\omega(0, s) = 0$), we get

$$\omega(\cdot, s) = 0 \ \text{ in } \ (0, \beta) \times (0, \infty). \tag{3.3.10}$$

From (3.3.2), (3.3.6) and (3.3.10), we obtain

$$u = v = 0 \ \text{ in } \ (0, \beta). \tag{3.3.11}$$

Inserting (3.3.2) and (3.3.4) in (3.3.3) and (3.3.5), then using (3.3.8) together with the definition of $S_{\tilde{b}(\cdot)}(u, \omega)$ and $\widetilde{b}(\cdot)$, we obtain the following system

$$\begin{aligned} \lambda^2 u + (\widetilde{b}(\cdot)u_x)_x - c(\cdot)i\lambda y &= 0, \ \text{ in } \ (0, L), && (3.3.12)\\ \lambda^2 y + y_{xx} + c(\cdot)i\lambda u &= 0, \ \text{ in } \ (0, L), && (3.3.13)\\ u(0) = u(L) = y(0) = y(L) &= 0. && (3.3.14) \end{aligned}$$

From (3.3.11), (3.3.12), the definition of $c(\cdot)$ and (3.3.4), we obtain

$$y = z = 0 \ \text{ in } \ (\alpha, \beta). \tag{3.3.15}$$

Now, from (3.3.15) and the fact that $y \in C^1([0, L])$, we get

$$y(\alpha) = y_x(\alpha) = 0. \tag{3.3.16}$$

Next, from (3.3.13), (3.3.16) and the definition of $c(\cdot)$, we obtain the following system

$$\begin{aligned} \lambda^2 y + y_{xx} &= 0, \ \text{ in } \ (0, \alpha), && (3.3.17)\\ y(0) = y(\alpha) = y_x(\alpha) &= 0. && (3.3.18) \end{aligned}$$

Thus, from the above system and by using Holmgren uniqueness theorem, we obtain

$$y = 0 \ \text{ in } \ (0, \alpha). \tag{3.3.19}$$

Therefore, from (3.3.4) and (3.3.19), we obtain

$$y = z = 0 \ \text{ in } \ (0, \alpha). \tag{3.3.20}$$

According to the definition of $S_{\tilde{b}(\cdot)}(u, \omega)$ and the fact that $\widetilde{b_0} = a - b_0\widetilde{g}$, we obtain

$$S_{\tilde{b}(\cdot)}(u, \omega) = \begin{cases} au_x - b_0\widetilde{g}u_x + b_0\displaystyle\int_0^\infty g(s)\omega_x(x, s)ds, & \text{in } (0, \beta), \\ au_x, & \text{in } (\beta, L). \end{cases} \tag{3.3.21}$$

From (3.3.8), (3.3.11) and (3.3.21), we get

$$S_{\tilde{b}(\cdot)}(u,\omega) = au_x \text{ in } (0,L) \text{ and consequently } \left(S_{\tilde{b}(\cdot)}(u,\omega)\right)_x = au_{xx} \text{ in } (0,L). \tag{3.3.22}$$

Thus, from (3.3.22) and the fact that $U \in D(\mathcal{A})$, we obtain

$$u_{xx} \in L^2(0,L) \text{ and consequently } u \in C^1([0,L]). \tag{3.3.23}$$

Now, from (3.3.11), (3.3.15), (3.3.23) and the fact that $y \in C^1([0,L])$, we obtain

$$u(\beta) = u_x(\beta) = y(\beta) = y_x(\beta) = 0. \tag{3.3.24}$$

Next, from the definition of $\widetilde{b}(\cdot)$ and $c(\cdot)$, the System (3.3.12)-(3.3.13) can be written in (β,γ) as the following system

$$\begin{aligned} \lambda^2 u + au_{xx} - c_0 i\lambda y &= 0, \text{ in } (\beta,\gamma), && (3.3.25)\\ \lambda^2 y + y_{xx} + c_0 i\lambda u &= 0, \text{ in } (\beta,\gamma), && (3.3.26)\\ u(\beta) = u_x(\beta) = y(\beta) = y_x(\beta) &= 0. && (3.3.27) \end{aligned}$$

Let $V = (u, u_x, y, y_x)^\top$, then system (3.3.25)-(3.3.27) can be written as the following

$$V_x = BV, \quad V(\beta) = 0. \tag{3.3.28}$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -a^{-1}\lambda^2 & 0 & a^{-1}i\lambda c_0 & 0 \\ 0 & 0 & 0 & 1 \\ -i\lambda c_0 & 0 & -\lambda^2 & 0 \end{pmatrix}.$$

The solution of the differential equation (3.3.28) is given by

$$V(x) = e^{B(x-\beta)}V(\beta). \tag{3.3.29}$$

Thus, from (3.3.29) and the fact that $V(\beta) = 0$, we get

$$V = 0 \text{ in } (\beta,\gamma) \text{ and consequently } u = u_x = y = y_x = 0 \text{ in } (\beta,\gamma). \tag{3.3.30}$$

So, from (3.3.2), (3.3.4) and (3.3.30), we get

$$u = v = 0 \text{ in } (\beta,\gamma) \quad \text{and} \quad y = z = 0 \text{ in } (\beta,\gamma). \tag{3.3.31}$$

Now, from (3.3.30) and the fact that $u, y \in C^1([0,L])$, we obtain

$$u(\gamma) = u_x(\gamma) = y(\gamma) = y_x(\gamma) = 0. \tag{3.3.32}$$

Next, from the definition of $\widetilde{b}(\cdot)$ and $c(\cdot)$, the system (3.3.12)-(3.3.13) can be written in (γ, L) as the following system

$$\begin{aligned} \lambda^2 u + au_{xx} &= 0, \text{ in } (\gamma,L), && (3.3.33)\\ \lambda^2 y + y_{xx} &= 0, \text{ in } (\gamma,L), && (3.3.34)\\ u(L) = u(\gamma) = u_x(\gamma) &= 0, && (3.3.35)\\ y(L) = y(\gamma) = y_x(\gamma) &= 0. && (3.3.36) \end{aligned}$$

From the above system and by using Holmgren uniqueness theorem, we deduce that

$$u = y = 0 \quad \text{in} \quad (\gamma, L). \tag{3.3.37}$$

Thus, from (3.3.2), (3.3.4) and (3.3.37), we obtain

$$u = v = 0 \quad \text{in} \quad (\gamma, L) \quad \text{and} \quad y = z = 0 \quad \text{in} \quad (\gamma, L). \tag{3.3.38}$$

Finally, from (3.3.10) (3.3.11), (3.3.15), (3.3.20), (3.3.31) and (3.3.38), we obtain

$$U = 0. \tag{3.3.39}$$

The proof is thus complete. □

Lemma 3.3.2. Under the hypotheses (H), for all $\lambda \in \mathbb{R}$, we have

$$\mathcal{R}(i\lambda I - \mathcal{A}) = \mathcal{H}.$$

Proof. From Proposition 3.2.1, we have $0 \in \rho(\mathcal{A})$. We still need to show the result for $\lambda \in \mathbb{R}^\star$. For this aim, let $F = (f^1, f^2, f^3, f^4, f^5(\cdot, s))^\top \in \mathcal{H}$, we want to find $U = (u, v, y, z, \omega(\cdot, s))^\top \in D(\mathcal{A})$ solution of

$$(i\lambda I - \mathcal{A})U = F. \tag{3.3.40}$$

Equivalently, we have the following system

$$\begin{aligned}
i\lambda u - v &= f^1, && (3.3.41)\\
i\lambda v - \left(S_{\tilde{b}(\cdot)}(u, \omega)\right)_x + c(\cdot) z &= f^2, && (3.3.42)\\
i\lambda y - z &= f^3, && (3.3.43)\\
i\lambda z - y_{xx} - c(\cdot) v &= f^4, && (3.3.44)\\
i\lambda \omega(\cdot, s) + \omega_s(\cdot, s) - v &= f^5(\cdot, s), && (3.3.45)
\end{aligned}$$

with the following boundary conditions

$$u(0) = u(L) = y(0) = y(L) = 0, \quad \omega(\cdot, 0) = 0 \quad \text{in} \quad (0, \beta) \quad \text{and} \quad \omega(0, s) = 0 \quad \text{in} \quad (0, \infty). \tag{3.3.46}$$

From (3.3.41), (3.3.45) and (3.3.46), we have

$$\omega(x, s) = \frac{1}{i\lambda}(i\lambda u - f^1)(1 - e^{-i\lambda s}) + \int_0^s f^5(x, \xi) e^{i\lambda(\xi - s)} d\xi, \quad (x, s) \in (0, \beta) \times (0, \infty). \tag{3.3.47}$$

Due the definition of $S_{\tilde{b}(\cdot)}(u, \omega)$, inserting (3.3.41), (3.3.43) and (3.3.47) in (3.3.42) and (3.3.44), we obtain the following system

$$\left\{\begin{aligned}
-\lambda^2 u - \left(S_{\tilde{b}(\cdot)}(u, \omega)\right)_x + i\lambda c(\cdot) y &= F_1 := f^2 + c(\cdot) f^3 + i\lambda f^1,\\
-\lambda^2 y - y_{xx} - i\lambda c(\cdot) u &= F_2 := f^4 - c(\cdot) f^1 + i\lambda f^3,\\
u(0) = u(L) = y(0) = y(L) &= 0,
\end{aligned}\right. \tag{3.3.48}$$

where here $S_{\tilde{b}(\cdot)}(u, \omega)$ takes the form

$$S_{\tilde{b}(\cdot)}(u, \omega) = \left\{\begin{array}{ll}
\widehat{b_0} u_x + \dfrac{b_0}{i\lambda} \displaystyle\int_0^\infty g(s)(1 - e^{-i\lambda s}) f^1_x ds + b_0 \int_0^\infty g(s) \int_0^s f^5_x(x, \xi) e^{i\lambda(\xi - s)} d\xi ds, & \text{in } (0, \beta),\\
a u_x, & \text{in } (\beta, L),
\end{array}\right.$$

and

$$\widehat{b_0} := a - b_0 \int_0^\infty g(s) e^{-i\lambda s} ds.$$

Let $(\phi, \psi) \in H_0^1(0, L) \times H_0^1(0, L)$. Multiplying the first equation of (3.3.48) and the second equation of (3.3.48) by $\overline{\phi}$ and $\overline{\psi}$ respectively, integrating over $(0, L)$, then using formal integrations by parts, we obtain

$$\begin{aligned} &-\lambda^2 \int_0^L u\overline{\phi} dx + \int_0^L \widehat{b}(\cdot) u_x \overline{\phi_x} dx + \frac{b_0}{i\lambda} \int_0^\beta \int_0^\infty g(s)(1 - e^{-i\lambda s}) f_x^1 \overline{\phi_x} ds dx \\ &+ b_0 \int_0^\beta \int_0^\infty g(s) \int_0^s e^{i\lambda(\xi - s)} f_x^5(\cdot, \xi) \overline{\phi_x} d\xi ds dx + i\lambda c_0 \int_\alpha^\gamma y\overline{\phi} dx = \int_0^L F_1 \overline{\phi} dx \end{aligned} \tag{3.3.49}$$

and

$$-\lambda^2 \int_0^L y\overline{\psi} dx + \int_0^L y_x \overline{\psi_x} dx - i\lambda c_0 \int_\alpha^\gamma u\overline{\psi} dx = \int_0^L F_2 \overline{\psi} dx, \tag{3.3.50}$$

where

$$\widehat{b}(x) = \begin{cases} \widehat{b_0}, & x \in (0, \beta), \\ a, & x \in (\beta, L). \end{cases}$$

Adding (3.3.49) and (3.3.50), we get

$$\mathcal{B}((u, y), (\phi, \psi)) = \mathcal{L}(\phi, \psi), \quad \forall (\phi, \psi) \in \mathbb{V} := H_0^1(0, L) \times H_0^1(0, L), \tag{3.3.51}$$

where

$$\mathcal{B}((u, y), (\phi, \psi)) = \mathcal{B}_1((u, y), (\phi, \psi)) + \mathcal{B}_2((u, y), (\phi, \psi))$$

with

$$\begin{cases} \mathcal{B}_1((u, y), (\phi, \psi)) = \displaystyle\int_0^L \widehat{b}(\cdot) u_x \overline{\phi_x} dx + \int_0^L y_x \overline{\psi_x} dx, \\ \mathcal{B}_2((u, y), (\phi, \psi)) = -\lambda^2 \displaystyle\int_0^L (u\overline{\phi} + y\overline{\psi}) dx - i\lambda c_0 \int_0^L (u\overline{\psi} - y\overline{\phi}) dx \end{cases} \tag{3.3.52}$$

and

$$\begin{aligned} \mathcal{L}(\phi, \psi) = &\int_0^L (F_1 \overline{\phi} + F_2 \overline{\psi}) dx - \frac{b_0}{i\lambda} \int_0^\beta \int_0^\infty g(s)(1 - e^{-i\lambda s}) f_x^1 \overline{\phi_x} ds dx \\ &- b_0 \int_0^\beta \int_0^\infty g(s) \left(\int_0^s e^{i\lambda(\xi - s)} f_x^5(\cdot, \xi) d\xi \right) \overline{\phi_x} ds dx. \end{aligned}$$

Let $\mathbb{V}'$ be the dual space of $\mathbb{V}$. Let us define the following operators

$$\begin{array}{rcll} \mathbb{B}: & \mathbb{V} & \longmapsto & \mathbb{V}' \\ & (u, y) & \longmapsto & \mathbb{B}(u, y) \end{array} \quad \text{and} \quad \begin{array}{rcll} \mathbb{B}_i: & \mathbb{V} & \longmapsto & \mathbb{V}' \\ & (u, y) & \longmapsto & \mathbb{B}_i(u, y) \end{array}, \quad i \in \{1, 2\}, \tag{3.3.53}$$

such that

$$\begin{cases} (\mathbb{B}(u, y))(\phi, \psi) = \mathcal{B}((u, y), (\phi, \psi)), \quad \forall (\phi, \psi) \in \mathbb{V}, \\ (\mathbb{B}_i(u, y))(\phi, \psi) = \mathcal{B}_i((u, y), (\phi, \psi)), \quad \forall (\phi, \psi) \in \mathbb{V}, \ i \in \{1, 2\}. \end{cases} \tag{3.3.54}$$

We need to prove that the operator $\mathbb{B}$ is an isomorphism. For this aim, we divide the proof into three steps:

Step 1. In this step, we want to prove that the operator $\mathbb{B}_1$ is an isomorphism. For this aim, it is easy to see that $\mathcal{B}_1$ is sesquilinear, continuous and coercive form on $\mathbb{V}$. Then, from (3.3.54) and Lax-Milgram theorem, the operator $\mathbb{B}_1$ is an isomorphism.

Step 2. In this step, we want to prove that the operator $\mathbb{B}_2$ is compact. For this aim, from (3.3.52) and (3.3.54) we have

$$|\mathcal{B}_2((u,y),(\phi,\psi))| \lesssim \|(u,y)\|_{(L^2(0,L))^2}\|(\phi,\psi)\|_{(L^2(0,L))^2}, \tag{3.3.55}$$

and consequently, using the compact embedding from $\mathbb{V}$ into $\left(L^2(0,L)\right)^2$, we deduce that $\mathbb{B}_2$ is a compact operator.

Therefore, from the above steps, we obtain that the operator $\mathbb{B} = \mathbb{B}_1 + \mathbb{B}_2$ is a Fredholm operator of index zero. Now, following Fredholm alternative, we still need to prove that the operator $\mathbb{B}$ is injective to obtain that the operator $\mathbb{B}$ is an isomorphism.

Step 3. In this step, we want to prove that the operator $\mathbb{B}$ is injective (i.e. $\ker(\mathbb{B}) = \{0\}$). For this aim, let $(\widetilde{u},\widetilde{y}) \in \ker(\mathbb{B})$ which gives

$$\mathcal{B}((\widetilde{u},\widetilde{y}),(\phi,\psi)) = 0, \quad \forall(\phi,\psi) \in \mathbb{V}.$$

Equivalently, we have

$$\int_0^L \widehat{b}(\cdot)\widetilde{u}_x\overline{\phi_x}dx + \int_0^L \widetilde{y}_x\overline{\psi_x}dx - \lambda^2\int_0^L (\widetilde{u}\overline{\phi} + \widetilde{y}\overline{\psi})dx - i\lambda\int_0^L c(\cdot)(\widetilde{u}\overline{\psi} - \widetilde{y}\overline{\phi})dx = 0, \quad \forall(\phi,\psi) \in \mathbb{V}.$$

Thus, we find that

$$\begin{cases} -\lambda^2\widetilde{u} - (\widehat{b}(\cdot)\widetilde{u}_x)_x + i\lambda c(\cdot)\widetilde{y} = 0, \\ -\lambda^2\widetilde{y} - \widetilde{y}_{xx} - i\lambda c(\cdot)\widetilde{u} = 0, \\ \widetilde{u}(0) = \widetilde{u}(L) = \widetilde{y}(0) = \widetilde{y}(L) = 0. \end{cases}$$

Therefore, the vector $\widetilde{U}$ defined by

$$\widetilde{U} = (\widetilde{u}, i\lambda\widetilde{u}, \widetilde{y}, i\lambda\widetilde{y}, (1-e^{-i\lambda s})\widetilde{u})^\top$$

belongs to $D(\mathcal{A})$ and satisfies

$$i\lambda\widetilde{U} - \mathcal{A}\widetilde{U} = 0,$$

and consequently $\widetilde{U} \in \ker(i\lambda I - \mathcal{A})$. Then, according to Lemma 3.3.1, we obtain $\widetilde{U} = 0$ and consequently $\widetilde{u} = \widetilde{y} = 0$ and $\ker(\mathbb{B}) = \{0\}$.

Finally, from Step 3 and Fredholm alternative, we deduce that the operator $\mathbb{B}$ is isomorphism. It is easy to see that the operator $\mathcal{L}$ is a antilinear and continuous form on $\mathbb{V}$. Consequently, (3.3.51) admits a unique solution $(u,y) \in \mathbb{V}$. By using the classical elliptic regularity, we deduce that $U \in D(\mathcal{A})$ is a unique solution of (3.3.40). The proof is thus complete. $\square$

Proof of Theorem 3.3.1. From Lemma 3.3.1, we obtain the the operator $\mathcal{A}$ has no pure imaginary eigenvalues (i.e. $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset$). Moreover, from Lemma 3.3.2 and with the help of the closed graph theorem of Banach, we deduce that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Therefore, according to Theorem 1.3.3, we get that the C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable. The proof is thus complete. $\square$

Remark 3.3.1. We mention [8] for a direct approach of the strong stability of Timoshenko system in the absence of compactness of the resolvent.

3.4 Exponential and Polynomial Stability

In this section, under the hypotheses (H), we show the influence of the ratio of the wave propagation speed on the stability of system (3.2.2)-(3.2.6). Our main result in this part are the following theorems.

Theorem 3.4.1. Assume that $a = 1$ and the hypotheses (H) hold. Then, the C_0-semigroup $e^{t\mathcal{A}}$ is exponentially stable; i.e., for all $U_0 \in \mathcal{H}$, there exist constants $M \geq 1$ and $\epsilon > 0$ independent of U_0 such that

$$\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} \leq Me^{-\epsilon t}\|U_0\|_{\mathcal{H}}, \ t > 0.$$

Theorem 3.4.2. Assume that $a \neq 1$ and the hypotheses (H) hold. Then, for all $U_0 \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of U_0 such that

$$E(t) \leq \frac{C}{t}\|U_0\|^2_{D(\mathcal{A})}, \quad t > 0.$$

Since $i\mathbb{R} \subset \rho(\mathcal{A})$ (see Section 3.3), according to Theorem 1.3.6 and Theorem 1.3.7, to prove Theorem 3.4.1 and Theorem 3.4.2, we still need to prove the following condition

$$\limsup_{\lambda\in\mathbb{R},\ |\lambda|\to\infty} \frac{1}{|\lambda|^\ell}\left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty, \quad \text{with} \quad \begin{cases} \ell = 0 & \text{for Theorem 3.4.1,} \\ \ell = 2 & \text{for Theorem 3.4.2.} \end{cases} \tag{H$_1$}$$

We will prove condition (H$_1$) by a contradiction argument. For this purpose, suppose that (H$_1$) is false, then there exists $\left\{(\lambda^n, U^n := (u^n, v^n, y^n, z^n, \omega^n(\cdot,s))^\top)\right\}_{n\geq 1} \subset \mathbb{R}^* \times D(\mathcal{A})$ with

$$|\lambda^n| \to \infty \text{ as } n \to \infty \quad \text{and} \quad \|U^n\|_{\mathcal{H}} = \|(u^n, v^n, y^n, z^n, \omega^n(\cdot,s))^\top\|_{\mathcal{H}} = 1, \forall n \geq 1, \tag{3.4.1}$$

such that

$$(\lambda^n)^\ell(i\lambda^n I - \mathcal{A})U^n = F^n := (f^{1,n}, f^{2,n}, f^{3,n}, f^{4,n}, f^{5,n}(\cdot,s))^\top \to 0 \quad \text{in} \quad \mathcal{H}, \quad \text{as } n \to \infty. \tag{3.4.2}$$

For simplicity, we drop the index n. Equivalently, from (3.4.2), we have

$$\begin{aligned}
i\lambda u - v &= \lambda^{-\ell}f^1 \to 0 && \text{in} \quad H_0^1(0,L), && (3.4.3)\\
i\lambda v - \left(S_{\tilde{b}(\cdot)}(u,\omega)\right)_x + c(\cdot)z &= \lambda^{-\ell}f^2 \to 0 && \text{in} \quad L^2(0,L), && (3.4.4)\\
i\lambda y - z &= \lambda^{-\ell}f^3 \to 0 && \text{in} \quad H_0^1(0,L), && (3.4.5)\\
i\lambda z - y_{xx} - c(\cdot)v &= \lambda^{-\ell}f^4 \to 0 && \text{in} \quad L^2(0,L), && (3.4.6)\\
i\lambda\omega(\cdot,s) + \omega_s(\cdot,s) - v &= \lambda^{-\ell}f^5(\cdot,s) \to 0 && \text{in} \quad \mathcal{W}_g. && (3.4.7)
\end{aligned}$$

Here we will check the condition (H$_1$) by finding a contradiction with (3.4.1) by showing $\|U\|_{\mathcal{H}} = o(1)$. For clarity, we divide the proof into several lemmas.

Lemma 3.4.1. Under the hypotheses (H), the solution $U = (u, v, y, z, \omega(\cdot,s))^\top \in D(\mathcal{A})$ of system (3.4.3)-(3.4.7) satisfies the following estimations

$$-\int_0^\beta\int_0^\infty g'(s)|\omega_x(\cdot,s)|^2 ds dx = \frac{o(1)}{|\lambda|^\ell} \quad \text{and} \quad \int_0^\beta\int_0^\infty g(s)|\omega_x(\cdot,s)|^2 ds dx = \frac{o(1)}{|\lambda|^\ell}, \tag{3.4.8}$$

$$\int_0^\beta |u_x|^2 dx = o(|\lambda|^{-\ell}) \quad \text{and} \quad \int_0^\beta \left|S_{\tilde{b_0}}(u,\omega)\right|^2 dx = o(|\lambda|^{-\ell}). \tag{3.4.9}$$

Proof. First, taking the inner product of (3.4.2) with U in $\mathcal{H}$ and using (3.2.16), we get

$$-\frac{b_0}{2}\int_0^\beta\int_0^\infty g'(s)|\omega_x(\cdot,s)|^2dsdx = -\Re\left(\mathcal{A}U,U\right)_{\mathcal{H}} = \frac{1}{\lambda^\ell}\Re\left(F,U\right)_{\mathcal{H}} \leq \frac{1}{|\lambda|^\ell}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \quad (3.4.10)$$

Thus, from (3.4.10), (H) and the fact that $\|F\|_{\mathcal{H}} = o(1)$ and $\|U\|_{\mathcal{H}} = 1$, we obtain the first estimation in (3.4.8). From hypotheses (H), we obtain

$$\int_0^\beta\int_0^\infty g(s)|\omega_x(\cdot,s)|^2dsdx \leq -\frac{1}{m}\int_0^\beta\int_0^\infty g'(s)|\omega_x(\cdot,s)|^2dsdx. \quad (3.4.11)$$

Then, from the first estimation in (3.4.8) and (3.4.11), we obtain the second estimation in (3.4.8). Next, inserting (3.4.3) in (3.4.7), then deriving the resulting equation with respect to x, we get

$$i\lambda\omega_x(\cdot,s) + \omega_{sx}(\cdot,s) - i\lambda u_x = \lambda^{-\ell}f_x^5(\cdot,s) - \lambda^{-\ell}f_x^1 \ \text{ in } \ (0,\beta)\times(0,\infty). \quad (3.4.12)$$

Multiplying (3.4.12) by $\lambda^{-1}g(s)\overline{u_x}$, integrating over $(0,\beta)\times(0,\infty)$, then taking the imaginary part, we obtain

$$\begin{aligned}
&\int_0^\beta\int_0^\infty g(s)|u_x|^2dsdx = \Im\left\{i\int_0^\beta\int_0^\infty g(s)\omega_x(\cdot,s)\overline{u_x}dsdx\right\}\\
&+\Im\left\{\lambda^{-1}\int_0^\beta\int_0^\infty g(s)\omega_{xs}(\cdot,s)\overline{u_x}dsdx\right\} - \Im\left\{\lambda^{-(\ell+1)}\int_0^\beta\int_0^\infty g(s)f_x^5(\cdot,s)\overline{u_x}dsdx\right\}\\
&+\Im\left\{\lambda^{-(\ell+1)}\int_0^\beta\int_0^\infty g(s)f_x^1\overline{u_x}dsdx\right\}.
\end{aligned}$$

Using integration by parts with respect to s in the above equation, then using hypotheses (H) and the fact that $\omega(\cdot,0)=0$ in $(0,\beta)$, we get

$$\begin{aligned}
&\widetilde{g}\int_0^\beta |u_x|^2dx = \Im\left\{i\int_0^\beta\int_0^\infty g(s)\omega_x(\cdot,s)\overline{u_x}dsdx\right\}\\
&+\Im\left\{\frac{1}{\lambda}\int_0^\beta\int_0^\infty -g'(s)\omega_x(\cdot,s)\overline{u_x}dsdx\right\} - \Im\left\{\frac{1}{\lambda^{\ell+1}}\int_0^\beta\int_0^\infty g(s)f_x^5(\cdot,s)\overline{u_x}dsdx\right\} \quad (3.4.13)\\
&+\Im\left\{\widetilde{g}\lambda^{-(\ell+1)}\int_0^\beta f_x^1\overline{u_x}dx\right\}.
\end{aligned}$$

Using Young's inequality and Cauchy-Schwarz inequality in (3.4.13) with the help of hypotheses (H), we obtain

$$\begin{aligned}
\widetilde{g}\int_0^\beta |u_x|^2dx \ \leq\ & \frac{\widetilde{g}}{2}\int_0^\beta |u_x|^2dx + \frac{1}{2}\int_0^\beta\int_0^\infty g(s)|\omega_x(\cdot,s)|^2dsdx\\
&+|\lambda|^{-1}\sqrt{g_0}\left(\int_0^\beta\int_0^\infty -g'(s)|\omega_x(\cdot,s)|^2dsdx\right)^{\frac{1}{2}}\left(\int_0^\beta |u_x|^2dx\right)^{\frac{1}{2}}\\
&+|\lambda|^{-(\ell+1)}\sqrt{\widetilde{g}}\left(\int_0^\beta\int_0^\infty g(s)|f_x^5(\cdot,s)|^2dsdx\right)^{\frac{1}{2}}\left(\int_0^\beta |u_x|^2dx\right)^{\frac{1}{2}}\\
&+\widetilde{g}|\lambda|^{-(\ell+1)}\left(\int_0^\beta |f_x^1|^2dx\right)^{\frac{1}{2}}\left(\int_0^\beta |u_x|^2dx\right)^{\frac{1}{2}}.
\end{aligned}$$

From the above inequality, (3.4.8) and the fact that u_x is uniformly bounded in $L^2(0,L)$ and $f_x^1 \to 0$ in $L^2(0,L)$, $f^5(\cdot,s) \to 0$ in $\mathcal{W}_g$, we obtain the first estimation in (3.4.9). Now, by using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\int_0^\beta \left|S_{\widetilde{b_0}}(u,\omega)\right|^2 dx &= \int_0^\beta \left|\widetilde{b_0}u_x + b_0\int_0^\infty g(s)\omega_x(\cdot,s)ds\right|^2 dx \\
&\le 2(\widetilde{b_0})^2\int_0^\beta |u_x|^2 + 2b_0^2\int_0^\beta \left(\int_0^\infty g(s)|\omega_x(\cdot,s)|ds\right)^2 dx \\
&\le 2(\widetilde{b_0})^2\int_0^\beta |u_x|^2 + 2b_0^2\widetilde{g}\int_0^\beta\int_0^\infty g(s)|\omega_x(\cdot,s)|^2 ds dx.
\end{aligned}$$

Finally, from the above inequality, (3.4.8) and the first estimation in (3.4.9), we obtain the second estimation in (3.4.9). The proof is thus complete. $\square$

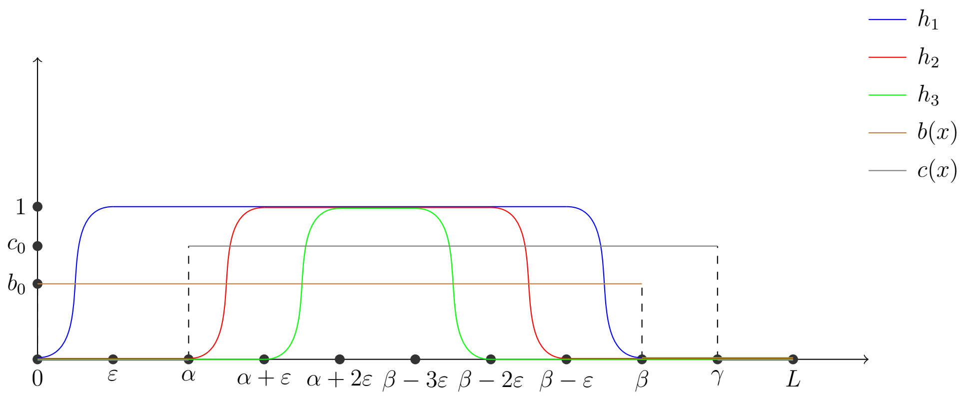

Figure 3.2: Geometric description of the functions h_1, h_2 and h_3.

Lemma 3.4.2. Let $0 < \varepsilon < \min\left(\alpha, \frac{\beta-\alpha}{5}\right)$. Under the hypotheses (H), the solution $U = (u,v,y,z,\omega(\cdot,s))^\top \in D(\mathcal{A})$ of (3.4.3)-(3.4.7) satisfies the following estimation

$$\int_\varepsilon^{\beta-\varepsilon} |v|^2 dx = o\left(|\lambda|^{-\frac{\ell}{2}}\right). \tag{3.4.14}$$

Proof. First, we fix a cut-off function $h_1 \in C^1([0,\beta])$ (see Figure 3.2) such that $0 \le h_1(x) \le 1$, for all $x \in [0,\beta]$ and

$$h_1(x) = \begin{cases} 1 & \text{if} \quad x \in [\varepsilon, \beta-\varepsilon], \\ 0 & \text{if} \quad x \in \{0,\beta\}, \end{cases} \tag{h_1}$$

and set

$$\max_{x\in[0,\beta]} |h_1'(x)| = M_{h_1'}.$$

From (3.4.4), we deduce that

$$i\lambda v - \left(S_{\widetilde{b_0}}(u,\omega)\right)_x + c(\cdot)z = \lambda^{-\ell} f^2 \ \text{ in } \ (0,\beta).$$

Multiplying the above equation by $-h_1\displaystyle\int_0^\infty g(s)\overline{\omega}(\cdot,s)ds$ and integrate over $(0,\beta)$, using integration by parts with the help of the properties of h_1 (i.e. $h_1(0)=h_1(\beta)=0$), then using the definition of $c(\cdot)$, we obtain

$$\begin{aligned}
&-i\lambda\int_0^\beta h_1 v\int_0^\infty g(s)\overline{\omega}(\cdot,s)dsdx=\int_0^\beta S_{\widetilde{b_0}}(u,\omega)\left(h_1\int_0^\infty g(s)\overline{\omega}(\cdot,s)\right)_x dsdx\\
&+c_0\int_\alpha^\beta h_1 z\int_0^\infty g(s)\overline{\omega}(\cdot,s)dsdx-\lambda^{-\ell}\int_0^\beta h_1 f^2\int_0^\infty g(s)\overline{\omega}(\cdot,s)dsdx.
\end{aligned}\tag{3.4.15}$$

From (3.4.7), we deduce that

$$-i\lambda\overline{\omega}(\cdot,s)=-\overline{\omega_s}(\cdot,s)+\overline{v}+\lambda^{-\ell}\overline{f^5}(\cdot,s)\quad\text{in}\quad(0,\beta)\times(0,\infty).$$

Inserting the above equation in the left hand side of (3.4.15), then using the definition of $c(\cdot)$ and h_1, we get

$$\begin{aligned}
&\widetilde{g}\int_0^\beta h_1|v|^2dx=\int_0^\beta h_1 v\int_0^\infty g(s)\overline{\omega_s}(\cdot,s)dsdx-\lambda^{-\ell}\int_0^\beta h_1 v\int_0^\infty g(s)\overline{f^5}(\cdot,s)dsdx\\
&+\int_0^\beta S_{\widetilde{b_0}}(u,\omega)h_1'\int_0^\infty g(s)\overline{\omega}(\cdot,s)dsdx+\int_0^\beta S_{\widetilde{b_0}}(u,\omega)h_1\int_0^\infty g(s)\overline{\omega_x}(\cdot,s)dsdx\\
&+c_0\int_\alpha^\beta h_1 z\int_0^\infty g(s)\overline{\omega}(\cdot,s)dsdx-\lambda^{-\ell}\int_0^\beta h_1 f^2\int_0^\infty g(s)\overline{\omega}(\cdot,s)dsdx.
\end{aligned}\tag{3.4.16}$$

Using integration by parts with respect to s with the help of $\omega(\cdot,0)=0$ in $(0,\beta)$ and hypotheses (H), Cauchy-Schwarz inequality, Poincaré inequality, v is uniformly bounded in $L^2(0,L)$, and (3.4.9), we get

$$\begin{aligned}
&\left|\int_0^\beta h_1 v\int_0^\infty g(s)\overline{\omega_s}(\cdot,s)dsdx\right|=\left|\int_0^\beta h_1 v\int_0^\infty -g'(s)\overline{\omega}(\cdot,s)dsdx\right|\\
&\leq\sqrt{g_0}\left(\int_0^\beta|v|^2dx\right)^{\frac{1}{2}}\left(\int_0^\beta\int_0^\infty -g'(s)|\omega(\cdot,s)|^2dsdx\right)^{\frac{1}{2}}\\
&\lesssim\sqrt{g_0}\left(\int_0^\beta|v|^2dx\right)^{\frac{1}{2}}\left(\int_0^\beta\int_0^\infty -g'(s)|\omega_x(\cdot,s)|^2dsdx\right)^{\frac{1}{2}}=o\left(|\lambda|^{-\frac{\ell}{2}}\right).
\end{aligned}\tag{3.4.17}$$

Using the definition of h_1, Cauchy-Schwarz inequality, Poincaré inequality, (3.4.8) and the fact that v, z are uniformly bounded in $L^2(0,L)$ and $\|f^2\|_{L^2(0,L)} = o(1)$, $f^5 \to 0$ in $\mathcal{W}_g$, we get

$$\left\{\begin{array}{l} \left|\dfrac{1}{\lambda^\ell}\displaystyle\int_0^\beta h_1 v \int_0^\infty g(s)\overline{f^5}(\cdot,s)dsdx\right| \\ \lesssim \dfrac{1}{|\lambda|^\ell}\sqrt{\widetilde{g}}\left(\displaystyle\int_0^\beta |v|^2dx\right)^{\frac{1}{2}}\left(\displaystyle\int_0^\beta\int_0^\infty g(s)|f_x^5(\cdot,s)|^2dsdx\right)^{\frac{1}{2}} = \dfrac{o(1)}{|\lambda|^\ell}, \\ \\ \left|c_0\displaystyle\int_\alpha^\beta h_1 z\int_0^\infty g(s)\overline{\omega}(\cdot,s)dsdx\right| \leq |c_0|\sqrt{\widetilde{g}}\left(\displaystyle\int_\alpha^\beta |z|^2dx\right)\left(\displaystyle\int_\alpha^\beta\int_0^\infty g(s)|\omega(\cdot,s)|^2dsdx\right)^{\frac{1}{2}} \\ \qquad\qquad\qquad\qquad\qquad\qquad \lesssim |c_0|\sqrt{\widetilde{g}}\left(\displaystyle\int_\alpha^\beta |z|^2dx\right)\left(\displaystyle\int_0^\beta\int_0^\infty g(s)|\omega_x(\cdot,s)|^2dsdx\right)^{\frac{1}{2}} \\ \qquad\qquad\qquad\qquad\qquad\qquad = \dfrac{o(1)}{|\lambda|^{\frac{\ell}{2}}}, \\ \\ \left|\dfrac{1}{\lambda^\ell}\displaystyle\int_0^\beta h_1 f_2\int_0^\infty g(s)\overline{\omega}(\cdot,s)dsdx\right| \\ \lesssim \dfrac{1}{|\lambda|^\ell}\sqrt{\widetilde{g}}\left(\displaystyle\int_0^\beta |f^2|^2dx\right)^{\frac{1}{2}}\left(\displaystyle\int_0^\beta\int_0^\infty g(s)|\omega_x(\cdot,s)|^2dsdx\right)^{\frac{1}{2}} = \dfrac{o(1)}{|\lambda|^{\frac{3\ell}{2}}}. \end{array}\right. \tag{3.4.18}$$

On the other hand, we have

$$\left\{\begin{array}{l} |S_{\widetilde{b_0}}(u,\omega)||h_1'|g(s)|\omega(\cdot,s)| \leq \dfrac{1}{2}|h_1'||S_{\widetilde{b_0}}(u,\omega)|^2g(s) + \dfrac{1}{2}|h_1'||\omega(\cdot,s)|^2g(s), \\ |S_{\widetilde{b_0}}(u,\omega)||h_1|g(s)|\omega_x(\cdot,s)| \leq \dfrac{1}{2}|h_1||S_{\widetilde{b_0}}(u,\omega)|^2g(s) + \dfrac{1}{2}|h_1||\omega_x(\cdot,s)|^2g(s). \end{array}\right.$$

Then from the above inequalities, the definition of $S_{\tilde{b}(\cdot)}(u,\omega)$ and h_1, Poincaré inequality and estimations (3.4.8) and (3.4.9), we obtain

$$\left\{\begin{array}{l} \left|\displaystyle\int_0^\beta S_{\widetilde{b_0}}(u,\omega)h_1'\int_0^\infty g(s)\overline{\omega}(\cdot,s)dsdx\right| \\ \leq \dfrac{M_{h_1'}}{2}\left(\widetilde{g}\displaystyle\int_0^\beta |S_{\widetilde{b_0}}(u,\omega)|^2dx + C_p\int_0^\beta\int_0^\infty g(s)|\omega_x(\cdot,s)|^2dsdx\right) = o(|\lambda|^{-\ell}), \\ \\ \left|\displaystyle\int_0^\beta S_{\widetilde{b_0}}(u,\omega)h_1\int_0^\infty g(s)\overline{\omega_x}(\cdot,s)dsdx\right| \\ \leq \dfrac{\widetilde{g}}{2}\displaystyle\int_0^\beta |S_{\widetilde{b_0}}(u,\omega)|^2dx + \int_0^\beta\int_0^\infty g(s)|\omega_x(\cdot,s)|^2dsdx = o(|\lambda|^{-\ell}), \end{array}\right. \tag{3.4.19}$$

where $C_p > 0$ is a Poincaré constant. Inserting inequalities (3.4.17)-(3.4.19) in (3.4.16), we obtain

$$\int_0^\beta h_1|v|^2dx = o\left(|\lambda|^{-\frac{\ell}{2}}\right).$$

Finally, from the above estimation and the definition of h_1, we obtain the desired result (3.4.14). The proof is thus complete. ☐

Lemma 3.4.3. Let $0 < \varepsilon < \min\left(\alpha, \frac{\beta-\alpha}{5}\right)$. Under the hypotheses (H), the solution the solution $U = (u, v, y, z, \omega(\cdot, s))^\top \in D(\mathcal{A})$ of (3.4.3)-(3.4.7) satisfies the following estimation

$$\int_{\alpha+\varepsilon}^{\beta-2\varepsilon} |y_x|^2 dx \leq \frac{|a-1||\lambda|}{|c_0|} \int_\alpha^{\beta-\varepsilon} |u_x||y_x| dx + o(1). \tag{3.4.20}$$

Proof. First, we fix a cut-off function $h_2 \in C^1([0, L])$ (see Figure 3.2) such that $0 \leq h_2(x) \leq 1$, for all $x \in [0, L]$ and

$$h_2(x) = \begin{cases} 0 & \text{if} \quad x \in [0, \alpha] \cup [\beta - \varepsilon, L], \\ 1 & \text{if} \quad x \in [\alpha + \varepsilon, \beta - 2\varepsilon], \end{cases} \tag{h_2}$$

From (3.4.6), $i\lambda^{-1} h_2 \overline{y_{xx}}$ is uniformly bounded in $L^2(0, L)$. Multiplying (3.4.4) by $i\lambda^{-1} h_2 \overline{y_{xx}}$, using integration by parts over $(0, L)$ and over $(\alpha, \beta - \varepsilon)$, the definitions of $c(\cdot)$ and h_2, and using the fact that $\|f^2\|_{L^2(0,L)} = o(1)$, we get

$$\begin{aligned} &\int_0^L h_2' v \overline{y_x} dx + \int_0^L h_2 v_x \overline{y_x} dx - \frac{i}{\lambda} \int_0^L h_2 \left(S_{\tilde{b}(\cdot)}(u, \omega)\right)_x \overline{y_{xx}} dx \\ &- \frac{ic_0}{\lambda} \int_\alpha^{\beta-\varepsilon} \left(h_2' z \overline{y_x} + h_2 z_x \overline{y_x}\right) dx = \frac{o(1)}{|\lambda|^\ell}. \end{aligned} \tag{3.4.21}$$

From (3.4.3) and (3.4.5), we obtain

$$v_x = i\lambda u_x - \lambda^{-\ell} f_x^1 \quad \text{and} \quad -\frac{i}{\lambda} z_x = y_x + i\lambda^{-(\ell+1)} f_x^3.$$

Inserting the above equations in (3.4.21) and taking the real part, we get

$$\begin{aligned} &c_0 \int_0^L h_2 |y_x|^2 dx + \Re\left\{i\lambda \int_0^L h_2 u_x \overline{y_x} dx\right\} - \Re\left\{\frac{i}{\lambda} \int_0^L \left(S_{\tilde{b}(\cdot)}(u, \omega)\right)_x h_2 \overline{y_{xx}} dx\right\} = -\Re\left\{\int_0^L h_2' v \overline{y_x} dx\right\} \\ &+ \Re\left\{\frac{1}{\lambda^\ell} \int_0^L h_2 f_x^1 \overline{y_x} dx\right\} + \Re\left\{i\frac{c_0}{\lambda} \int_\alpha^{\beta-\varepsilon} h_2' z \overline{y_x} dx\right\} - \Re\left\{\frac{ic_0}{\lambda^{\ell+1}} \int_\alpha^{\beta-\varepsilon} h_2 f_x^3 \overline{y_x} dx\right\} + \frac{o(1)}{|\lambda|^\ell}. \end{aligned} \tag{3.4.22}$$

Using the fact that y_x is uniformly bounded in $L^2(0, L)$, $\|f_x^1\|_{L^2(0,L)} = o(1)$ and $\|f_x^3\|_{L^2(0,L)} = o(1)$, we get

$$\Re\left\{\lambda^{-\ell} \int_0^L h_2 f_x^1 \overline{y_x} dx\right\} = \frac{o(1)}{|\lambda|^\ell} \quad \text{and} \quad -\Re\left\{ic_0 \lambda^{-(\ell+1)} \int_\alpha^{\beta-\varepsilon} h_2 f_x^3 \overline{y_x} dx\right\} = \frac{o(1)}{|\lambda|^{\ell+1}}. \tag{3.4.23}$$

Using Cauchy-Schwarz inequality, the definition of h_2, y_x and z are uniformly bounded in $L^2(0, L)$, and estimation (3.4.14), we get

$$-\Re\left\{\int_0^L h_2' v \overline{y_x} dx\right\} = o\left(|\lambda|^{-\frac{\ell}{4}}\right) \quad \text{and} \quad \Re\left\{i\frac{c_0}{\lambda} \int_\alpha^{\beta-\varepsilon} h_2' z \overline{y_x} dx\right\} = O\left(|\lambda|^{-1}\right) = o(1). \tag{3.4.24}$$

Inserting (3.4.23) and (3.4.24) in (3.4.22), then using the definition of h_2, we get

$$\begin{aligned} &c_0 \int_\alpha^{\beta-\varepsilon} h_2 |y_x|^2 dx + \Re\left\{i\lambda \int_\alpha^{\beta-\varepsilon} h_2 u_x \overline{y_x} dx\right\} - \Re\left\{\frac{i}{\lambda} \int_\alpha^{\beta-\varepsilon} \left(S_{\tilde{b_0}}(u, \omega)\right)_x h_2 \overline{y_{xx}} dx\right\} \\ &= o(1). \end{aligned} \tag{3.4.25}$$

From (3.4.4), $i\lambda^{-1}h_2\left(S_{\tilde{b}(\cdot)}(u,\omega)\right)_x$ is uniformly bounded in $L^2(0,L)$. Multiplying (3.4.6) by $i\lambda^{-1}\left(\overline{S_{\tilde{b}(\cdot)}}(u,\omega)\right)_x$, using integration by parts over $(0,L)$ and over $(\alpha,\beta-\varepsilon)$, the definitions of $c(\cdot)$, h_2 and $S_{\tilde{b}(\cdot)}(u,\omega)$, and the fact that $\|f^4\|_{L^2(0,L)}=o(1)$, we get

$$\begin{aligned}
&\int_0^L h_2' z\overline{S_{b(\cdot)}}(u,\omega)dx+\int_0^L h_2 z_x\overline{S_{\tilde{b}(\cdot)}}(u,\omega)dx-\frac{i}{\lambda}\int_0^L h_2 y_{xx}\left(\overline{S_{\tilde{b}(\cdot)}}(u,\omega)\right)_x dx\\
&+\frac{ic_0}{\lambda}\int_\alpha^{\beta-\varepsilon}\left(h_2' v+h_2 v_x\right)\overline{S_{\widetilde{b_0}}}(u,\omega)dx=o(|\lambda|^{-\ell}).
\end{aligned}\tag{3.4.26}$$

From (3.4.5), we have

$$z_x=i\lambda y_x-\lambda^{-\ell}f_x^3.$$

Using the above equation and the definition of $S_{\tilde{b}(\cdot)}(u,\omega)$ and h_2, we get

$$\begin{aligned}
&\int_0^L h_2 z_x\overline{S_{\tilde{b}(\cdot)}}(u,\omega)dx=\int_\alpha^{\beta-\varepsilon} h_2\left(i\lambda y_x-\lambda^{-\ell}f_x^3\right)\overline{S_{\widetilde{b_0}}}(u,\omega)dx\\
&=i\lambda\widetilde{b_0}\int_\alpha^{\beta-\varepsilon}h_2 y_x\overline{u_x}dx+i\lambda b_0\int_\alpha^{\beta-\varepsilon}h_2 y_x\int_0^\infty g(s)\overline{\omega_x}(\cdot,s)dsdx\\
&-\widetilde{b_0}\lambda^{-\ell}\int_\alpha^{\beta-\varepsilon}h_2 f_x^3\overline{u_x}dx-\lambda^{-\ell}b_0\int_\alpha^{\beta-\varepsilon}h_2 f_x^3\int_0^\infty g(s)\overline{\omega_x}(\cdot,s)dsdx.
\end{aligned}\tag{3.4.27}$$

From (3.4.7), we have

$$i\lambda\overline{\omega_x}(\cdot,s)=\overline{\omega_{xs}}(\cdot,s)+i\lambda\overline{u_x}+\lambda^{-\ell}\overline{f_x^1}-\lambda^{-\ell}\overline{f_x^5}(\cdot,s)\quad\text{in}\quad(0,\beta)\times(0,\infty).$$

From the above equation and by using integration by parts with respect to s, we get

$$\begin{aligned}
&i\lambda b_0\int_\alpha^{\beta-\varepsilon}h_2 y_x\int_0^\infty g(s)\overline{\omega_x}(\cdot,s)dsdx=b_0\int_\alpha^{\beta-\varepsilon}h_2 y_x\int_0^\infty -g'(s)\overline{\omega_x}(\cdot,s)dsdx\\
&+i\lambda b_0\widetilde{g}\int_\alpha^{\beta-\varepsilon}h_2 y_x\overline{u_x}dx+b_0\widetilde{g}\lambda^{-\ell}\int_\alpha^{\beta-\varepsilon}h_2 y_x\overline{f_x^1}dx\\
&-b_0\lambda^{-\ell}\int_\alpha^{\beta-\varepsilon}h_2 y_x\int_0^\infty g(s)\overline{f_x^5}(\cdot,s)dsdx.
\end{aligned}\tag{3.4.28}$$

Inserting (3.4.28) in (3.4.27), then using the fact that $\widetilde{b_0}=a-b_0\widetilde{g}$, we get

$$\begin{aligned}
&\int_0^L h_2 z_x\overline{S_{\tilde{b}(\cdot)}}(u,\omega)dx=i\lambda a\int_\alpha^{\beta-\varepsilon}h_2 y_x\overline{u_x}dx+b_0\int_\alpha^{\beta-\varepsilon}h_2 y_x\int_0^\infty -g'(s)\overline{\omega_x}(\cdot,s)dsdx\\
&+b_0\widetilde{g}\lambda^{-\ell}\int_\alpha^{\beta-\varepsilon}h_2 y_x\overline{f_x^1}dx-b_0\lambda^{-\ell}\int_\alpha^{\beta-\varepsilon}h_2 y_x\int_0^\infty g(s)\overline{f_x^5}(\cdot,s)dsdx\\
&-\widetilde{b_0}\lambda^{-\ell}\int_\alpha^{\beta-\varepsilon}h_2 f_x^3\overline{u_x}dx-\lambda^{-\ell}b_0\int_\alpha^{\beta-\varepsilon}h_2 f_x^3\int_0^\infty g(s)\overline{\omega_x}(\cdot,s)dsdx.
\end{aligned}\tag{3.4.29}$$

Using Cauchy-Schwarz inequality, the facts that y_x, u_x is uniformly bounded in $L^2(0,L)$, and estimation (3.4.8), $\|F\|_{\mathcal{H}} = o(1)$, we get

$$\begin{cases} b_0 \displaystyle\int_\alpha^{\beta-\varepsilon} h_2 y_x \int_0^\infty -g'(s)\overline{\omega_x}(\cdot,s)dsdx = o\left(|\lambda|^{-\frac{\ell}{2}}\right), \quad b_0\widetilde{g}\lambda^{-\ell}\int_\alpha^{\beta-\varepsilon} h_2 y_x \overline{f_x^1} = o\left(|\lambda|^{-\ell}\right), \\ -b_0\lambda^{-\ell}\displaystyle\int_\alpha^{\beta-\varepsilon} h_2 y_x \int_0^\infty g(s)\overline{f_x^5}(\cdot,s)dsdx = o\left(|\lambda|^{-\ell}\right), \quad -\widetilde{b_0}\lambda^{-\ell}\int_\alpha^{\beta-\varepsilon} h_2 f_x^3 \overline{u_x}dx = o(|\lambda|^{-\ell}), \\ -\lambda^{-\ell} b_0 \displaystyle\int_\alpha^{\beta-\varepsilon} h_2 f_x^3 \int_0^\infty g(s)\overline{\omega_x}(\cdot,s)dsdx = o\left(|\lambda|^{-\frac{3\ell}{2}}\right). \end{cases}$$

Inserting the above estimations in (3.4.29), we get

$$\int_0^L h_2 z_x \overline{S_{\tilde{b}(\cdot)}}(u,\omega)dx = i\lambda a \int_\alpha^{\beta-\varepsilon} h_2 y_x \overline{u_x}dx + o\left(|\lambda|^{-\frac{\ell}{2}}\right). \tag{3.4.30}$$

From (3.4.3), we have

$$i\lambda^{-1}v_x = -u_x - i\lambda^{-(\ell+1)}f_x^1.$$

Then from the above equation and the definition of $S_{\tilde{b}(\cdot)}(u,\omega)$ and h_2, we get

$$i\frac{c_0}{\lambda}\int_\alpha^{\beta-\varepsilon} h_2 v_x \overline{S_{\widetilde{b_0}}}(u,\omega)dx = -\int_\alpha^{\beta-\varepsilon} u_x \overline{S_{\widetilde{b_0}}}(u,\omega)dx - i\lambda^{-(\ell+1)}\int_\alpha^{\beta-\varepsilon} f_x^1 \overline{S_{\widetilde{b_0}}}(u,\omega)dx. \tag{3.4.31}$$

Using Cauchy-Schwarz inequality, the definition of h_2, the fact that u_x is uniformly bounded in $L^2(0,L)$ and $\|f_x^1\| = o(1)$, and estimation (3.4.9), we get

$$-\int_\alpha^{\beta-\varepsilon} u_x \overline{S_{\widetilde{b_0}}}(u,\omega)dx = o\left(|\lambda|^{-\frac{\ell}{2}}\right) \quad \text{and} \quad -i\lambda^{-(\ell+1)}\int_\alpha^{\beta-\varepsilon} f_x^1 \overline{S_{\widetilde{b_0}}}(u,\omega)dx = o\left(|\lambda|^{-\frac{3\ell}{2}-1}\right)$$

Inserting the above estimations in (3.4.31), we get

$$i\frac{c_0}{\lambda}\int_\alpha^{\beta-\varepsilon} h_2 v_x \overline{S_{\widetilde{b_0}}}(u,\omega)dx = o\left(|\lambda|^{-\frac{\ell}{2}}\right). \tag{3.4.32}$$

Now, using the definition of h_2 and $S_{\tilde{b}(\cdot)}(u,\omega)$, (3.4.9), and the fact that v and z are uniformly bounded in $L^2(0,L)$, we get

$$\begin{cases} \displaystyle\int_0^L h_2' z \overline{S_{\tilde{b}(\cdot)}}(u,\omega)dx = \int_\alpha^{\beta-\varepsilon} h_2' z \overline{S_{\widetilde{b_0}}}(u,\omega)dx = o\left(|\lambda|^{-\frac{\ell}{2}}\right), \\ i\dfrac{c_0}{\lambda}\displaystyle\int_\alpha^{\beta-\varepsilon} h_2' v \overline{S_{\widetilde{b_0}}}(u,\omega)dx = o\left(|\lambda|^{-\frac{3\ell}{2}}\right). \end{cases} \tag{3.4.33}$$

Inserting (3.4.30), (3.4.32) and (3.4.33) in (3.4.26), using the definition of h_2, then taking the real part, we get

$$\Re\left\{i\lambda a\int_\alpha^{\beta-\varepsilon} h_2 y_x \overline{u_x}dx\right\} - \Re\left\{\frac{i}{\lambda}\int_\alpha^{\beta-\varepsilon} h_2 y_{xx}\left(\overline{S_{\widetilde{b_0}}}(u,\omega)\right)_x dx\right\} = o\left(|\lambda|^{-\frac{\ell}{2}}\right). \tag{3.4.34}$$

Now, adding (3.4.25) and (3.4.34) and using the fact that $\ell \geq 0$, we get

$$\int_\alpha^{\beta-\varepsilon} h_2|y_x|^2dx = \Re\left\{\frac{i\lambda(a-1)}{c_0}\int_\alpha^{\beta-\varepsilon} h_2 u_x \overline{y_x}dx\right\} + o(1).$$

Using the definition of h_2 in the above equation, we get the desired estimation (3.4.20). The proof is thus complete. □

Lemma 3.4.4. Let $0 < \varepsilon < \min\left(\alpha, \frac{\beta-\alpha}{5}\right)$. Under the hypotheses (H), the solution $U = (u, v, y, z, \omega(\cdot, s))^\top \in D(\mathcal{A})$ of (3.4.3)-(3.4.7) satisfies the following estimation

$$\int_{\alpha+2\varepsilon}^{\beta-3\varepsilon} |z|^2 dx \leq \frac{3|a-1||\lambda|}{|c_0|} \int_{\alpha}^{\beta-\varepsilon} |u_x||y_x| dx + o(1). \tag{3.4.35}$$

Proof. First, we fix a cut-off function $h_3 \in C^1([0, L])$ (see Figure 3.2) such that $0 \leq h_3(x) \leq 1$, for all $x \in [0, L]$ and

$$h_3(x) = \begin{cases} 0 & \text{if} \quad x \in [0, \alpha+\varepsilon] \cup [\beta - 2\varepsilon, L], \\ 1 & \text{if} \quad x \in [\alpha + 2\varepsilon, \beta - 3\varepsilon], \end{cases} \tag{h_3}$$

Multiplying (3.4.6) by $-i\lambda^{-1} h_3 \overline{z}$, using integration by parts over $(0, L)$, the fact that z is uniformly bounded in $L^2(0, L)$ and $\|f^4\| = o(1)$, and the definition of $c(\cdot)$, we get

$$\int_0^L h_3 |z|^2 dx - \frac{i}{\lambda} \int_0^L h_3' \overline{z} y_x dx - \frac{i}{\lambda} \int_0^L h_3 \overline{z_x} y_x dx + i\frac{c_0}{\lambda} \int_{\alpha+\varepsilon}^{\beta-2\varepsilon} h_3 v \overline{z} dx = o\left(|\lambda|^{-(\ell+1)}\right). \tag{3.4.36}$$

From (3.4.5), we have

$$-\frac{i}{\lambda} \overline{z_x} = -\overline{y_x} + i\lambda^{-(\ell+1)} \overline{f_x^3}.$$

Inserting the above equation in (3.4.36), we get

$$\begin{aligned} \int_0^L h_3 |z|^2 dx &= \int_0^L h_3 |y_x|^2 dx - i\lambda^{-(\ell+1)} \int_0^L h_3 \overline{f_x^3} y_x dx \\ &\quad + \frac{i}{\lambda} \int_0^L h_3' \overline{z} y_x dx - i\frac{c_0}{\lambda} \int_{\alpha+\varepsilon}^{\beta-2\varepsilon} h_3 v \overline{z} dx + o\left(|\lambda|^{-(\ell+1)}\right). \end{aligned} \tag{3.4.37}$$

Using the fact that $\|f_x^3\|_{L^2(0,L)} = o(1)$, y_x and z are uniformly bounded in $L^2(0, L)$, and the definition of h_3, we get

$$\begin{cases} -i\lambda^{-(\ell+1)} \displaystyle\int_0^L h_3 \overline{f_x^3} y_x dx = o\left(|\lambda|^{-(\ell+1)}\right), \quad \dfrac{i}{\lambda} \displaystyle\int_0^L h_3' \overline{z} y_x dx = o(1) \quad \text{and} \\ -i\dfrac{c_0}{\lambda} \displaystyle\int_{\alpha+\varepsilon}^{\beta-2\varepsilon} h_3 v \overline{z} dx = o(1). \end{cases} \tag{3.4.38}$$

Using (3.4.20) and the definition of h_3, we get

$$\int_0^L h_3 |y_x|^2 dx \leq 3 \int_{\alpha+\varepsilon}^{\beta-2\varepsilon} |y_x|^2 dx \leq \frac{3|a-1|}{|c_0|} |\lambda| \int_{\alpha}^{\beta-\varepsilon} |u_x||y_x| dx + o(1). \tag{3.4.39}$$

Inserting (3.4.38) and (3.4.39) in (3.4.37) and using the definition of h_3, we get the desired estimation (3.4.35). The proof has been completed. $\square$

Now, we fix a function $\chi \in C^1([\beta - 3\varepsilon, \gamma])$ by

$$\chi(\beta - 3\varepsilon) = -\chi(\gamma) = 1, \quad \text{and set} \quad \max_{x \in [\beta-3\varepsilon, \gamma]} |\chi(x)| = M_\chi \quad \text{and} \quad \max_{x \in [\beta-3\varepsilon, \gamma]} |\chi'(x)| = M_{\chi'}. \tag{χ}$$

Remark 3.4.1. It is easy to see the existence of $\chi(x)$. For example, we can take

$$\chi(x) = \frac{1}{(\gamma - \beta + 3\varepsilon)^2} \left(-2x^2 + 4(\beta - 3\varepsilon)x + \gamma^2 - (\beta - 3\varepsilon)^2 - 2\gamma(\beta - 3\varepsilon)\right),$$

to get $\chi(\beta - 3\varepsilon) = -\chi(\gamma) = 1$, $\chi \in C^1([\beta - 3\varepsilon, \gamma])$, $M_\chi = 1$ and $M_{\chi'} = \frac{4}{\gamma - \beta + 3\varepsilon}$. $\square$

Lemma 3.4.5. Let $0<\varepsilon<\min\left(\alpha,\frac{\beta-\alpha}{5}\right)$. Under the hypotheses (H), the solution $U=(u,v,y,z,\omega(\cdot,s))^\top\in D(\mathcal{A})$ of (3.4.3)-(3.4.7) satisfies the following estimations

$$|v(\gamma)|^2+|v(\beta-3\varepsilon)|^2=O(|\lambda|),\quad |z(\gamma)|^2+|z(\beta-3\varepsilon)|^2+|y_x(\gamma)|^2+|y_x(\beta-3\varepsilon)|^2=O(1). \tag{3.4.40}$$

Proof. First, deriving Equation (3.4.3) with respect to x, we obtain

$$i\lambda u_x-v_x=\lambda^{-\ell}f_x^1\ \text{ in }\ (\beta-3\varepsilon,\gamma).$$

Multiplying the above equation by $2\chi\overline{v}$, integrating over $(\beta-3\varepsilon,\gamma)$, then taking the real part, we obtain

$$\Re\left\{2i\lambda\int_{\beta-3\varepsilon}^{\gamma}\chi u_x\overline{v}dx\right\}-\int_{\beta-3\varepsilon}^{\gamma}\chi(|v|^2)_xdx=\Re\left\{2\lambda^{-\ell}\int_{\beta-3\varepsilon}^{\gamma}\chi f_x^1\overline{v}dx\right\}. \tag{3.4.41}$$

Using integration by parts in (3.4.41), we obtain

$$\left[-\chi|v|^2\right]_{\beta-3\varepsilon}^{\gamma}=-\int_{\beta-3\varepsilon}^{\gamma}\chi'|v|^2dx-\Re\left\{2i\lambda\int_{\beta-3\varepsilon}^{\gamma}\chi u_x\overline{v}dx\right\}+\Re\left\{2\lambda^{-\ell}\int_{\beta-3\varepsilon}^{\gamma}\chi f_x^1\overline{v}dx\right\}. \tag{3.4.42}$$

Using the definition of χ and Cauchy-Schwarz inequality in (3.4.42), we obtain

$$\begin{aligned}|v(\gamma)|^2+|v(\beta-3\varepsilon)|^2\ &\leq\ M_{\chi'}\int_{\beta-3\varepsilon}^{\gamma}|v|^2dx\\ &\quad+2|\lambda|M_\chi\left(\int_{\beta-3\varepsilon}^{\gamma}|u_x|^2dx\right)^{\frac12}\left(\int_{\beta-3\varepsilon}^{\gamma}|v|^2dx\right)^{\frac12}\\ &\quad+2|\lambda|^{-\ell}M_\chi\left(\int_{\beta-3\varepsilon}^{\gamma}|f_x^1|^2dx\right)^{\frac12}\left(\int_{\beta-3\varepsilon}^{\gamma}|v|^2dx\right)^{\frac12}.\end{aligned} \tag{3.4.43}$$

Thus, from (3.4.43) and the fact that u_x,v are uniformly bounded in $L^2(0,L)$ and $\|f_x^1\|_{L^2(0,L)}=o(1)$, we obtain the first estimation in (3.4.40). From (3.4.5), (3.4.6) and the definition of $c(\cdot)$, we have

$$i\lambda y_x-z_x=\lambda^{-\ell}f_x^3\ \text{ in }\ (\beta-3\varepsilon,\gamma)$$

and

$$i\lambda z-y_{xx}-c_0v=\lambda^{-\ell}f^4\ \text{ in }\ (\beta-3\varepsilon,\gamma).$$

Multiplying the above equations by $2\chi\overline{z}$ and $2\chi\overline{y_x}$ respectively, integrating over $(\beta-3\varepsilon,\gamma)$, taking the real part, then using the fact that y_x,z are uniformly bounded in $L^2(0,L)$ and $\|f^2\|_{L^2(0,L)}=o(1)$ and $\|f_x^3\|_{L^2(0,L)}=o(1)$, we obtain

$$\Re\left\{2i\lambda\int_{\beta-3\varepsilon}^{\gamma}\chi y_x\overline{z}dx\right\}-\int_{\beta-3\varepsilon}^{\gamma}\chi(|z|^2)_xdx=o(|\lambda|^{-\ell}) \tag{3.4.44}$$

and

$$\Re\left\{2i\lambda\int_{\beta-3\varepsilon}^{\gamma}\chi z\overline{y_x}dx\right\}-\int_{\beta-3\varepsilon}^{\gamma}\chi(|y_x|^2)_xdx-\Re\left\{2c_0\int_{\beta-3\varepsilon}^{\gamma}\chi v\overline{y_x}dx\right\}=o(|\lambda|^{-\ell}). \tag{3.4.45}$$

Adding (3.4.44) and (3.4.45), then using integration by parts, we obtain

$$\left[-\chi(|z|^2+|y_x|^2)\right]_{\beta-3\varepsilon}^{\gamma}=-\int_{\beta-3\varepsilon}^{\gamma}\chi'(|z|^2+|y_x|^2)dx+\Re\left\{2c_0\int_{\beta-3\varepsilon}^{\gamma}\chi v\overline{y_x}dx\right\}+o(|\lambda|^{-\ell}).$$

Using the definition of χ and Cauchy-Schwarz inequality in the above equation, we obtain

$$\begin{aligned}
&|z(\gamma)|^2+|z(\beta-3\varepsilon)|^2+|y_x(\gamma)|^2+|y_x(\beta-3\varepsilon)|^2\\
\leq\;& M_{\chi'}\int_{\beta-3\varepsilon}^{\gamma}(|z|^2+|y_x|^2)dx+2c_0M_\chi\left(\int_{\beta-3\varepsilon}^{\gamma}|v|^2dx\right)^{\frac12}\left(\int_{\beta-3\varepsilon}^{\gamma}|y_x|^2dx\right)^{\frac12}\\
&+o(|\lambda|^{-\ell}).
\end{aligned}\tag{3.4.46}$$

Finally, from (3.4.46) and the fact that v, y_x, z are uniformly bounded in $L^2(0,L)$, we obtain the second estimation in (3.4.40). The proof is thus complete. ☐

Lemma 3.4.6. Let $\theta\in C^1([0,L])$ be a function with $\theta(0)=\theta(L)=0$. Under the hypotheses (H), the solution $U=(u,v,y,z,\omega(\cdot,s))^\top\in D(\mathcal{A})$ of (3.4.3)-(3.4.7) satisfies the following estimation

$$\begin{aligned}
&\int_0^L\theta'\left(|v|^2+a^{-1}\left|S_{\tilde{b}(\cdot)}(u,\omega)\right|^2+|z|^2+|y_x|^2\right)dx\\
&+\Re\left\{2a^{-1}\int_0^Lc(\cdot)\theta z\overline{S_{\tilde{b}(\cdot)}}(u,\omega)dx\right\}-\Re\left\{2\int_0^Lc(\cdot)\theta v\overline{y_x}dx\right\}=o\left(|\lambda|^{-\frac{\ell}{2}}\right).
\end{aligned}\tag{3.4.47}$$

Proof. First, from (3.4.3), we deduce that

$$i\lambda\overline{u_x}=-\overline{v_x}-\lambda^{-\ell}\overline{f_x^1}.\tag{3.4.48}$$

Multiplying (3.4.4) by $2a^{-1}\theta\overline{S_{\tilde{b}(\cdot)}}(u,\omega)$, integrating over $(0,L)$, taking the real part, then using (3.4.9) and the fact that $\|f^2\|_{L^2(0,L)}=o(1)$, we get

$$\begin{aligned}
&\Re\left\{2i\lambda a^{-1}\int_0^L\theta v\overline{S_{\tilde{b}(\cdot)}}(u,\omega)dx\right\}-a^{-1}\int_0^L\theta\left(\left|S_{\tilde{b}(\cdot)}(u,\omega)\right|^2\right)_xdx\\
&+\Re\left\{2a^{-1}\int_0^Lc(\cdot)\theta z\overline{S_{\tilde{b}(\cdot)}}(u,\omega)dx\right\}=o(|\lambda|^{-\ell}).
\end{aligned}\tag{3.4.49}$$

Using the definition of $S_{\tilde{b}(\cdot)}(u,\omega)$ and the fact that $\widetilde{b_0}=a-b_0\widetilde{g}$, we obtain

$$\begin{aligned}
&\Re\left\{2i\lambda a^{-1}\int_0^L\theta v\overline{S_{\tilde{b}(\cdot)}}(u,\omega)dx\right\}=\Re\left\{2i\lambda\int_0^L\theta v\overline{u_x}dx\right\}-\Re\left\{2i\lambda a^{-1}b_0\widetilde{g}\int_0^\beta\theta v\overline{u_x}dx\right\}\\
&+\Re\left\{2i\lambda a^{-1}b_0\int_0^\beta\theta v\int_0^\infty g(s)\overline{\omega_x}(\cdot,s)dsdx\right\}.
\end{aligned}$$

Inserting (3.4.48) in the above equation and using the fact that v is uniformly bounded in $L^2(0,L)$, $\|f_x^1\|_{L^2(0,L)}=o(1)$, we get

$$\begin{aligned}
&\Re\left\{2i\lambda a^{-1}\int_0^L\theta v\overline{S_{\tilde{b}(\cdot)}}(u,\omega)dx\right\}=-\int_0^L\theta\left(|v|^2\right)_xdx+\Re\left\{2a^{-1}b_0\widetilde{g}\int_0^\beta\theta v\overline{v_x}dx\right\}\\
&+\Re\left\{2i\lambda a^{-1}b_0\int_0^\beta\theta v\int_0^\infty g(s)\overline{\omega_x}(\cdot,s)dsdx\right\}+o(|\lambda|^{-\ell}).
\end{aligned}$$

Now, inserting the above equation in (3.4.49), we obtain

$$
\begin{aligned}
&-\int_0^L \theta\left(|v|^2+a^{-1}\left|S_{\tilde{b}(\cdot)}(u,\omega)\right|^2\right)_x dx+\Re\left\{2a^{-1}\int_0^L c(\cdot)\theta z\overline{S_{\tilde{b}(\cdot)}}(u,\omega)dx\right\}\\
&=-\Re\left\{2a^{-1}b_0\tilde{g}\int_0^\beta \theta v\overline{v_x}dx\right\}-\Re\left\{2i\lambda a^{-1}b_0\int_0^\beta \theta v\int_0^\infty g(s)\overline{\omega_x}(\cdot,s)dsdx\right\}\\
&\quad+o\left(|\lambda|^{-\ell}\right).
\end{aligned}
\tag{3.4.50}
$$

From (3.4.7), we deduce that

$$
i\lambda\overline{\omega_x}(\cdot,s)=\overline{\omega_{xs}}(\cdot,s)-\overline{v_x}-\lambda^{-\ell}\overline{f_x^5}(\cdot,s)\ \text{ in }\ (0,\beta)\times(0,\infty). \tag{3.4.51}
$$

Inserting (3.4.51) in the right hand side of (3.4.50), then using integration by parts with respect to s with the help of hypotheses (H) and the fact that $\omega(\cdot,0)=0$, we get

$$
\begin{aligned}
&-\int_0^L \theta\left(|v|^2+a^{-1}\left|S_{\tilde{b}(\cdot)}(u,\omega)\right|^2\right)_x dx+\Re\left\{2a^{-1}\int_0^L c(\cdot)\theta z\overline{S_{\tilde{b}(\cdot)}}(u,\omega)dx\right\}\\
&=-\Re\left\{\frac{2b_0}{a}\int_0^\beta \theta v\int_0^\infty -g'(s)\overline{\omega_x}(\cdot,s)dsdx\right\}-\Re\left\{\frac{2b_0}{a\lambda^\ell}\int_0^\beta \theta v\int_0^\infty g(s)\overline{f_x^5}(\cdot,s)dsdx\right\}\\
&\quad+o\left(|\lambda|^{-\ell}\right).
\end{aligned}
\tag{3.4.52}
$$

Using Cauchy-Schwarz inequality, the fact that v is uniformly bounded in $L^2(0,L)$, the definition of g and (3.4.8), we obtain

$$
\begin{cases}
-\Re\left\{2a^{-1}b_0\displaystyle\int_0^\beta \theta v\int_0^\infty -g'(s)\overline{\omega_x}(\cdot,s)dsdx\right\}=o\left(|\lambda|^{-\frac{\ell}{2}}\right),\\
\Re\left\{2a^{-1}\lambda^{-\ell}b_0\displaystyle\int_0^\beta \theta v\int_0^\infty g(s)\overline{f_x^5}(\cdot,s)dsdx\right\}=o\left(|\lambda|^{-\ell}\right).
\end{cases}
\tag{3.4.53}
$$

Inserting (3.4.53) in (3.4.52), then using integration by parts and the fact that $\theta(0)=\theta(L)=0$, we obtain

$$
\int_0^L \theta'\left(|v|^2+a^{-1}\left|S_{\tilde{b}(\cdot)}(u,\omega)\right|^2\right)dx+\Re\left\{2a^{-1}\int_0^L c(\cdot)\theta z\overline{S_{\tilde{b}(\cdot)}}(u,\omega)dx\right\}=o\left(|\lambda|^{-\frac{\ell}{2}}\right). \tag{3.4.54}
$$

Next, multiplying (3.4.6) by $2h\overline{y_x}$, integrating over $(0,L)$, taking the real part, then using the fact that y_x is uniformly bounded in $L^2(0,L)$ and $\|f^4\|_{L^2(0,L)}=o(1)$, we obtain

$$
\Re\left\{2i\lambda\int_0^L \theta z\overline{y_x}dx\right\}-\int_0^L \theta(|y_x|^2)_x dx-\Re\left\{2\int_0^L c(\cdot)\theta v\overline{y_x}dx\right\}=o(|\lambda|^{-\ell}). \tag{3.4.55}
$$

From (3.4.5), we deduce that

$$
i\lambda\overline{y_x}=-\overline{z_x}-\lambda^{-\ell}\overline{f_x^3}. \tag{3.4.56}
$$

Inserting (3.4.56) in (3.4.55), then using the fact that z is uniformly bounded in $L^2(0,L)$ and $\|f_x^3\|_{L^2(0,L)}=o(1)$, we obtain

$$
-\int_0^L \theta(|z|^2+|y_x|^2)_x dx-\Re\left\{2\int_0^L c(\cdot)\theta v\overline{y_x}dx\right\}=o(|\lambda|^{-\ell}). \tag{3.4.57}
$$

Using integration by parts in (3.4.57) and the fact that $\theta(0) = \theta(L) = 0$, we obtain

$$\int_0^L \theta'(|z|^2 + |y_x|^2)_x dx - \Re\left\{2\int_0^L c(\cdot)\theta v\overline{y_x}dx\right\} = o(|\lambda|^{-\ell}). \tag{3.4.58}$$

Finally, adding (3.4.54) and (3.4.58), we obtain the desired estimation (3.4.47). The proof is thus complete. $\square$

Let $0 < \varepsilon < \min\left(\alpha, \frac{\beta-\alpha}{5}\right)$, we fix cut-off functions $h_4, h_5 \in C^1([0, L])$ (see Figure 3.3) such that $0 \le h_4(x) \le 1$, $0 \le h_5(x) \le 1$, for all $x \in [0, L]$ and

$$h_4(x) = \begin{cases} 1 & \text{if} \quad x \in [0, \alpha + 2\varepsilon], \\ 0 & \text{if} \quad x \in [\beta - 3\varepsilon, L], \end{cases} \quad \text{and} \quad h_5(x) = \begin{cases} 0 & \text{if} \quad x \in [0, \alpha + 2\varepsilon], \\ 1 & \text{if} \quad x \in [\beta - 3\varepsilon, L], \end{cases}$$

and set $\max_{x\in[0,L]} |h_4'(x)| = M_{h_4'}$ and $\max_{x\in[0,L]} |h_5'(x)| = M_{h_5'}$.

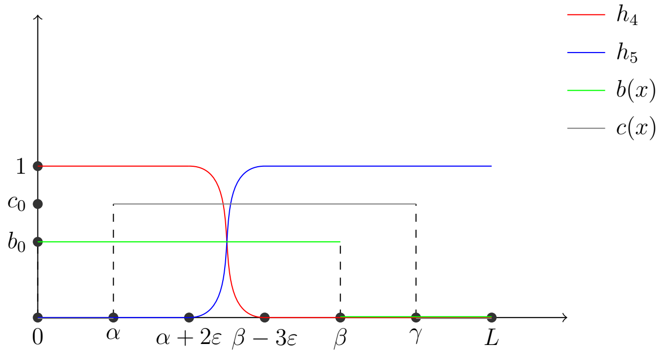

Figure 3.3: Geometric description of the functions h_4 and h_5.

Lemma 3.4.7. Let $0 < \varepsilon < \min\left(\alpha, \frac{\beta-\alpha}{5}\right)$. Under the hypotheses (H), the solution $U = (u, v, y, z, \omega(\cdot, s))^\top \in D(\mathcal{A})$ of System (3.4.3)-(3.4.7) satisfies the following estimations

$$\int_0^{\alpha+2\varepsilon} \left(|v|^2 + |y_x|^2 + |z|^2\right) dx \le K_1 \,|a-1||\lambda| \int_\alpha^{\beta-\varepsilon} |u_x||y_x| dx + o(1), \tag{3.4.59}$$

$$a\int_\beta^L |u_x|^2 dx + \int_{\beta-3\varepsilon}^L \left(|v|^2 + |y_x|^2 + |z|^2\right) dx \le K_2 \,|a-1||\lambda| \int_\alpha^{\beta-\varepsilon} |u_x||y_x| dx + o(1), \tag{3.4.60}$$

where $K_1 = \dfrac{4}{|c_0|}\left(1 + (\beta - 3\varepsilon) M_{h_4'}\right)$ and $K_2 = \dfrac{4}{|c_0|}(1 + (L - \alpha + 2\varepsilon)M_{h_5'})$.

Proof. First, using the result of Lemma 3.4.6 with $\theta = xh_4$ and the definition of $S_{\tilde{b}(\cdot)}(u, \omega)$ and $c(\cdot)$, we obtain

$$\begin{aligned} \int_0^{\alpha+2\varepsilon} \left(|v|^2 + |y_x|^2 + |z|^2\right) dx &= -\,a^{-1}\int_0^{\alpha+2\varepsilon} \left|S_{\tilde{b_0}}(u,\omega)\right|^2 dx \\ &- \int_{\alpha+2\varepsilon}^{\beta-3\varepsilon} (h_4 + xh_4')\left(|v|^2 + a^{-1}\left|S_{\tilde{b_0}}(u,\omega)\right|^2 + |y_x|^2 + |z|^2\right) dx \\ &- \Re\left\{2a^{-1}c_0\int_\alpha^{\beta-3\varepsilon} xh_4 z\overline{S_{\tilde{b_0}}}(u,\omega)dx\right\} + \Re\left\{2c_0\int_\alpha^{\beta-3\varepsilon} xh_4 v\overline{y_x}dx\right\} + o\left(|\lambda|^{-\frac{\ell}{2}}\right). \end{aligned}$$

Using Cauchy-Schwarz inequality in the above equation, we obtain

$$\begin{aligned}
&\int_0^{\alpha+2\varepsilon}\left(|v|^2+|y_x|^2+|z|^2\right)dx \leq a^{-1}\int_0^{\alpha+2\varepsilon}\left|S_{\widetilde{b_0}}(u,\omega)\right|^2 dx\\
&+\left(1+(\beta-3\varepsilon)M_{h_4'}\right)\int_{\alpha+2\varepsilon}^{\beta-3\varepsilon}\left(|v|^2+a^{-1}\left|S_{\widetilde{b_0}}(u,\omega)\right|^2+|y_x|^2+|z|^2\right)dx\\
&+2c_0(\beta-3\varepsilon)a^{-1}\left(\int_\alpha^{\beta-3\varepsilon}|z|^2dx\right)^{\frac12}\left(\int_\alpha^{\beta-3\varepsilon}|S_{\widetilde{b_0}}(u,\omega)|^2dx\right)^{\frac12}\\
&+2c_0(\beta-3\varepsilon)\left(\int_\alpha^{\beta-3\varepsilon}|v|^2dx\right)^{\frac12}\left(\int_\alpha^{\beta-3\varepsilon}|y_x|^2dx\right)^{\frac12}+o\left(|\lambda|^{-\frac{\ell}{2}}\right).
\end{aligned}$$

Thus, from the above inequality, Lemmas 3.4.1-3.4.4 and the fact that y_x, z are uniformly bounded in $L^2(0,L)$, we obtain (3.4.59). Next, using the result of Lemma 3.4.6 with $\theta=(x-L)h_5$ and the definition of $S_{\tilde{b}(\cdot)}(u,\omega)$ and $c(\cdot)$, we obtain

$$\begin{aligned}
&a\int_\beta^L|u_x|^2dx+\int_{\beta-3\varepsilon}^L\left(|v|^2+|z|^2+|y_x|^2\right)dx=-a^{-1}\int_{\beta-3\varepsilon}^\beta\left|S_{\widetilde{b_0}}(u,\omega)\right|^2dx\\
&-\int_{\alpha+2\varepsilon}^{\beta-3\varepsilon}(h_5+(x-L)h_5')\left(|v|^2+a^{-1}\left|S_{\widetilde{b_0}}(u,\omega)\right|^2+|y_x|^2+|z|^2\right)dx\\
&-\Re\left\{2a^{-1}c_0\int_{\alpha+2\varepsilon}^{\beta-3\varepsilon}(x-L)h_5z\overline{S_{\widetilde{b_0}}}(u,\omega)dx\right\}+\Re\left\{2c_0\int_{\alpha+2\varepsilon}^{\beta-3\varepsilon}(x-L)h_5v\overline{y_x}dx\right\}\\
&-\Re\left\{2a^{-1}b_0c_0\int_{\beta-3\varepsilon}^\beta(x-L)z\left(-\widetilde{g}u_x+\int_0^\infty g(s)\overline{\omega_x}(\cdot,s)ds\right)dx\right\}\\
&-\Re\left\{2c_0\int_{\beta-3\varepsilon}^\gamma(x-L)z\overline{u_x}dx\right\}+\Re\left\{2c_0\int_{\beta-3\varepsilon}^\gamma(x-L)v\overline{y_x}dx\right\}.
\end{aligned}$$

Using Cauchy-Schwarz inequality in the above equation, Lemmas 3.4.1-3.4.4 and the fact that y_x, z are uniformly bounded in $L^2(0,L)$, we obtain

$$\begin{aligned}
&a\int_\beta^L|u_x|^2dx+\int_{\beta-3\varepsilon}^L\left(|v|^2+|z|^2+|y_x|^2\right)dx\\
\leq\ &\frac{4}{|c_0|}\left(1+(L-\alpha-2\varepsilon)M_{h_5'}\right)|a-1||\lambda|\int_\alpha^{\beta-\varepsilon}|u_x||y_x|dx+\mathcal{I}+o(1),
\end{aligned}\tag{3.4.61}$$

where

$$\mathcal{I}:=\Re\left\{2c_0\int_{\beta-3\varepsilon}^\gamma(x-L)v\overline{y_x}dx\right\}-\Re\left\{2c_0\int_{\beta-3\varepsilon}^\gamma(x-L)z\overline{u_x}dx\right\}.\tag{3.4.62}$$

From (3.4.3) and (3.4.5), we have

$$\overline{u_x}=i\lambda^{-1}\overline{v_x}+i\lambda^{-(\ell+1)}\overline{f_x^1}\quad\text{and}\quad\overline{y_x}=i\lambda^{-1}\overline{z_x}+i\lambda^{-(\ell+1)}\overline{f_x^3}.\tag{3.4.63}$$

Inserting (3.4.63) in (3.4.62), then using the fact that v, z are uniformly bounded in $L^2(0,L)$ and $\|f_x^1\|_{L^2(0,L)}=o(1)$, $\|f_x^3\|_{L^2(0,L)}=o(1)$, we obtain

$$\mathcal{I}=\Re\left\{2c_0i\lambda^{-1}\int_{\beta-3\varepsilon}^\gamma(x-L)v\overline{z_x}dx\right\}-\Re\left\{2c_0i\lambda^{-1}\int_{\beta-3\varepsilon}^\gamma(x-L)z\overline{v_x}dx\right\}+o(|\lambda|^{-(\ell+1)}).\tag{3.4.64}$$

Using integration by parts to the second term in (3.4.64), we obtain

$$\mathcal{I} = \Re\left\{2c_0 i\lambda^{-1}\int_{\beta-3\varepsilon}^{\gamma} z\overline{v}dx\right\} - \Re\left\{2c_0 i\lambda^{-1}\left[(x-L)z\overline{v}\right]_{\beta-3\varepsilon}^{\gamma}\right\} + o(|\lambda|^{-(\ell+1)}). \tag{3.4.65}$$

From Lemma 3.4.5, we deduce that

$$|v(\gamma)| = O(\sqrt{|\lambda|}),\quad |v(\beta-3\varepsilon)| = O(\sqrt{|\lambda|}),\quad |z(\gamma)| = O(1)\quad \text{and}\quad |z(\beta-3\varepsilon)| = O(1). \tag{3.4.66}$$

Using Cauchy-Schwarz inequality, (3.4.66) and the fact that v, z are uniformly bounded in $L^2(0,L)$, we obtain

$$\begin{cases} \Re\left\{2c_0 i\lambda^{-1}\displaystyle\int_{\beta-3\varepsilon}^{\gamma} z\overline{v}dx\right\} = O\left(|\lambda|^{-1}\right) = o(1), \\ -\Re\left\{2c_0 i\lambda^{-1}\left[(x-L)z\overline{v}\right]_{\beta-3\varepsilon}^{\gamma}\right\} = O\left(|\lambda|^{-\frac{1}{2}}\right) = o(1). \end{cases}$$

Inserting the above estimations in (3.4.65), we get

$$\mathcal{I} = o(1).$$

Finally, from the above estimation and (3.4.61), we obtain the desired estimation (3.4.60). The proof is thus complete. $\square$

Proof of Theorem 3.4.1. The proof of Theorem 3.4.1 is divided into three steps.
Step 1. Under the hypotheses (H), by taking $a = 1$ and $\ell = 0$ in Lemmas 3.4.1-3.4.4, we obtain

$$\begin{cases} \displaystyle\int_0^\beta\int_0^\infty g(s)|\omega_x(\cdot,s)|^2 ds dx = o(1),\ \int_0^\beta |u_x|^2 dx = o(1),\ \int_\varepsilon^{\beta-\varepsilon} |v|^2 dx = o(1), \\ \displaystyle\int_{\alpha+\varepsilon}^{\beta-2\varepsilon} |y_x|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha+2\varepsilon}^{\beta-3\varepsilon} |z|^2 dx = o(1). \end{cases} \tag{3.4.67}$$

Step 2. Using the fact that $a = 1$ and (3.4.67) in Lemma 3.4.7, we obtain

$$\begin{cases} \displaystyle\int_0^\varepsilon |v|^2 dx = o(1),\ \int_{\beta-\varepsilon}^L |v|^2 dx = o(1),\ \int_\beta^L |u_x|^2 dx = o(1),\ \int_0^{\alpha+\varepsilon} |y_x|^2 dx = o(1), \\ \displaystyle\int_{\beta-2\varepsilon}^L |y_x|^2 dx = o(1),\ \int_0^{\alpha+2\varepsilon} |z|^2 dx = o(1) \quad \text{and} \quad \int_{\beta-3\varepsilon}^L |z|^2 dx = o(1). \end{cases} \tag{3.4.68}$$

Step 3. According to **Step 1** and **Step 2**, we obtain $\|U\|_{\mathcal{H}} = o(1)$, which contradicts ($\mathrm{H}_1$). Therefore, ($\mathrm{H}_1$) holds, and so by Theorem 1.3.6, we deduce that system (3.2.2)-(3.2.6) is exponentially stable. $\square$

Proof of Theorem 3.4.2. The proof of Theorem 3.4.2 is divided into three steps.
Step 1. Under the hypotheses (H) and $a \neq 1$, using the fact that y_x is uniformly bounded in $L^2(0,L)$ and (3.4.9) in estimation (3.4.20), we get

$$\int_{\alpha+\varepsilon}^{\beta-2\varepsilon} |y_x|^2 dx = o(|\lambda|^{-\frac{\ell}{2}+1}) \quad \text{and} \quad \int_{\alpha+2\varepsilon}^{\beta-3\varepsilon} |z|^2 dx = o(|\lambda|^{-\frac{\ell}{2}+1}).$$

Taking $\ell = 2$ in the above estimations, we obtain

$$\int_{\alpha+\varepsilon}^{\beta-2\varepsilon} |y_x|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha+2\varepsilon}^{\beta-3\varepsilon} |z|^2 dx = o(1). \tag{3.4.69}$$

Taking $\ell = 2$ in Lemmas 3.4.1, 3.4.2, we obtain

$$\int_0^\beta \int_0^\infty g(s)|\omega_x(\cdot, s)|^2 ds dx = o(\lambda^{-2}), \; \int_0^\beta |u_x|^2 dx = o(\lambda^{-2}), \; \int_\varepsilon^{\beta-\varepsilon} |v|^2 dx = o(|\lambda|^{-1}). \tag{3.4.70}$$

Step 2. Using the fact that $a \neq 1$, y_x is uniformly bounded in $L^2(0, L)$ and (3.4.70) in Lemma 3.4.7, we obtain

$$\int_0^{\alpha+2\varepsilon} \left(|v|^2 + |y_x|^2 + |z|^2\right) dx = o(1), \tag{3.4.71}$$

$$a \int_\beta^L |u_x|^2 dx + \int_{\beta-3\varepsilon}^L \left(|v|^2 + |y_x|^2 + |z|^2\right) dx = o(1). \tag{3.4.72}$$

Using (3.4.69) and (3.4.70) in (3.4.71) and (3.4.72), we obtain

$$\begin{cases} \int_0^\varepsilon |v|^2 dx = o(1), \; \int_{\beta-\varepsilon}^L |v|^2 dx = o(1), \; \int_\beta^L |u_x|^2 dx = o(1), \; \int_0^{\alpha+\varepsilon} |y_x|^2 dx = o(1), \\ \int_{\beta-2\varepsilon}^L |y_x|^2 dx = o(1), \; \int_0^{\alpha+2\varepsilon} |z|^2 dx = o(1) \quad \text{and} \quad \int_{\beta-3\varepsilon}^L |z|^2 dx = o(1). \end{cases} \tag{3.4.73}$$

Step 3. According to **Step 1** and **Step 2**, we obtain $\|U\|_{\mathcal{H}} = o(1)$, which contradicts ($\mathrm{H}_1$). This implies that

$$\limsup_{\lambda \in \mathbb{R}, \; |\lambda| \to \infty} \frac{1}{\lambda^2} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} < \infty.$$

Finally, according to Theorem 1.3.7, we obtain the desired result. The proof is thus complete. □

3.5 Lack of exponential stability with global past history damping in case of different speed propagation waves ($a \neq 1$)

This section is independent from the previous ones, here we prove the lack of exponential stability with global past history damping and global coupling. For this aim, we consider the following system:

$$\begin{cases} u_{tt} - a u_{xx} + \int_0^\infty g(s) u_{xx}(x, t-s) ds + y_t = 0, & (x,t) \in (0, L) \times (0, \infty), \\ y_{tt} - y_{xx} - u_t = 0, & (x,t) \in (0, L) \times (0, \infty), \\ u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, & t > 0, \\ (u(x, -s), u_t(x, 0)) = (u_0(x, s), u_1(x)), & (x, s) \in (0, L) \times (0, \infty), \\ (y(x, 0), y_t(x, 0)) = (y_0(x), y_1(x)), & x \in (0, L), \end{cases} \tag{3.5.1}$$

and the general integral term represent a history term with the relaxation function g that is supposed to satisfy the following hypotheses

$$\begin{cases} g \in L^1([0,\infty)) \cap C^1([0,\infty)) \text{ is a positive function such that} \\ g(0) := g_0 > 0, \quad \displaystyle\int_0^\infty g(s)ds := \widetilde{g}, \quad \widetilde{a} := a - \widetilde{g} > 0, \quad \text{and} \\ g'(s) \leq -mg(s), \quad \text{for some } m > 0, \ \forall s \geq 0. \end{cases} \tag{H$_\mathrm{G}$}$$

Now, as in [35], we introduce the following auxiliary change of variable

$$\omega(x,s,t) := u(x,t) - u(x,t-s), \quad (x,s,t) \in (0,L) \times (0,\infty) \times (0,\infty). \tag{3.5.2}$$

Then system (3.5.1) becomes

$$u_{tt} - \widetilde{a}u_{xx} - \int_0^\infty g(s)\omega_{xx}(\cdot,s,t) + y_t = 0, \qquad (x,t) \in (0,L) \times (0,\infty), \tag{3.5.3}$$

$$y_{tt} - y_{xx} - u_t = 0, \qquad (x,t) \in (0,L) \times (0,\infty), \tag{3.5.4}$$

$$\omega_t(\cdot,s,t) + \omega_s(\cdot,s,t) - u_t = 0, \quad (x,s,t) \in (0,L) \times (0,\infty) \times (0,\infty), \tag{3.5.5}$$

with the following boundary conditions

$$\begin{cases} u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, \quad t > 0, \\ \omega(\cdot,0,t) = 0, \quad (x,t) \in (0,L) \times (0,\infty), \\ \omega(0,s,t) = \omega(L,s,t) = 0, \quad (s,t) \in (0,\infty) \times (0,\infty), \end{cases} \tag{3.5.6}$$

and the following initial conditions

$$\begin{cases} u(\cdot,-s) = u_0(\cdot,s), & u_t(\cdot,0) = u_1(\cdot), & (x,s) \in (0,L) \times (0,\infty), \\ y(\cdot,0) = y_0(\cdot), & y_t(\cdot,0) = y_1(\cdot), & x \in (0,L), \\ \omega_0(\cdot,s) := \omega(\cdot,s,0) = u_0(\cdot,0) - u_0(\cdot,s), & & (x,s) \in (0,L) \times (0,\infty). \end{cases} \tag{3.5.7}$$

The energy of system (3.5.3)-(3.5.7) is given by

$$E_G(t) = \frac{1}{2}\int_0^L \left(|u_t|^2 + \widetilde{a}|u_x|^2 + |y_t|^2 + |y_x|^2\right) dx + \frac{1}{2}\int_0^L \int_0^\infty g(s)|\omega_x(\cdot,s,t)|^2 ds dx. \tag{3.5.8}$$

Under the hypotheses (H$_\mathrm{G}$) and by letting $U = (u,v,y,z,\omega)$ be a regular solution of system (3.5.3)-(3.5.7), then we get with the help of (3.5.6) that

$$\frac{d}{dt}E_G(t) = \frac{1}{2}\int_0^L \int_0^\infty g'(s)|\omega_x(\cdot,s,t)|^2 ds dx \leq 0,$$

which implies that the system (3.5.3)-(3.5.7) is dissipative in the sense that its energy is non-increasing with respect to time. Now, we define the following Hilbert space $\mathcal{H}_G$ by

$$\mathcal{H}_G = \left(H_0^1(0,L) \times L^2(0,L)\right)^2 \times L_g^2((0,\infty); H_0^1(0,L)),$$

and it is equipped with the following inner product

$$\left(U, U^1\right)_{\mathcal{H}_G} = \int_0^L \left(\widetilde{a}u_x\overline{u_x^1} + v\overline{v^1} + y_x\overline{y_x^1} + z\overline{z^1}\right) dx + \int_0^L \int_0^\infty g(s)\omega_x(\cdot,s)\overline{\omega_x^1}(\cdot,s) ds dx,$$

where $U=(u,v,y,z,\omega(\cdot,s))^\top\in\mathcal{H}_G$ and $U^1=(u^1,v^1,y^1,z^1,\omega^1(\cdot,s))^\top\in\mathcal{H}_G$. We define the linear unbounded operator $\mathcal{A}_G : D(\mathcal{A}_G)\subset\mathcal{H}_G\longmapsto\mathcal{H}_G$ by:

$$D(\mathcal{A}_G)=\left\{\begin{array}{l} U=(u,v,y,z,\omega(\cdot,s))^\top\in\mathcal{H}_G \mid y\in H^2(0,L)\cap H_0^1(0,L),\quad v,z\in H_0^1(0,L)\\ \left(\widetilde{a}u_x+\displaystyle\int_0^\infty g(s)\omega_x(\cdot,s)ds\right)_x\in L^2(0,L),\quad \omega_s(\cdot,s)\in L_g^2((0,\infty);H_0^1(0,L)),\\ \omega(\cdot,0)=0.\end{array}\right\}$$

and

$$\mathcal{A}_G\begin{pmatrix}u\\ v\\ y\\ z\\ \omega(\cdot,s)\end{pmatrix}=\begin{pmatrix} v\\ \left(\widetilde{a}u_x+\displaystyle\int_0^\infty g(s)\omega_x(\cdot,s)ds\right)_x-z\\ z\\ y_{xx}+v\\ -\omega_s(\cdot,s)+v\end{pmatrix}.$$

Now, if $U=(u,u_t,y,y_t,\omega(\cdot,s))^\top$, then system (3.5.3)-(3.5.7) can be written as the following first order evolution equation

$$U_t=\mathcal{A}_GU,\quad U(0)=U_0, \tag{3.5.9}$$

where $U_0=(u_0(\cdot,0),u_1,y_0,y_1,\omega_0(\cdot,s))^\top\in\mathcal{H}_G$.

Theorem 3.5.1. Under the hypotheses ($\mathrm{H_G}$). If $a\neq 1$, then for any $0<\epsilon<2$, we can not expect the energy decay rate $t^{-\frac{2}{2-\epsilon}}$ for every $U_0\in D(\mathcal{A}_G)$.

Proof. Following Huang [67] and Pruss [94] (see also Theorem 1.3.6), it is sufficient to show the existence of sequences $(\lambda_n)_n\subset\mathbb{R}_+^*$ with $\lambda_n\to\infty$, $(U_n)_n\subset D(\mathcal{A}_G)$ and $(F_n)_n\subset\mathcal{H}_G$ such that $(i\lambda_nI-\mathcal{A}_G)U_n=F_n$ is bounded in $\mathcal{H}_G$ and

$$\lim_{n\to\infty}\lambda_n^{-2+\epsilon}\|U_n\|_{\mathcal{H}_G}=\infty. \tag{3.5.10}$$

For this aim, take

$$F_n=\left(0,0,0,\sin\left(\frac{n\pi x}{L}\right),0\right)\quad\text{and}\quad U_n=(u_n,i\lambda_nu_n,y_n,i\lambda_ny_n,\omega_n)$$

such that

$$\begin{cases}\lambda_n=\dfrac{n\pi}{L}-\dfrac{L}{2n\pi(a-1)}\ \text{ such that }\ n^2>\dfrac{L^2}{2\pi^2(a-1)},\\ u_n(x)=A_n\sin\left(\dfrac{n\pi x}{L}\right),\ y_n(x)=B_n\sin\left(\dfrac{n\pi x}{L}\right),\\ \omega_n(x,s)=A_n(1-e^{-i\lambda_ns})\sin\left(\dfrac{n\pi x}{L}\right),\end{cases} \tag{3.5.11}$$

where A_n and B_n are complex numbers depending on n and determined explicitly in the sequel. Note that this choice is compatible with the boundary conditions. So, its is clear that $\lambda_n>0$, $\lim\limits_{n\to\infty}\lambda_n=\infty$, F_n is uniformly bounded in $\mathcal{H}$ and $U_n\in D(\mathcal{A}_G)$. Next, detailing $i\lambda_nU_n-\mathcal{A}_GU_n=F_n$, we get

$$\begin{cases} iA_nL^2\lambda_n+\left(\lambda_n^2L^2-\pi^2n^2\right)B_n=-L^2,\\ \left(n^2\pi^2(a-g_{\lambda_n})-\lambda_n^2L^2\right)A_n+iL^2\lambda_nB_n=0,\end{cases} \tag{3.5.12}$$

where $g_{\lambda_n} = \displaystyle\int_0^\infty g(s)e^{-i\lambda_n s}ds$. From the first equation of (3.5.12), we get

$$A_n = \frac{i}{\lambda_n} + \frac{i(L^2\lambda_n^2 - \pi^2 n^2)B_n}{L^2\lambda_n}. \tag{3.5.13}$$

Inserting (3.5.13) in the second equation of (3.5.12), we get

$$B_n = \frac{\left(\lambda_n^2 L^2 - (a - g_{\lambda_n})n^2\pi^2\right)L^2}{-n^4(a - g_{\lambda_n})\pi^4 + L^2\pi^2 n^2\lambda_n^2(a + 1 - g_{\lambda_n}) + L^4(\lambda_n^2 - \lambda_n^4)}.$$

Consequently, the solution of (3.5.12) is given by

$$A_n = \frac{i}{\lambda_n} + \frac{i(L^2\lambda_n^2 - \pi^2 n^2)B_n}{L^2\lambda_n} \quad \text{and} \quad B_n = B_{1,n}\left(1 + \frac{B_{2,n}}{\lambda_n g_{\lambda_n} + B_{3,n}}\right), \tag{3.5.14}$$

where

$$\begin{cases} B_{1,n} = \dfrac{L^2}{(n^2\pi^2 - L^2\lambda_n^2)}, \quad B_{2,n} = \dfrac{L^4\lambda_n^3}{n^2\pi^2\left(\lambda_n^2 L^2 - n^2\pi^2\right)} \\ B_{3,n} = \dfrac{\left(-\pi^4 a n^4 + L^2 n^2\lambda_n^2(a+1)\pi^2 + L^4(\lambda_n^2 - \lambda_n^4)\right)\lambda_n}{n^2\pi^2\left(n^2\pi^2 - L^2\lambda_n^2\right)}. \end{cases}$$

Now, inserting λ_n given in (3.5.11) in the above equation, then using asymptotic expansion, we get

$$B_{1,n} = a - 1 + O(n^{-2}), \quad B_{2,n} = \frac{1-a}{L}\pi n + O(n^{-1}), \quad B_{3,n} = O(n^{-1}). \tag{3.5.15}$$

On the other hand, using hypotheses ($\mathrm{H_G}$) and integration by parts, we obtain

$$\lambda_n g_{\lambda_n} = -ig_0 - i\int_0^\infty g'(s)e^{-i\lambda_n s}ds.$$

It is clear from Riemann-Lebesgue Lemma that the second term in the above equation goes to zero as $\lambda_n \to \infty$. Thus, we obtain

$$\lambda_n g_{\lambda_n} = -ig_0 + o(1). \tag{3.5.16}$$

Substituting (3.5.15) and (3.5.16) in (3.5.14), we get

$$A_n = O(1) \quad \text{and} \quad B_n = \left(-\frac{i(a-1)^2}{g_0 L} + o(1)\right)n\pi.$$

Therefore, from the above equation and (3.5.16), we get

$$z_n(x) = i\lambda_n B_n \sin\left(\frac{n\pi x}{L}\right) = \left(\frac{(a-1)^2}{g_0 L^2} + o(1)\right)n^2\pi^2 \sin\left(\frac{n\pi x}{L}\right).$$

Consequently,

$$\left(\int_0^L |z_n|^2 dx\right)^{\frac{1}{2}} \sim \sqrt{\frac{L}{2}}\left(\frac{(a-1)^2}{g_0 L^2} + o(1)\right)n^2\pi^2.$$

Since

$$\|U_n\|_{\mathcal{H}} \geq \left(\int_0^L |z_n|^2 dx\right)^{\frac{1}{2}} \sim \sqrt{\frac{L}{2}}\left(\frac{(a-1)^2}{g_0 L^2} + o(1)\right)n^2\pi^2 \sim \lambda_n^2,$$

then for all $0 < \epsilon < 2$, we have

$$\lambda_n^{-2+\epsilon}\|U_n\|_{\mathcal{H}_1} \sim \lambda_n^{\epsilon} \to \infty \quad \text{as} \quad n \to \infty,$$

hence, we get (3.5.10). Consequently, we cannot expect the energy decay rate $t^{-\frac{2}{2-\epsilon}}$. The proof is thus complete. $\square$

Remark 3.5.1. In [16] and [33], the authors proved the lack of exponential stability of a coupled wave equations system with past history damping by taking a particular relaxation function $g(s) = e^{-\mu s}$ such that $s \in \mathbb{R}_+$ and $\mu > 1$. $\square$

3.6 Conclusion and Future Works

We have studied the stabilization of a locally coupled wave equations with local viscoelastic damping of past history type acting only in one equation via non-smooth coefficients. We proved the strong stability of the system by using Arendt-Batty criteria. We established the exponential stability of the solution if the waves have the same speed propagation (i.e. $a = 1$). In the case $a \neq 1$, we proved that the energy of our system decays polynomially with the rate t^{-1}. Lack of exponential stability result has been proved in case that the speeds of wave propagation are different with a global damping and a global coupling (i.e. $a \neq 1$ and $b = c = 1$). According to Theorem 3.5.1, we can conjecture that the energy decay rate t^{-1} is optimal but this question remains open. Moreover, it would be interesting to

1. study system (3.1.1) in the multidimensional case,

2. obtain the decay rates of system (3.1.1) for a much wider class of relaxation functions g, like in [54, 55] and using the recent results from [99],

3. study system (3.1.1) with local internal past history damping, in other words, by only assuming that b is positive on a non empty subinterval of $(0, L)$ that could be away from the boundary.

Chapter 4

On the Stability of Bresse system with one discontinuous local internal Kelvin-Voigt damping on the axial force

In this chapter, we investigate the stabilization of a linear Bresse system with one discontinuous local internal viscoelastic damping of Kelvin-Voigt type acting on the axial force, under fully Dirichlet boundary conditions. First, using a general criteria of Arendt-Batty, we prove the strong stability of our system. Finally, using a frequency domain approach combined with the multiplier method, we prove that the energy of our system decays polynomially with different rates. This chapter is published in [5].

4.1 Introduction

In this chapter, we investigate the stability of a Bresse system with only one discontinuous local internal Kelvin-Voigt damping on the axial force. More precisely, we consider the following system in $(0, L) \times (0, \infty)$:

$$\begin{cases} \rho_1 \varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) - ld(x)(w_{tx} - l\varphi_t) = 0, \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi + lw) = 0, \\ \rho_1 w_{tt} - [k_3(w_x - l\varphi) + d(x)(w_{tx} - l\varphi_t)]_x + lk_1(\varphi_x + \psi + lw) = 0, \end{cases} \tag{4.1.1}$$

with the following Dirichlet boundary conditions

$$\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = w(0,t) = w(L,t) = 0, \quad t > 0, \tag{4.1.2}$$

and the following initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \ \varphi_t(x,0) = \varphi_1(x), \ \psi(x,0) = \psi_0(x), \ x \in (0,L), \\ \psi_t(x,0) = \psi_1(x), \ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x), \ x \in (0,L), \end{cases} \tag{4.1.3}$$

where $\rho_1, \rho_2, k_1, k_2, k_3, l$ and L are positive real numbers. We suppose that there exist $0 < \alpha < \beta < L$ and a positive constant d_0, such that

$$d(x) = \begin{cases} d_0 & \text{if} \quad x \in (\alpha, \beta), \\ 0 & \text{if} \quad x \in (0, \alpha) \cup (\beta, L). \end{cases} \tag{4.1.4}$$

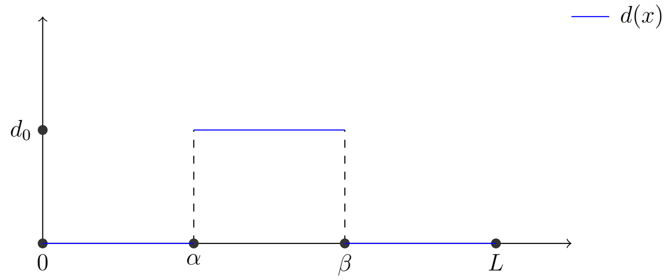

Figure 4.1: Geometric description of the function $d(x)$.

The Bresse system is a model for arched beams (see Fig. 4.2 for an illustration), see [74, Chap. 6]. It can be expressed by the equations of motion:

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + lN, \\ \rho_2\psi_{tt} = M_x - Q, \\ \rho_1 w_{tt} = N_x - lQ, \end{cases} \tag{4.1.5}$$

where $N = k_3(w_x - l\varphi) + d(x)(w_{tx} - l\varphi_t)$ is the axial force, $Q = k_1(\varphi_x + \psi + lw)$ is the shear force, and $M = k_2\psi_x$ is the bending moment. The functions φ, ψ, and w are respectively the vertical, shear angle, and longitudinal displacements. Here $\rho_1 = \rho A$, $\rho_2 = \rho I$, $k_1 = kGA$, $k_3 = EA$, $k_2 = EI$ and $l = R^{-1}$, in which ρ is the density of the material, E the modulus of the elasticity, G the shear modulus, k the shear factor, A the cross-sectional area, I the second moment of area of the cross section, R the radius of the curvature, and l the curvature.

There are several publications concerning the stabilization of Bresse system with different kinds of damping (see [1], [4], [14], [40], [43], [46], [47], [48], [56], [57] [79], [86], [90] and [106]). We note that by neglecting w ($l \to 0$) in (4.1.5), the Bresse system reduces to the following conservative Timoshenko system:

$$\rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x = 0,$$
$$\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) = 0.$$

There are also several publications concerning the stabilization of Timoshenko system with different kinds of damping (see [9], [25], [26] and [105]).

In the recent years, many researchers showed interest in problems involving Kelvin-Voigt damping where different types of stability, depending on the smoothness of the damping coefficients, has been showed (see [17], [18], [62], [63], [66], [76], [80], [91] and [98]). Moreover, there is a number of new results concerning systems with local Kelvin-Voigt damping and non-smooth coefficients at the interface (see [7], [103], [50], [51], [52], [65] and [104]).

Among this vast literature let us recall some specific results on the Bresse systems.

In 2017, Guesmia in [56] studied the stability of Bresse system with one infinite memory in the longitudinal displacement (i.e. third equation) under Dirichlet-Neumann-Neumann boundary conditions, he established some stability results under a smallness condition on l and on $\int_0^\infty g(s)ds$, where l is the curvature and g is the memory kernel. In

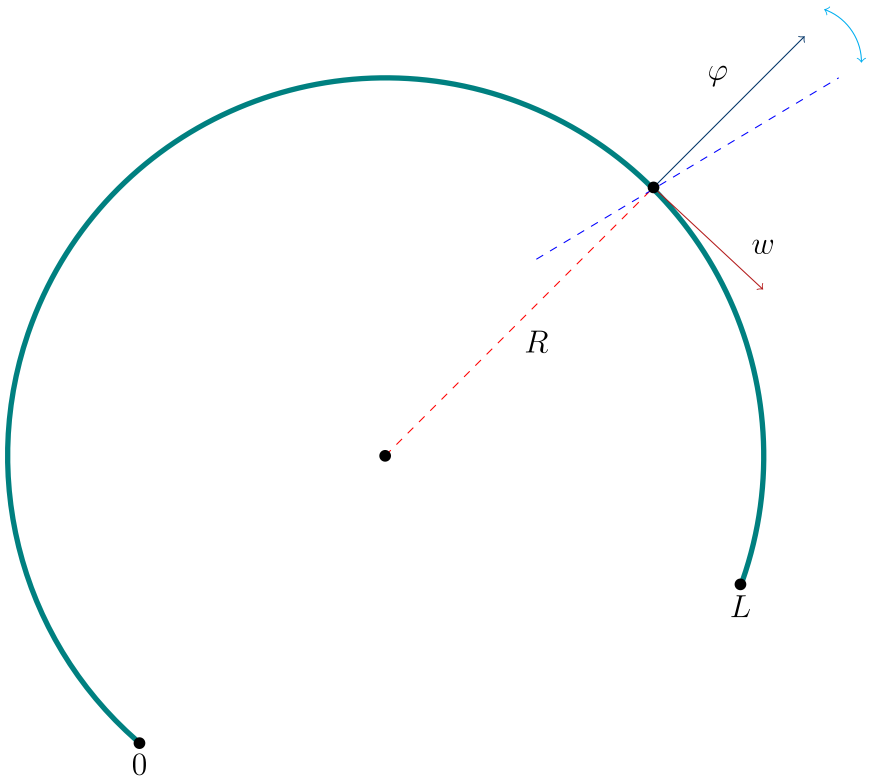

Figure 4.2: The circular arch

2018, Afilal *et al.* in [4] studied the stability of Bresse system with global frictional damping in the longitudinal displacement, by considering the following system on $(0,1)\times(0,\infty)$:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) = 0, \\ \rho_1 w_{tt} - k_3(w_x - l\varphi) + lk_1(\varphi_x + \psi + lw) + \delta w_t = 0, \end{cases} \tag{4.1.6}$$

with the initial conditions (4.1.3) where $L = 1$ and under mixed boundary conditions of the form:

$$\begin{cases} \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = 0, & \text{in } (0,\infty), \\ \varphi_x(1,t) = \psi(1,t) = w(1,t) = 0, & \text{in } (0,\infty), \end{cases}$$

where δ is a positive real number, they assumed that:

$$l \neq \frac{\pi}{2} + m\pi, \quad \forall m \in \mathbb{N}. \tag{4.1.7}$$

They proved under (4.1.7), the strong stability of system (4.1.6) provided that the curvature l satisfies:

$$l^2 \neq \frac{\rho_2 k_3 + \rho_1 k_2}{\rho_2 k_3}\left(\frac{\pi}{2} + m\pi\right)^2 + \frac{\rho_1 k_1}{\rho_2(k_1 + k_3)}, \quad \forall m \in \mathbb{Z}. \tag{4.1.8}$$

Also, they established under (4.1.7) and (4.1.8), the exponential stability of system (4.1.6) if and only if $\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} = \frac{k_3}{\rho_1}$. Otherwise, they established polynomial energy decay rate of order

$t^{-\frac{1}{4}}$. In 2019, Fatori *et al.* in [46] proved under

$$lL \quad \text{is not a multiple of} \quad \pi, \tag{4.1.9}$$

the strong stability of system (4.1.6) on $(0, L) \times (0, \infty)$ under Dirichlet-Neumann-Neumann boundary conditions provided that:

$$\begin{aligned} & k_1\rho_1 - \rho_2(k_3 + k_1)l^2 \geq 0 \quad \text{or} \\ & 0 < \rho_2(k_3 + k_1)l^2 - k_1\rho_1 \neq \frac{\rho_1\rho_2(k_3 + k_1)}{k_3}\left(\frac{k_3}{\rho_1}n^2 + \frac{k_2}{\rho_2}m^2\right)\frac{\pi^2}{L^2}, \end{aligned} \tag{4.1.10}$$

for all $m \in \mathbb{N}$ and $n \in \mathbb{N}^\star$. Also, they established under (4.1.9) and (4.1.10) the exponential stability of system (4.1.6) on $(0, L) \times (0, \infty)$ if and only if

$$\frac{\rho_1}{\rho_2} = \frac{k_1}{k_2} \quad \text{and} \quad k_1 = k_3. \tag{4.1.11}$$

Moreover, they used the previous results (i.e. strong and exponential stability of (4.1.6) on $(0, L) \times (0, \infty)$) to obtain under (4.1.9), (4.1.10) and (4.1.11) the exponential stability of Bresse system with indefinite memory in the longitudinal displacement under Dirichlet-Neumann-Neumann boundary conditions.

In 2019, El Arwadi and Youssef in [43] studied the stabilization of the Bresse beam with three global Kelvin-Voigt damping under fully Dirichlet boundary conditions, they established an exponential energy decay rate. In 2020, Gerbi *et al.* in [49] studied the stabilization of non-smooth transmission problem involving Bresse systems with fully Dirichlet or Dirichlet-Neumann-Neumann boundary conditions, by considering system (4.1.5) on $(0, L) \times (0, \infty)$ with

$$N = k_3(w_x - l\varphi) + D_3(w_{xt} - l\varphi_t), \quad Q = k_1(\varphi_x + \psi + lw) + D_1(\varphi_{xt} + \psi_t + lw_t), \quad M = k_2\psi_x + D_2\psi_{xt},$$

where D_1, D_2 and D_3 are bounded positive functions over $(0, L)$. They established:

- Analytic stability in the case of three global Kelvin-Voigt dampings (i.e. $D_i \in L^\infty(0, L)$, $D_i \geq d_0 > 0$ in $(0, L)$, $i = 1, 2, 3$).

- Exponential stability in the case of three local Kelvin-Voigt dampings with smooth coefficients at the interface (i.e. $D_i \in W^{1,\infty}(0, L)$, $D_i \geq d_0 > 0$ in $\emptyset \neq \omega := (\alpha, \beta) \subset (0, L)$, $i = 1, 2, 3$).

- Polynomial energy decay rate of order t^{-1} in the case of three local Kelvin-Voigt dampings with non-smooth coefficients at the interface (i.e. $D_i \in L^\infty(0, L)$, $D_i \geq d_0^i > 0$ in $(\alpha_i, \beta_i) \subset (0, L)$, $i = 1, 2, 3$, and $\bigcap_{i=1}^{3}(\alpha_i, \beta_i) = \omega$).

- Polynomial stability energy decay rate of order $t^{-\frac{1}{2}}$ in the case of one local Kelvin-Voigt damping on the bending moment with non-smooth coefficient at the interface (i.e. $D_1 = D_3 = 0$, $D_2 \in L^\infty(0, L)$ and $D_2 \geq d_0 > 0$ in ω).

.

But to the best of our knowledge, it seems that no result in the literature exists concerning the case of Bresse system with only one discontinuous local internal Kelvin-Voigt damping on the axial force, especially under fully Dirichlet boundary conditions and without any condition on the curvature l. The goal of the present chapter is to fill this gap by studying the stability of system (4.1.1)-(4.1.3).

This chapter is organized as follows: In Section 4.2, we prove the well-posedness of our system by using semigroup approach. In Section 4.3, following a general criteria of Arendt Batty, we show the strong stability of our system in the absence of the compactness of the resolvent. Finally, in Section 4.4, by using the frequency domain approach combining with a specific multiplier method, we prove that the energy of our system decays polynomially with the rates:

$$\begin{cases} t^{-1} & \text{if } \dfrac{k_1}{\rho_1} = \dfrac{k_2}{\rho_2}, \\ t^{-\frac{1}{2}} & \text{if } \dfrac{k_1}{\rho_1} \neq \dfrac{k_2}{\rho_2}. \end{cases}$$

4.2 Well-posedness of the system

In this section, we will establish the well-posedness of system (4.1.1)-(4.1.3) by using semigroup approach. The energy of system (4.1.1)-(4.1.3) is given by

$$E(t) = \frac{1}{2}\int_0^L \left(\rho_1 |\varphi_t|^2 + \rho_2|\psi_t|^2 + \rho_1|w_t|^2 + k_1|\varphi_x + \psi + lw|^2 + k_2|\psi_x|^2 + k_3|w_x - l\varphi|^2\right) dx.$$

Let $(\varphi, \varphi_t, \psi, \psi_t, w, w_t)$ be a regular solution of system (4.1.1)-(4.1.3). Multiplying the equations in (4.1.1) by $\overline{\varphi_t}$, $\overline{\psi_t}$ and $\overline{w_t}$ respectively, Then using the boundary conditions (4.2.14) and the definition of $d(x)$ (see (4.1.4) and Figure 4.1), we obtain

$$E'(t) = -\int_0^L d(x)|w_{tx} - l\varphi_t|^2 dx = -d_0\int_\alpha^\beta |w_{tx} - l\varphi_t|^2 dx \leq 0. \tag{4.2.1}$$

From (4.2.1), system (4.1.1)-(4.1.3) is dissipative in the sense that its energy is non-increasing with respect to time. Now, we define the following Hilbert space $\mathcal{H}$ by:

$$\mathcal{H} := \left(H_0^1(0, L) \times L^2(0, L)\right)^3.$$

The Hilbert space $\mathcal{H}$ is equipped with the following inner product and norm

$$(U, U^1)_{\mathcal{H}} = \int_0^L \Big\{k_1(v_x^1 + v^3 + lv^5)\overline{(\widetilde{v_x^1} + \widetilde{v^3} + l\widetilde{v^5})} + \rho_1 v^2\overline{\widetilde{v^2}} + k_2 v_x^3\overline{\widetilde{v_x^3}} + \rho_2 v^4\overline{\widetilde{v^4}} \\ + k_3(v_x^5 - lv^1)\overline{(\widetilde{v_x^5} - l\widetilde{v^1})}dx + \rho_1 v^6\overline{\widetilde{v^6}}\Big\}\, dx$$

and

$$\|U\|_{\mathcal{H}}^2 = \int_0^L \left\{k_1|v_x^1 + v^3 + lv^5|^2 + \rho_1|v^2|^2 + k_2|v_x^3|^2 + \rho_2|v^4|^2 \right. \\ \left. + k_3|v_x^5 - lv^1|^2 + \rho_1|v^6|^2\right\} dx. \tag{4.2.2}$$

Where $U=(v^1,v^2,v^3,v^4,v^5,v^6)^\top\in\mathcal{H}$ and $\widetilde{U}=(\widetilde{v^1},\widetilde{v^1},\widetilde{v^2},\widetilde{v^3},\widetilde{v^4},\widetilde{v^5},\widetilde{v^6})^\top\in\mathcal{H}$. Now, we define the linear unbounded operator $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\longmapsto\mathcal{H}$ by:

$$D(\mathcal{A})=\left\{\begin{array}{c} U=(v^1,v^2,v^3,v^4,v^5,v^6)^\top\in\mathcal{H}\mid v^1,v^3\in H^2(0,L)\cap H_0^1(0,L)\\ v^2,v^4,v^6\in H_0^1(0,L),\quad \left[k_3v_x^5+d(x)(v_x^6-lv^2)\right]_x\in L^2(0,L)\end{array}\right\} \tag{4.2.3}$$

and

$$\mathcal{A}\begin{pmatrix}v^1\\v^2\\v^3\\v^4\\v^5\\v^6\end{pmatrix}=\begin{pmatrix} v^2\\ \dfrac{k_1}{\rho_1}(v_x^1+v^3+lv^5)_x+\dfrac{lk_3}{\rho_1}(v_x^5-lv^1)+\dfrac{ld(x)}{\rho_1}(v_x^6-lv^2)\\ v^4\\ \dfrac{k_2}{\rho_2}v_{xx}^3-\dfrac{k_1}{\rho_2}(v_x^1+v^3+lv^5)\\ v^6\\ \dfrac{1}{\rho_1}\left[k_3(v_x^5-lv^1)+d(x)(v_x^6-lv^2)\right]_x-\dfrac{lk_1}{\rho_1}(v_x^1+v^3+lv^5)\end{pmatrix}, \tag{4.2.4}$$

for all $U=(v^1,v^2,v^3,v^4,v^5,v^6)^\top\in D(\mathcal{A})$.
In this sequel, $\|\cdot\|$ will denote the usual norm of $L^2(0,L)$.

Remark 4.2.1. From Poincaré inequality, we deduce that there exists a positive constant c_1 such that

$$k_1\|v_x^1+v^3+lv^5\|^2+k_2\|v_x^3\|^2+k_3\|v_x^5-lv^1\|^2\le c_1\left(\|v_x^1\|^2+\|v_x^3\|^2+\|v_x^5\|^2\right),$$

for all $(v^1,v^3,v^5)\in\left(H_0^1(0,L)\right)^3$. Moreover, we can show by a contradiction argument that there exists a positive constant c_2 such that

$$c_2\left(\|v_x^1\|^2+\|v_x^3\|^2+\|v_x^5\|^2\right)\le k_1\|v_x^1+v^3+lv^5\|^2+k_2\|v_x^3\|^2+k_3\|v_x^5-lv^1\|^2,$$

for all $(v^1,v^3,v^5)\in\left(H_0^1(0,L)\right)^3$. Therefore, the norm defined in (4.2.2) is equivalent to the usual norm of $\mathcal{H}$. □

Now, if $U=(\varphi,\varphi_t,\psi,\psi_t,w,w_t)^\top$, then system (4.1.1)-(4.1.3) can be written as the following first order evolution equation

$$U_t=\mathcal{A}U,\quad U(0)=U_0, \tag{4.2.5}$$

where $U_0=(\varphi_0,\varphi_1,\psi_0,\psi_1,w_0,w_1)^\top\in\mathcal{H}$.

Proposition 4.2.1. The unbounded linear operator $\mathcal{A}$ is m-dissipative in the Hilbert space $\mathcal{H}$.

Proof. For all $U=(v^1,v^2,v^3,v^4,v^5,v^6)^\top\in D(\mathcal{A})$, we have

$$\Re(\mathcal{A}U,U)_\mathcal{H}=-\int_0^L d(x)\left|v_x^6-lv^2\right|^2dx=-d_0\int_\alpha^\beta\left|v_x^6-lv^2\right|^2dx\le 0. \tag{4.2.6}$$

which implies that $\mathcal{A}$ is dissipative. Let us prove that $\mathcal{A}$ is maximal. To this aim, let $F=(f^1,f^2,f^3,f^4,f^5,f^6)^\top\in\mathcal{H}$, we look for $U=(v^1,v^2,v^3,v^4,v^5,v^6)^\top\in D(\mathcal{A})$ unique solution of

$$-\mathcal{A}U=F. \tag{4.2.7}$$

Detailing (4.2.7), we obtain

$$
\begin{aligned}
-v^2 &= f^1, && (4.2.8)\\
-k_1\left(v_x^1+v^3+lv^5\right)_x - lk_3(v_x^5-lv^1) - ld(x)(v_x^6-lv^2) &= \rho_1 f^2, && (4.2.9)\\
-v^4 &= f^3, && (4.2.10)\\
-k_2 v_{xx}^3 + k_1(v_x^1+v^3+lv^5) &= \rho_2 f^4, && (4.2.11)\\
-v^6 &= f^5, && (4.2.12)\\
-\left[k_3\left(v_x^5-lv^1\right)+d(x)(v_x^6-lv^2)\right]_x + lk_1(v_x^1+v^3+lv^5) &= \rho_1 f^6, && (4.2.13)
\end{aligned}
$$

with the following boundary conditions

$$
v^1(0)=v^1(L)=v^3(0)=v^3(L)=v^5(0)=v^5(L)=0. \tag{4.2.14}
$$

By inserting (4.2.8) and (4.2.12) in (4.2.9) and (4.2.13), system (4.2.8)-(4.2.13) implies:

$$
\begin{aligned}
-k_1\left(v_x^1+v^3+lv^5\right)_x - lk_3(v_x^5-lv^1) = \rho_1 f^2 + ld(x)(-f_x^5+lf^1), && (4.2.15)\\
-k_2 v_{xx}^3 + k_1(v_x^1+v^3+lv^5) = \rho_2 f^4, && (4.2.16)\\
-\left[k_3\left(v_x^5-lv^1\right)+d(x)(-f_x^5+lf^1)\right]_x + lk_1(v_x^1+v^3+lv^5) = \rho_1 f^6. && (4.2.17)
\end{aligned}
$$

Let $(\phi^1,\phi^2,\phi^3)\in\left(H_0^1(0,L)\right)^3$. Multiplying (4.2.15), (4.2.16) and (4.2.17) by $\overline{\phi^1}$, $\overline{\phi^2}$ and $\overline{\phi^3}$ respectively, integrating over $(0,L)$, then using formal integrations by parts, we obtain

$$
\mathcal{B}((v^1,v^3,v^5),(\phi^1,\phi^2,\phi^3)) = \mathcal{L}((\phi^1,\phi^2,\phi^3)),\quad \forall(\phi^1,\phi^2,\phi^3)\in\left(H_0^1(0,L)\right)^3, \tag{4.2.18}
$$

where

$$
\begin{aligned}
\mathcal{B}((v^1,v^3,v^5),(\phi^1,\phi^2,\phi^3)) = {} & k_1\int_0^L (v_x^1+v^3+lv^5)\overline{\phi_x^1}dx - lk_3\int_0^L (v_x^5-lv^1)\overline{\phi^1}dx\\
& + k_2\int_0^L v_x^3\overline{\phi_x^2}dx + k_1\int_0^L (v_x^1+v^3+lv^5)\overline{\phi^2}dx\\
& + k_3\int_0^L (v_x^5-lv^1)\overline{\phi_x^3}dx + lk_1\int_0^L (v_x^1+v^3+lv^5)\overline{\phi^3}dx
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}((\phi^1,\phi^2,\phi^3)) = {} & \rho_1\int_0^L f^2\overline{\phi^1}dx + l\int_0^L d(x)(-f_x^5+lf^1)\overline{\phi^1}dx + \rho_2\int_0^L f^4\overline{\phi^2}dx\\
& + \int_0^L d(x)(f_x^5-lf^1)\overline{\phi_x^3}dx + \rho_1\int_0^L f^6\overline{\phi^3}dx.
\end{aligned}
$$

It is easy to see that $\mathcal{B}$ is a sesquilinear and continuous form on $\left(H_0^1(0,L)\right)^3\times\left(H_0^1(0,L)\right)^3$ and $\mathcal{L}$ is an antilinear and continuous form on $\left(H_0^1(0,L)\right)^3$. In fact, from Remark 4.2.1, we deduce that there exists a positive constant c such that

$$
\begin{aligned}
\mathcal{B}((v^1,v^3,v^5),(v^1,v^3,v^5)) &= k_1\|v_x^1+v^3+lv^5\|^2 + k_2\|v_x^3\|^2 + k_3\|v_x^5-lv^1\|^2\\
&\geq c\left(\|v_x^1\|^2+\|v_x^3\|^2+\|v_x^5\|^2\right)\\
&= c\,\|(v^1,v^3,v^5)\|^2_{\left(H_0^1(0,L)\right)^3}.
\end{aligned} \tag{4.2.19}
$$

Thus, $\mathcal{B}$ is a coercive form on $\left(H_0^1(0,L)\right)^3 \times \left(H_0^1(0,L)\right)^3$. Then, it follows by Lax-Milgram theorem that (4.2.18) admits a unique solution $(v^1,v^3,v^5) \in \left(H_0^1(0,L)\right)^3$. By taking test-functions $(\phi^1,\phi^2,\phi^3) \in \left(\mathcal{D}(0,L)\right)^3$, we see that (4.2.15)-(4.2.17) hold in the distributional sense, from which we deduce that $(v^1,v^3) \in \left(H^2(0,L)\cap H_0^1(0,L)\right)^2$, while $[k_3v_x^5 + d(x)(v_x^6 - lv^2)]_x \in L^2(0,L)$. Consequently, $U = (v^1,-f^1,v^3,-f^3,v^5,-f^5)^\top \in D(\mathcal{A})$ is the unique solution of (4.2.7). Then, $\mathcal{A}$ is an isomorphism and since $\rho\left(\mathcal{A}\right)$ is open set of $\mathbb{C}$ (see Theorem 1.1.13), we easily get $\mathcal{R}(\lambda I - \mathcal{A}) = \mathcal{H}$ for a sufficiently small $\lambda > 0$. This, together with the dissipativeness of $\mathcal{A}$, imply that $D\left(\mathcal{A}\right)$ is dense in $\mathcal{H}$ and that $\mathcal{A}$ is m-dissipative in $\mathcal{H}$ (see Theorems 1.2.6, 1.2.9). The proof is thus complete. $\square$

According to Lumer-Philips theorem (see Theorem 1.2.8), Proposition 4.2.1 implies that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in $\mathcal{H}$ which gives the well-posedness of (4.2.5). Then, we have the following result:

Theorem 4.2.1. For all $U_0 \in \mathcal{H}$, system (4.2.5) admits a unique weak solution

$$U(t) = e^{t\mathcal{A}}U_0 \in C^0(\mathbb{R}_+,\mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, then the system (4.2.5) admits a unique strong solution

$$U(t) = e^{t\mathcal{A}}U_0 \in C^0(\mathbb{R}_+,D(\mathcal{A})) \cap C^1(\mathbb{R}_+,\mathcal{H}).$$

4.3 Strong Stability

In this section, we will prove the strong stability of system (4.1.1)-(4.1.3). The main result of this section is the following theorem.

Theorem 4.3.1. The C_0-semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is strongly stable in $\mathcal{H}$; i.e., for all $U_0 \in \mathcal{H}$, the solution of (4.2.5) satisfies

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

According to Theorem 1.3.3, to prove Theorem 4.3.1, we need to prove that the operator $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable. The proof of Theorem 4.3.1 has been divided into the following two Lemmas.

Lemma 4.3.1. For all $\lambda \in \mathbb{R}$, $i\lambda I - \mathcal{A}$ is injective i.e.

$$\ker(i\lambda I - \mathcal{A}) = \{0\}, \quad \forall \lambda \in \mathbb{R}.$$

Proof. From Proposition 4.2.1, we have $0 \in \rho(\mathcal{A})$. We still need to show the result for $\lambda \in \mathbb{R}^*$. For this aim, suppose that $\lambda \neq 0$ and let $U = (v^1,v^2,v^3,v^4,v^5,v^6)^\top \in D(\mathcal{A})$ such that

$$\mathcal{A}U = i\lambda U. \tag{4.3.1}$$

Equivalently, we have the following system

$$\begin{aligned}
v^2 &= i\lambda v^1, && (4.3.2)\\
k_1(v_x^1 + v^3 + lv^5)_x + lk_3(v_x^5 - lv^1) + ld(x)(v_x^6 - lv^2) &= i\lambda\rho_1 v^2, && (4.3.3)\\
v^4 &= i\lambda v^3, && (4.3.4)\\
k_2v_{xx}^3 - k_1(v_x^1 + v^3 + lv^5) &= i\lambda\rho_2 v^4, && (4.3.5)\\
v^6 &= i\lambda v^5, && (4.3.6)\\
\left[k_3(v_x^5 - lv^1) + d(x)(v_x^6 - lv^2)\right]_x - lk_1(v_x^1 + v^3 + lv^5) &= i\lambda\rho_1 v^6. && (4.3.7)
\end{aligned}$$

From (4.2.6), (4.3.1) and the definition of $d(x)$, we obtain

$$0 = \Re\left(i\lambda U, U\right)_{\mathcal{H}} = \Re\left(\mathcal{A}U, U\right)_{\mathcal{H}} = -\int_0^L d(x)\left|v_x^6 - lv^2\right|^2 dx = -d_0\int_\alpha^\beta \left|v_x^6 - lv^2\right|^2 dx. \quad (4.3.8)$$

Thus, we have

$$v_x^6 - lv^2 = 0 \quad \text{in} \quad (\alpha, \beta). \quad (4.3.9)$$

Inserting (4.3.2) and (4.3.6) in (4.3.9) and using the fact that $\lambda \neq 0$, we get

$$v_x^5 - lv^1 = 0 \quad \text{in} \quad (\alpha, \beta). \quad (4.3.10)$$

Now, inserting (4.3.9) and (4.3.10) in (4.3.3) and (4.3.7), then inserting (4.3.2), (4.3.4) and (4.3.6) in (4.3.3), (4.3.5) and (4.3.7) respectively, we deduce that

$$\rho_1\lambda^2 v^1 + k_1(v_x^1 + v^3 + lv^5)_x = 0 \quad \text{in} \quad (\alpha, \beta), \quad (4.3.11)$$

$$\rho_2\lambda^2 v^3 + k_2 v_{xx}^3 - k_1(v_x^1 + v^3 + lv^5) = 0 \quad \text{in} \quad (\alpha, \beta), \quad (4.3.12)$$

$$\rho_1\lambda^2 v^5 - lk_1(v_x^1 + v^3 + lv^5) = 0 \quad \text{in} \quad (\alpha, \beta). \quad (4.3.13)$$

Deriving (4.3.13) with respect to x, we get

$$\rho_1\lambda^2 v_x^5 - lk_1(v_x^1 + v^3 + lv^5)_x = 0 \quad \text{in} \quad (\alpha, \beta).$$

Inserting (4.3.11) in the above equation, we get

$$\rho_1\lambda^2(v_x^5 + lv^1) = 0 \quad \text{in} \quad (\alpha, \beta) \quad \text{and consequently as} \quad \lambda \neq 0, \quad \text{we get} \quad v_x^5 + lv^1 = 0 \quad \text{in} \quad (\alpha, \beta). \quad (4.3.14)$$

Now, adding (4.3.10) and (4.3.14), we obtain

$$v_x^5 = 0 \quad \text{in} \quad (\alpha, \beta) \quad \text{and consequently} \quad v^1 = 0 \quad \text{in} \quad (\alpha, \beta). \quad (4.3.15)$$

Inserting (4.3.15) in (4.3.11), we get

$$v_x^3 = 0 \quad \text{in} \quad (\alpha, \beta). \quad (4.3.16)$$

Now, system (4.3.2)-(4.3.7) can be written in $(0, \alpha) \cup (\beta, L)$ as the following:

$$\rho_1\lambda^2 v^1 + k_1(v_x^1 + v^3 + lv^5)_x + lk_3(v_x^5 - lv^1) = 0 \quad \text{in} \quad (0, \alpha) \cup (\beta, L), \quad (4.3.17)$$

$$\rho_2\lambda^2 v^3 + k_2 v_{xx}^3 - k_1(v_x^1 + v^3 + lv^5) = 0 \quad \text{in} \quad (0, \alpha) \cup (\beta, L), \quad (4.3.18)$$

$$\rho_1\lambda^2 v^5 + k_3(v_x^5 - lv^1)_x - lk_1(v_x^1 + v^3 + lv^5) = 0 \quad \text{in} \quad (0, \alpha) \cup (\beta, L). \quad (4.3.19)$$

Let $V = (v_x^1, v_{xx}^1, v_x^3, v_{xx}^3, v_x^5, v_{xx}^5)^\top$. From (4.3.15), (4.3.16) and the regularity of v^i, $i \in \{1, 3, 5\}$, we have $V(\alpha) = 0$. Now, by deriving system (4.3.17)-(4.3.19) with respect to x in $(0, \alpha)$, we deduce that

$$V_x = A_\lambda V \quad \text{in} \quad (0, \alpha), \quad (4.3.20)$$

where

$$A_\lambda = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{l^2k_3 - \lambda^2\rho_1}{k_1} & 0 & 0 & -1 & 0 & -l(1 + \frac{k_3}{k_1}) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{k_1}{k_2} & \frac{k_1 - \rho_2\lambda^2}{k_2} & 0 & \frac{lk_1}{k_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & l(\frac{k_1}{k_3} + 1) & l\frac{k_1}{k_3} & 0 & \frac{l^2k_1 - \rho_1\lambda^2}{k_3} & 0 \end{pmatrix}. \quad (4.3.21)$$

The solution of the differential equation (4.3.20) is given by

$$V(x) = e^{A_\lambda(x-\alpha)}V(\alpha). \tag{4.3.22}$$

Thus, from (4.3.22) and the fact that $V(\alpha) = 0$, we get

$$V = 0 \ \text{ in } \ (0,\alpha). \tag{4.3.23}$$

From (4.3.23) and the fact that $v^1(0) = v^3(0) = v^5(0) = 0$, we get

$$v^1 = 0 \ \text{ in } \ (0,\alpha), \quad v^3 = 0 \ \text{ in } \ (0,\alpha) \ \text{ and } \ v^5 = 0 \ \text{ in } \ (0,\alpha). \tag{4.3.24}$$

From (4.3.24), (4.3.2), (4.3.4), (4.3.6) and the fact that $\lambda \neq 0$, we obtain

$$U = 0 \ \text{ in } \ (0,\alpha). \tag{4.3.25}$$

From (4.3.25) and the regularity of v^i, $i \in \{3,5\}$, we obtain

$$v^3(\alpha) = 0 \ \text{ and } \ v^5(\alpha) = 0,$$

consequently, from (4.3.15) and (4.3.16), we get

$$v^1 = 0 \ \text{ in } \ (\alpha,\beta), \quad v^3 = 0 \ \text{ in } \ (\alpha,\beta) \ \text{ and } \ v^5 = 0 \ \text{ in } \ (\alpha,\beta),$$

consequently, from (4.3.2), (4.3.4), (4.3.6) and the fact that $\lambda \neq 0$, we obtain

$$U = 0 \ \text{ in } \ (\alpha,\beta). \tag{4.3.26}$$

Now, let $W = (v^1, v^1_x, v^3, v^3_x, v^5, v^5_x)^\top$. From (4.3.26) and the regularity of v^i, $i \in \{1,3,5\}$, we have $W(\beta) = 0$ and system (4.3.17)-(4.3.19) in (β, L) implies:

$$W_x = A_\lambda W \ \text{ in } \ (\beta, L),$$

where A_λ is defined before (see (4.3.21)). Thus, we have

$$W(x) = e^{A_\lambda(x-\beta)}W(\beta) = 0,$$

consequently, from (4.3.2), (4.3.4) and (4.3.6), we deduce that

$$U = 0 \ \text{ in } \ (\beta, L). \tag{4.3.27}$$

Finally, from (4.3.25), (4.3.26) and (4.3.27), we obtain

$$U = 0 \ \text{ in } \ (0, L).$$

The proof is thus complete. $\square$

Lemma 4.3.2. For all $\lambda \in \mathbb{R}$, we have

$$\mathcal{R}(i\lambda I - \mathcal{A}) = \mathcal{H}.$$

Proof. From Proposition 4.2.1, we have $0 \in \rho(\mathcal{A})$. We still need to show the result for $\lambda \in \mathbb{R}^*$. For this aim, let $F = (f^1, f^2, f^3, f^4, f^5, f^6)^\top \in \mathcal{H}$, we want to find $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A})$ solution of

$$(i\lambda I - \mathcal{A})U = F. \tag{4.3.28}$$

Detailing (4.3.28), we obtain

$$\begin{aligned}
i\lambda v^1 - v^2 &= f^1, & (4.3.29)\\
i\lambda v^2 - \frac{k_1}{\rho_1}\left(v^1_x + v^3 + lv^5\right)_x - \frac{lk_3}{\rho_1}(v^5_x - lv^1) - \frac{ld(x)}{\rho_1}(v^6_x - lv^2) &= f^2, & (4.3.30)\\
i\lambda v^3 - v^4 &= f^3, & (4.3.31)\\
i\lambda v^4 - \frac{k_2}{\rho_2}v^3_{xx} + \frac{k_1}{\rho_2}(v^1_x + v^3 + lv^5) &= f^4, & (4.3.32)\\
i\lambda v^5 - v^6 &= f^5, & (4.3.33)\\
i\lambda v^6 - \frac{1}{\rho_1}\left[k_3\left(v^5_x - lv^1\right) + d(x)(v^6_x - lv^2)\right]_x + \frac{lk_1}{\rho_1}(v^1_x + v^3 + lv^5) &= f^6, & (4.3.34)
\end{aligned}$$

with the following boundary conditions

$$v^1(0) = v^1(L) = v^3(0) = v^3(L) = v^5(0) = v^5(L) = 0. \tag{4.3.35}$$

Inserting $v^2 = i\lambda v^1 - f^1$, $v^4 = i\lambda v^3 - f^3$ and $v^6 = i\lambda v^5 - f^5$ in (4.3.30), (4.3.32) and (4.3.34) respectively, we obtain

$$\begin{aligned}
-\lambda^2 v^1 - \frac{k_1}{\rho_1}\left(v^1_x + v^3 + lv^5\right)_x - \frac{lk_3}{\rho_1}(v^5_x - lv^1) - \frac{i\lambda ld(x)}{\rho_1}(v^5_x - lv^1) &= g^1, & (4.3.36)\\
-\lambda^2 v^3 - \frac{k_2}{\rho_2}v^3_{xx} + \frac{k_1}{\rho_2}(v^1_x + v^3 + lv^5) &= g^2, & (4.3.37)\\
-\lambda^2 v^5 - \frac{1}{\rho_1}\left[k_3(v^5_x - lv^1) + i\lambda d(x)(v^5_x - lv^1)\right]_x + \frac{lk_1}{\rho_1}(v^1_x + v^3 + lv^5) &= g^3, & (4.3.38)
\end{aligned}$$

where

$$\begin{cases}
g^1 := i\lambda f^1 + f^2 + \dfrac{ld(x)}{\rho_1}(-f^5_x + lf^1) \in H^{-1}(0, L),\\
g^2 := i\lambda f^3 + f^4 \in H^{-1}(0, L),\\
g^3 := i\lambda f^5 + f^6 + \rho_1^{-1}\left[d(x)(-f^5_x + lf^1)\right]_x \in H^{-1}(0, L).
\end{cases} \tag{4.3.39}$$

For all $\mathsf{U} = (v^1, v^3, v^5)^\top \in \mathbb{H} := \left(H^1_0(0, L)\right)^3$, we define the linear operator $\mathbb{L} : \mathbb{H} \longmapsto \mathbb{H}' := \left(H^{-1}(0, L)\right)^3$ by:

$$\mathbb{L}\mathsf{U} = \begin{pmatrix}
-\dfrac{k_1}{\rho_1}\left(v^1_x + v^3 + lv^5\right)_x - \dfrac{lk_3}{\rho_1}(v^5_x - lv^1) - \dfrac{i\lambda ld(x)}{\rho_1}(v^5_x - lv^1)\\
-\dfrac{k_2}{\rho_2}v^3_{xx} + \dfrac{k_1}{\rho_2}(v^1_x + v^3 + lv^5)\\
-\dfrac{1}{\rho_1}\left[k_3(v^5_x - lv^1) + i\lambda d(x)(v^5_x - lv^1)\right]_x + \dfrac{lk_1}{\rho_1}(v^1_x + v^3 + lv^5)
\end{pmatrix}. \tag{4.3.40}$$

Let us prove that the operator $\mathbb{L}$ is an isomorphism. For this aim, take the duality bracket

$\langle\cdot,\cdot\rangle_{\mathbb{H}',\mathbb{H}}$ of (4.3.40) with $\Psi := (\rho_1\psi^1, \rho_2\psi^2, \rho_1\psi^3)^\top \in \mathbb{H}$, we obtain

$$\langle \mathbb{L}\mathsf{U}, \Psi\rangle_{\mathbb{H}',\mathbb{H}} = \left\langle -k_1\left(v_x^1 + v^3 + lv^5\right)_x - lk_3(v_x^5 - lv^1) - i\lambda l d(x)(v_x^5 - lv^1), \psi^1\right\rangle_{H^{-1}(0,L),H_0^1(0,L)}$$
$$+ \left\langle -k_2 v_{xx}^3 + k_1(v_x^1 + v^3 + lv^5), \psi^2\right\rangle_{H^{-1}(0,L),H_0^1(0,L)}$$
$$+ \left\langle -\left[k_3(v_x^5 - lv^1) + i\lambda d(x)(v_x^5 - lv^1)\right]_x + lk_1(v_x^1 + v^3 + lv^5), \psi^3\right\rangle_{H^{-1}(0,L),H_0^1(0,L)}.$$

Consequently, we obtain

$$\langle \mathbb{L}\mathsf{U}, \Psi\rangle_{\mathbb{H}',\mathbb{H}} = k_1\int_0^L (v_x^1 + v^3 + lv^5)\overline{\psi_x^1}dx - lk_3\int_0^L (v_x^5 - lv^1)\overline{\psi^1}dx$$
$$- i\lambda l\int_0^L d(x)(v_x^5 - lv^1)\overline{\psi^1}dx + k_2\int_0^L v_x^3\overline{\psi_x^2}dx + k_1\int_0^L (v_x^1 + v^3 + lv^5)\overline{\psi^2}dx$$
$$+ k_3\int_0^L (v_x^5 - lv^1)\overline{\psi_x^3}dx + i\lambda\int_0^L d(x)(v_x^5 - lv^1)\overline{\psi_x^3}dx + lk_1\int_0^L (v_x^1 + v^3 + lv^5)\overline{\psi^3}dx,$$

defines a continuous sesquilinear form which is coercive on $\mathbb{H}$. Indeed, from Remark 4.2.1, we deduce that there exists a positive constant c' such that

$$\begin{aligned}\Re\,\langle \mathbb{L}\mathsf{U}, \mathsf{U}\rangle_{\mathbb{H}',\mathbb{H}} &= k_1\|v_x^1 + v^3 + lv^5\|^2 + k_2\|v_x^3\|^2 + k_3\|v_x^5 - lv^1\|^2\\ &\geq c'\left(\|v_x^1\|^2 + \|v_x^3\|^2 + \|v_x^5\|^2\right)\\ &= c'\left\|(v^1, v^3, v^5)^\top\right\|_{\mathbb{H}}^2\\ &= c'\|\mathsf{U}\|_{\mathbb{H}}^2.\end{aligned}$$

Therefore, by using Lax-Milgram theorem, we deduce that $\mathbb{L}$ is an isomorphism from $\mathbb{H}$ onto $\mathbb{H}'$.

Now, let $\mathsf{U} = (v^1, v^3, v^5)^\top$ and $\mathsf{G} = (g^1, g^2, g^3)^\top$, then system (4.3.36)-(4.3.38) can be transformed into the following form:

$$(I - \lambda^2\mathbb{L}^{-1})\mathsf{U} = \mathbb{L}^{-1}\mathsf{G}. \tag{4.3.41}$$

Since I is compact operator from $\mathbb{H}$ onto $\mathbb{H}'$ and $\mathbb{L}^{-1}$ is an isomorphism from $\mathbb{H}'$ onto $\mathbb{H}$, the operator $I - \lambda^2\mathbb{L}^{-1}$ is Fredholm of index zero. Then, by Fredholm's alternative, (4.3.41) admits a unique solution $\mathsf{U} \in \mathbb{H}$ if and only if $I - \lambda^2\mathbb{L}^{-1}$ is injective. Let $\mathsf{V} = (\mathsf{v}^1, \mathsf{v}^3, \mathsf{v}^5)^\top \in \mathbb{H}$ such that

$$\mathsf{V} - \lambda^2\mathbb{L}^{-1}\mathsf{V} = 0 \iff \lambda^2\mathsf{V} - \mathbb{L}\mathsf{V} = 0. \tag{4.3.42}$$

Equivalently, we have

$$-\lambda^2\mathsf{v}^1 - \frac{k_1}{\rho_1}\left(\mathsf{v}_x^1 + \mathsf{v}^3 + l\mathsf{v}^5\right)_x - \frac{lk_3}{\rho_1}(\mathsf{v}_x^5 - l\mathsf{v}^1) - \frac{i\lambda l d(x)}{\rho_1}(\mathsf{v}_x^5 - l\mathsf{v}^1) = 0, \tag{4.3.43}$$
$$-\lambda^2\mathsf{v}^3 - \frac{k_2}{\rho_2}\mathsf{v}_{xx}^3 + \frac{k_1}{\rho_2}(\mathsf{v}_x^1 + \mathsf{v}^3 + l\mathsf{v}^5) = 0, \tag{4.3.44}$$
$$-\lambda^2\mathsf{v}^5 - \frac{1}{\rho_1}\left[(k_3 + i\lambda d(x))\mathsf{v}_x^5 - l(k_3 + i\lambda)\mathsf{v}^1\right]_x + \frac{lk_1}{\rho_1}(\mathsf{v}_x^1 + \mathsf{v}^3 + l\mathsf{v}^5) = 0. \tag{4.3.45}$$

It is easy to see that if $\mathsf{V} = (\mathsf{v}^1, \mathsf{v}^2, \mathsf{v}^3)^\top$ is a solution of (4.3.43)-(4.3.45), then the vector W defined by

$$\mathsf{W} = (\mathsf{v}^1, i\lambda\mathsf{v}^1, \mathsf{v}^3, i\lambda\mathsf{v}^3, \mathsf{v}^5, i\lambda\mathsf{v}^5)^\top$$

belongs to $D(\mathcal{A})$ and satisfies

$$i\lambda \mathsf{W} - \mathcal{A}\mathsf{W} = 0.$$

Thus, by using Lemma 4.3.1, we obtain $\mathsf{W} = 0$ and consequently $I - \lambda^2 \mathbb{L}^{-1}$ is injective. Thanks to Fredholm's alternative, (4.3.41) admits a unique solution $\mathsf{U} \in \mathbb{H}$ and

$$v^1, v^3 \in H^2(0,L), \quad [k_3 v_x^5 + d(x)(i\lambda v_x^5 - f_x^5 - l(i\lambda v^1 - f^1))]_x \in L^2(0,L).$$

Finally, by setting $v^2 = i\lambda v^1 - f^1$, $v^4 = i\lambda v^3 - f^3$ and $v^6 = i\lambda v^5 - f^5$, we deduce that $U \in D(\mathcal{A})$ is a unique solution of (4.3.28). The proof is thus complete $\square$

Proof of Theorem 4.3.1. From Lemma 4.3.1, we obtain that the operator $\mathcal{A}$ has no pure imaginary eigenvalues (i.e. $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset$). Moreover, from Lemma 4.3.1 and Lemma 4.3.2, $i\lambda I - \mathcal{A}$ is bijective for all $\lambda \in \mathbb{R}$ and since $\mathcal{A}$ is closed, we conclude with the help of the closed graph theorem that $i\lambda I - \mathcal{A}$ is an isomorphism for all $\lambda \in \mathbb{R}$, hence that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Therefore, according to Theorem 1.3.3, we get that the C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable. The proof is thus complete. $\square$

4.4 Polynomial Stability

In this section, we will prove the polynomial stability of system (4.1.1)-(4.1.3) with different rates. The main results of this section are the following theorems.

Theorem 4.4.1. If

$$\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2},$$

then, for all $U_0 \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of U_0 such that

$$E(t) \leq \frac{C}{t} \|U_0\|^2_{D(\mathcal{A})}, \quad t > 0.$$

Theorem 4.4.2. If

$$\frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2},$$

then, for all $U_0 \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of U_0 such that

$$E(t) \leq \frac{C}{\sqrt{t}} \|U_0\|^2_{D(\mathcal{A})}, \quad t > 0.$$

Since $i\mathbb{R} \subset \rho(\mathcal{A})$ (see Section 4.3), according to Theorem 1.3.7, to prove Theorem 4.4.1 and Theorem 4.4.2, we still need to prove the following condition

$$\limsup_{\lambda\in\mathbb{R},\ |\lambda|\to\infty} \frac{1}{|\lambda|^\ell} \left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty, \quad \text{with} \quad \begin{cases} \ell = 2 & \text{for Theorem 4.4.1}, \\ \ell = 4 & \text{for Theorem 4.4.2}. \end{cases} \tag{H}$$

We will prove condition (H) by a contradiction argument. For this purpose, suppose that (H) is false, then there exists $\left\{(\lambda^n, U^n := (v^{1,n}, v^{2,n}, v^{3,n}, v^{4,n}, v^{5,n}, v^{6,n})^\top)\right\}_{n\geq 1} \subset \mathbb{R}^* \times D(\mathcal{A})$ with

$$|\lambda^n| \to \infty \text{ as } n \to \infty \quad \text{and} \quad \|U^n\|_{\mathcal{H}} = \|(v^{1,n}, v^{2,n}, v^{3,n}, v^{4,n}, v^{5,n}, v^{6,n})^\top\|_{\mathcal{H}} = 1, \forall n \geq 1, \tag{4.4.1}$$

such that

$$(\lambda^n)^\ell (i\lambda^n I - \mathcal{A})U^n = F^n := (f^{1,n}, f^{2,n}, f^{3,n}, f^{4,n}, f^{5,n}, f^{6,n})^\top \to 0 \quad \text{in} \quad \mathcal{H} \text{ as } n \to \infty. \tag{4.4.2}$$

For simplicity, we drop the index n. Equivalently, from (4.4.2), we have

$$
\begin{aligned}
i\lambda v^1 - v^2 &= \lambda^{-\ell} f^1, && (4.4.3)\\
i\lambda\rho_1 v^2 - k_1(v_x^1 + v^3 + lv^5)_x - lk_3(v_x^5 - lv^1) - ld(x)(v_x^6 - lv^2) &= \rho_1\lambda^{-\ell} f^2, && (4.4.4)\\
i\lambda v^3 - v^4 &= \lambda^{-\ell} f^3, && (4.4.5)\\
i\lambda\rho_2 v^4 - k_2 v_{xx}^3 + k_1(v_x^1 + v^3 + lv^5) &= \rho_2\lambda^{-\ell} f^4, && (4.4.6)\\
i\lambda v^5 - v^6 &= \lambda^{-\ell} f^5, && (4.4.7)\\
i\lambda\rho_1 v^6 - \left[k_3(v_x^5 - lv^1) + d(x)(v_x^6 - lv^2)\right]_x + lk_1(v_x^1 + v^3 + lv^5) &= \rho_1\lambda^{-\ell} f^6. && (4.4.8)
\end{aligned}
$$

By inserting (4.4.3) in (4.4.4), (4.4.5) in (4.4.6) and (4.4.7) in (4.4.8), we deduce that

$$
\begin{aligned}
\lambda^2\rho_1 v^1 + k_1(v_x^1 + v^3 + lv^5)_x + lk_3(v_x^5 - lv^1) + ld(x)(v_x^6 - lv^2) &= h^1, && (4.4.9)\\
\lambda^2\rho_2 v^3 + k_2 v_{xx}^3 - k_1(v_x^1 + v^3 + lv^5) &= h^2, && (4.4.10)\\
\lambda^2\rho_1 v^5 + \left[k_3(v_x^5 - lv^1) + d(x)(v_x^6 - lv^2)\right]_x - lk_1(v_x^1 + v^3 + lv^5) &= h^3. && (4.4.11)
\end{aligned}
$$

where

$$
\begin{cases}
h^1 = -\rho_1\lambda^{-\ell} f^2 - i\rho_1\lambda^{-\ell+1} f^1, \quad h^2 = -\rho_2\lambda^{-\ell} f^4 - i\rho_2\lambda^{-\ell+1} f^3 \quad \text{and}\\
h^3 = -\rho_1\lambda^{-\ell} f^6 - i\rho_1\lambda^{-\ell+1} f^5.
\end{cases}
$$

Here we will check the condition (H) by finding a contradiction with (4.4.1) by showing $\|U\|_{\mathcal{H}} = o(1)$. For clarity, we divide the proof into several Lemmas. From the above system and the fact that $\ell \in \{2,4\}$, $\|U\|_{\mathcal{H}} = 1$ and $\|F\|_{\mathcal{H}} = o(1)$, we remark that

$$
\begin{cases}
\|v^1\| = O\left(|\lambda|^{-1}\right), \quad \|v^3\| = O\left(|\lambda|^{-1}\right), \quad \|v^5\| = O\left(|\lambda|^{-1}\right), \quad \|v_{xx}^1\| = O\left(|\lambda|\right),\\
\|v_{xx}^3\| = O\left(|\lambda|\right) \quad \text{and} \quad \left\|\left[k_3(v_x^5 - lv^1) + d(x)(v_x^6 - lv^2)\right]_x\right\| = O\left(|\lambda|\right).
\end{cases} \tag{4.4.12}
$$

Also, from Poincaré inequality and the fact that $\|F\|_{\mathcal{H}} = o(1)$, we remark that

$$\|f^1\| \lesssim \|f_x^1\| = o(1), \quad \|f^3\| \lesssim \|f_x^3\| = o(1) \quad \text{and} \quad \|f^5\| \lesssim \|f_x^5\| = o(1). \tag{4.4.13}$$

Lemma 4.4.1. If $\left(\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \text{ and } \ell = 2\right)$ or $\left(\frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2} \text{ and } \ell = 4\right)$. Then, the solution $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A})$ of (4.4.3)-(4.4.8) satisfies the following estimations

$$
\begin{aligned}
&\int_\alpha^\beta \left|v_x^6 - lv^2\right|^2 dx = \frac{o(1)}{\lambda^\ell}, \quad \int_\alpha^\beta \left|v_x^5 - lv^1\right|^2 dx = \frac{o(1)}{\lambda^{\ell+2}},\\
&\int_\alpha^\beta \left|v_x^6\right|^2 dx = O(1) \quad \text{and} \quad \int_\alpha^\beta \left|v_x^5\right|^2 dx = \frac{O(1)}{\lambda^2}.
\end{aligned} \tag{4.4.14}
$$

Proof. First, taking the inner product of (4.4.2) with U in $\mathcal{H}$ and using (4.2.6), we get

$$
\begin{aligned}
\int_0^L d(x)\left|v_x^6 - lv^2\right|^2 dx = d_0\int_\alpha^\beta \left|v_x^6 - lv^2\right|^2 dx = -\Re\left(\mathcal{A}U, U\right)_{\mathcal{H}} = \lambda^{-\ell}\Re\left(F, U\right)_{\mathcal{H}}\\
\leq \lambda^{-\ell}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.
\end{aligned} \tag{4.4.15}
$$

Thus, from (4.4.15) and the fact that $\|F\|_{\mathcal{H}} = o(1)$ and $\|U\|_{\mathcal{H}} = 1$, we obtain the first estimation in (4.4.14). Deriving (4.4.7) with respect to x and multiply (4.4.3) by l, then subtract the resulting equations, we deduce that

$$i\lambda(v_x^5 - lv^1) - (v_x^6 - lv^2) = \lambda^{-\ell}(f_x^5 - lf^1).$$

From the above equation, we obtain

$$\begin{aligned}\int_\alpha^\beta \left|v_x^5 - lv^1\right|^2 dx &\leq \frac{2}{\lambda^2}\int_\alpha^\beta \left|v_x^6 - lv^2\right|^2 dx + \frac{2}{\lambda^{2\ell+2}}\int_\alpha^\beta \left|f_x^5 - lf^1\right|^2 dx \\ &\leq \frac{2}{\lambda^2}\int_\alpha^\beta \left|v_x^6 - lv^2\right|^2 dx + \frac{4}{\lambda^{2\ell+2}}\|f_x^5\|^2 + \frac{4l^2}{\lambda^{2\ell+2}}\|f^1\|^2.\end{aligned} \tag{4.4.16}$$

From (4.4.16), the first estimation in (4.4.14) and the fact that $\ell \in \{2,4\}$, $\|f^1\| = o(1)$ (see (4.4.13)), $\|f_x^5\| = o(1)$, we get the second estimation in (4.4.14). Now, it is easy to see that

$$\int_\alpha^\beta |v_x^6|^2 dx = \int_\alpha^\beta |v_x^6 - lv^2 + lv^2|^2 dx \leq 2\int_\alpha^\beta |v_x^6 - lv^2|^2 dx + 2l^2\int_\alpha^\beta |v^2|^2 dx.$$

From the above estimation, the first estimation in (4.4.14) and the fact that v^2 is uniformly bounded in $L^2(0,L)$, we get the third estimation in (4.4.14). From (4.4.7), we deduce that

$$\int_\alpha^\beta \left|v_x^5\right|^2 dx \leq \frac{2}{\lambda^2}\int_\alpha^\beta \left|v_x^6\right|^2 dx + \frac{2}{\lambda^{2\ell+2}}\int_\alpha^\beta \left|f_x^5\right|^2 dx.$$

Finally, from the above estimation, the third estimation in (4.4.14) and the fact that $\|f_x^5\| = o(1)$, we obtain the fourth estimation in (4.4.14). The proof is thus complete. $\square$

For all $0 < \varepsilon < \dfrac{\beta-\alpha}{10}$, we fix the following cut-off functions

- $\mathsf{f}_j \in C^2([0,L])$, $j \in \{1,\cdots,5\}$ such that $0 \leq \mathsf{f}_j(x) \leq 1$, for all $x \in [0,L]$ and

$$\mathsf{f}_j(x) = \begin{cases} 1 & \text{if} \quad x \in [\alpha + j\varepsilon, \beta - j\varepsilon], \\ 0 & \text{if} \quad x \in [0, \alpha + (j-1)\varepsilon] \cup [\beta + (1-j)\varepsilon, L]. \end{cases}$$

- $\mathsf{q}_1, \mathsf{q}_2 \in C^1([0,L])$ such that $0 \leq \mathsf{q}_1(x) \leq 1,\ \ 0 \leq \mathsf{q}_2(x) \leq 1$, for all $x \in [0,L]$ and

$$\mathsf{q}_1(x) = \begin{cases} 1 & \text{if} \quad x \in [0,\gamma_1], \\ 0 & \text{if} \quad x \in [\gamma_2, L], \end{cases} \quad \text{and} \quad \mathsf{q}_2(x) = \begin{cases} 0 & \text{if} \quad x \in [0,\gamma_1], \\ 1 & \text{if} \quad x \in [\gamma_2, L], \end{cases}$$

  with $0 < \alpha < \gamma_1 < \gamma_2 < \beta < L$.

Lemma 4.4.2. If $\left(\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \text{ and } \ell = 2\right)$ or $\left(\frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2} \text{ and } \ell = 4\right)$. Then, the solution $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A})$ of (4.4.3)-(4.4.8) satisfies the following estimations

$$\int_{\alpha+\varepsilon}^{\beta-\varepsilon} |v^6|^2 dx = o(1) \ \text{ and } \ \int_{\alpha+\varepsilon}^{\beta-\varepsilon} |\lambda v^5|^2 dx = o(1). \tag{4.4.17}$$

Proof. First, multiplying (4.4.8) by $-i\lambda^{-1}\mathsf{f}_1\overline{v^6}$ and integrating over (α,β), then using the fact that v^6 is uniformly bounded in $L^2(0,L)$ and $\|f^6\|=o(1)$, we obtain

$$\rho_1\int_\alpha^\beta \mathsf{f}_1|v^6|^2dx=-\frac{i}{\lambda}\int_\alpha^\beta \mathsf{f}_1\left[k_3(v_x^5-lv^1)+d(x)(v_x^6-lv^2)\right]_x\overline{v^6}dx$$
$$+\frac{ilk_1}{\lambda}\int_\alpha^\beta \mathsf{f}_1(v_x^1+v^3+lv^5)\overline{v^6}dx+\frac{o(1)}{|\lambda|^{\ell+1}},$$

using the fact that $(v_x^1+v^3+lv^5)$, v^6 are uniformly bounded in $L^2(0,L)$, we get

$$\frac{ilk_1}{\lambda}\int_\alpha^\beta \mathsf{f}_1(v_x^1+v^3+lv^5)\overline{v^6}dx=o(1),$$

consequently, as $\ell\in\{2,4\}$, we obtain

$$\rho_1\int_\alpha^\beta \mathsf{f}_1|v^6|^2dx=\underbrace{\frac{i}{\lambda}\int_\alpha^\beta -\mathsf{f}_1\left[k_3(v_x^5-lv^1)+d(x)(v_x^6-lv^2)\right]_x\overline{v^6}dx}_{:=\mathtt{I}_1}+o(1). \tag{4.4.18}$$

Using integration by parts and the fact that $\mathsf{f}_1(\alpha)=\mathsf{f}_1(\beta)=0$, then using the definition of $d(x)$, we get

$$\mathtt{I}_1=\frac{i}{\lambda}\int_\alpha^\beta \mathsf{f}_1\left[k_3(v_x^5-lv^1)+d_0(v_x^6-lv^2)\right]\overline{v_x^6}dx+\frac{i}{\lambda}\int_\alpha^\beta \mathsf{f}_1'\left[k_3(v_x^5-lv^1)+d_0(v_x^6-lv^2)\right]\overline{v^6}dx,$$

using Lemma 4.4.1 and the fact that v^6 is uniformly bounded in $L^2(0,L)$, $\ell\in\{2,4\}$, we get

$$\mathtt{I}_1=\frac{o(1)}{|\lambda|^{\frac{\ell}{2}+1}}. \tag{4.4.19}$$

Inserting (4.4.19) in (4.4.18) and using the fact that $\ell\in\{2,4\}$, we obtain

$$\rho_1\int_\alpha^\beta \mathsf{f}_1|v^6|^2dx=o(1).$$

From the above estimation and the definition of f_1, we obtain the first estimation in (4.4.17). Next, from (4.4.7), we deduce that

$$\int_{\alpha+\varepsilon}^{\beta-\varepsilon}|\lambda v^5|^2dx\le 2\int_{\alpha+\varepsilon}^{\beta-\varepsilon}|v^6|^2dx+2\lambda^{-2\ell}\int_{\alpha+\varepsilon}^{\beta-\varepsilon}|f^5|^2dx.$$

Finally, from the above inequality, the first estimation in (4.4.17) and the fact that $\|f^5\|=o(1)$, $\ell\in\{2,4\}$, we obtain the second estimation in (4.4.17). The proof is thus complete. □

Lemma 4.4.3. If $\left(\frac{k_1}{\rho_1}=\frac{k_2}{\rho_2}\text{ and }\ell=2\right)$ or $\left(\frac{k_1}{\rho_1}\neq\frac{k_2}{\rho_2}\text{ and }\ell=4\right)$. Then, the solution $U=(v^1,v^2,v^3,v^4,v^5,v^6)^\top\in D(\mathcal{A})$ of (4.4.3)-(4.4.8) satisfies the following estimations

$$\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\left|v_x^1\right|^2dx=o(1)\quad\text{and}\quad\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\left|\lambda v^1\right|^2dx=o(1). \tag{4.4.20}$$

Proof. First, multiplying (4.4.8) by $\mathsf{p}_2\overline{v_x^1}$, integrating over $(\alpha+\varepsilon,\beta-\varepsilon)$, using the fact that v_x^1 is uniformly bounded in $L^2(0,L)$ and $\|f^6\|=o(1)$, we get

$$\underbrace{i\lambda\rho_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2 v^6\overline{v_x^1}dx}_{:=\mathtt{I}_2}+\underbrace{\int_{\alpha+\varepsilon}^{\beta-\varepsilon}-\mathsf{f}_2\left[k_3(v_x^5-lv^1)+d(x)(v_x^6-lv^2)\right]_x\overline{v_x^1}dx}_{:=\mathtt{I}_3}$$
$$+lk_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2\left|v_x^1\right|^2dx+lk_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2(v^3+lv^5)\overline{v_x^1}dx=o(\lambda^{-\ell}),$$

using the fact that v_x^1 is uniformly bounded in $L^2(0,L)$, $\|v^3\|=O(|\lambda|^{-1})$, $\|v^5\|=O(|\lambda|^{-1})$ (see (4.4.12)), we get

$$lk_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2(v^3+lv^5)\overline{v_x^1}dx=o(1),$$

consequently, as $\ell\in\{2,4\}$, we obtain

$$lk_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2\left|v_x^1\right|^2dx+\mathtt{I}_2+\mathtt{I}_3=o(1). \tag{4.4.21}$$

Now, using integration by parts and the definition of f_2, then using Lemma 4.4.2 and the fact that $\|v^1\|=O(|\lambda|^{-1})$, we get

$$\mathtt{I}_2=-i\lambda\rho_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2 v_x^6\overline{v^1}dx-i\lambda\rho_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2' v^6\overline{v^1}dx=-i\lambda\rho_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2 v_x^6\overline{v^1}dx+o(1). \tag{4.4.22}$$

Now, it is easy to see that

$$-i\lambda\rho_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2 v_x^6\overline{v^1}dx=-i\lambda\rho_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2(v_x^6-lv^2+lv^2)\overline{v^1}dx$$
$$=-i\lambda\rho_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2(v_x^6-lv^2)\overline{v^1}dx-i\lambda\rho_1 l\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2 v^2\overline{v^1}dx,$$

using Lemma 4.4.1 and the fact that $\|v^1\|=O(|\lambda|^{-1})$, we get

$$-i\lambda\rho_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2 v_x^6\overline{v^1}dx=-i\lambda\rho_1 l\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2 v^2\overline{v^1}dx+o(|\lambda|^{-\frac{\ell}{2}}).$$

Inserting $v^2=i\lambda v^1-\lambda^{-\ell}f^1$ in the above equation, we get

$$-i\lambda\rho_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2 v_x^6\overline{v^1}dx=l\rho_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2|\lambda v^1|^2dx+i\lambda^{-\ell+1}l\rho_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2 f^1\overline{v^1}dx+o(|\lambda|^{-\frac{\ell}{2}}),$$

using the fact that $\|v^1\|=O(|\lambda|^{-1})$ and $\|f^1\|=o(1)$, $\ell\in\{2,4\}$, we get

$$-i\lambda\rho_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2 v_x^6\overline{v^1}dx=l\rho_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2|\lambda v^1|^2dx+o(|\lambda|^{-\frac{\ell}{2}}). \tag{4.4.23}$$

Inserting (4.4.23) in (4.4.22) and using the fact that $\ell\in\{2,4\}$, we get

$$\mathtt{I}_2=l\rho_1\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\mathsf{f}_2|\lambda v^1|^2dx+o(1). \tag{4.4.24}$$

Next, using integration by parts and the definition of f_2, we get

$$\begin{aligned}\mathtt{I}_3 = & \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \mathsf{f}_2' \left[k_3(v_x^5 - lv^1) + d_0(v_x^6 - lv^2)\right] \overline{v_x^1} dx \\ & + \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \mathsf{f}_2 \left[k_3(v_x^5 - lv^1) + d_0(v_x^6 - lv^2)\right] \overline{v_{xx}^1} dx,\end{aligned} \tag{4.4.25}$$

using Lemma 4.4.1 and the fact that v_x^1 is uniformly bounded in $L^2(0,L)$, $\|v_{xx}^1\| = O(|\lambda|)$ (see (4.4.12)), $\ell \in \{2,4\}$, we get

$$\mathtt{I}_3 = o(|\lambda|^{-\frac{\ell}{2}+1}). \tag{4.4.26}$$

Inserting (4.4.24) and (4.4.26) in (4.4.21) and using the fact that $\ell \in \{2,4\}$, we get

$$lk_1 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \mathsf{f}_2 \left|v_x^1\right|^2 dx + l\rho_1 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \mathsf{f}_2 |\lambda v^1|^2 dx = o(1). \tag{4.4.27}$$

Finally, from the above estimation and the definition of f_2, we obtain (4.4.20). The proof is thus complete. $\square$

Lemma 4.4.4. If $\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}$ and $\ell = 2$. Then, the solution $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A})$ of (4.4.3)-(4.4.8) satisfies the following estimations

$$\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \left|v_x^3\right|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha+4\varepsilon}^{\beta-4\varepsilon} \left|\lambda v^3\right|^2 dx = o(1). \tag{4.4.28}$$

Proof. First, take $\ell = 2$ in (4.4.9) and multiply it by $\rho_1^{-1}\mathsf{f}_3\overline{v_x^3}$, integrating over $(\alpha+2\varepsilon, \beta-2\varepsilon)$, using the definition of $d(x)$ and fact that v_x^3 is uniformly bounded in $L^2(0,L)$, $\|f^1\| = o(1)$, $\|f^2\| = o(1)$, we obtain

$$\begin{aligned}\frac{k_1}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 \left|v_x^3\right|^2 dx = & -\lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 v^1 \overline{v_x^3} dx - \frac{k_1}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 v_{xx}^1 \overline{v_x^3} dx \\ & - \frac{lk_1}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 v_x^5 \overline{v_x^3} dx - \frac{lk_3}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 (v_x^5 - lv^1) \overline{v_x^3} dx \\ & - \frac{ld_0}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 (v_x^6 - lv^2) \overline{v_x^3} dx + o(|\lambda|^{-1}).\end{aligned} \tag{4.4.29}$$

Using Lemma 4.4.1 with $\ell = 2$, the definition of f_3 and the fact that v_x^3 is uniformly bounded in $L^2(0,L)$, we get

$$\begin{cases} -\dfrac{lk_1}{\rho_1} \displaystyle\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 v_x^5 \overline{v_x^3} dx = o(1), \quad -\dfrac{lk_3}{\rho_1} \displaystyle\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 (v_x^5 - lv^1) \overline{v_x^3} dx = o(\lambda^{-2}) \quad \text{and} \\ -\dfrac{ld_0}{\rho_1} \displaystyle\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 (v_x^6 - lv^2) \overline{v_x^3} dx = o(|\lambda|^{-1}). \end{cases} \tag{4.4.30}$$

Inserting (4.4.30) in (4.4.29), we get

$$\frac{k_1}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 \left|v_x^3\right|^2 dx = -\lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 v^1 \overline{v_x^3} dx - \frac{k_1}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 v_{xx}^1 \overline{v_x^3} dx + o(1). \tag{4.4.31}$$

Now, taking $\ell = 2$ in (4.4.10), we deduce that

$$\lambda^2\rho_2\overline{v^3} + k_2\overline{v^3_{xx}} - k_1(\overline{v^1_x} + \overline{v^3} + l\overline{v^5}) = -\rho_2\lambda^{-2}\overline{f^4} + i\rho_2\lambda^{-1}\overline{f^3}. \tag{4.4.32}$$

Multiplying (4.4.32) by $\rho_2^{-1}\mathsf{f}_3 v^1_x$, integrating over $(\alpha+2\varepsilon, \beta-2\varepsilon)$, we obtain

$$\lambda^2\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3 v^1_x\overline{v^3}dx + \frac{k_2}{\rho_2}\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3 v^1_x\overline{v^3_{xx}}dx - \frac{k_1}{\rho_2}\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3 v^1_x(\overline{v^1_x} + \overline{v^3} + l\overline{v^5})dx = o(|\lambda|^{-1}). \tag{4.4.33}$$

Using integration by parts to the first two terms in the above equation, we get

$$\begin{aligned} -\lambda^2\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3 v^1\overline{v^3_x}dx - \frac{k_2}{\rho_2}\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3 v^1_{xx}\overline{v^3_x}dx &= \lambda^2\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3' v^1\overline{v^3}dx \\ + \frac{k_2}{\rho_2}\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3' v^1_x\overline{v^3_x}dx &+ \frac{k_1}{\rho_2}\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3 v^1_x(\overline{v^1_x} + \overline{v^3} + l\overline{v^5})dx + o(|\lambda|^{-1}). \end{aligned} \tag{4.4.34}$$

Using Lemma 4.4.3 and the fact that v^3_x, $(v^1_x + v^3 + lv^5)$ are uniformly bounded in $L^2(0,L)$ and $\|v^3\| = O(|\lambda|^{-1})$, we get

$$\begin{cases} \lambda^2\displaystyle\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3' v^1\overline{v^3}dx = o(1), \quad \frac{k_2}{\rho_2}\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3' v^1_x\overline{v^3_x}dx = o(1) \quad \text{and} \\ \displaystyle\frac{k_1}{\rho_2}\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3 v^1_x(\overline{v^1_x} + \overline{v^3} + l\overline{v^5})dx = o(1). \end{cases} \tag{4.4.35}$$

Inserting (4.4.35) in (4.4.34), then using the fact that $\dfrac{k_2}{\rho_2} = \dfrac{k_1}{\rho_1}$, we get

$$-\lambda^2\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3 v^1\overline{v^3_x}dx - \frac{k_1}{\rho_1}\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3 v^1_{xx}\overline{v^3_x}dx = o(1).$$

Inserting the above estimation in (4.4.31), then using the definition of f_3, we obtain the first estimation in (4.4.28). Next, multiplying (4.4.32) by $\mathsf{f}_4 v^3$, integrating over $(\alpha+3\varepsilon, \beta-3\varepsilon)$, using integration by parts and the definition of f_4 and the fact that $\|v^3\| = O(|\lambda|^{-1})$, $\|f^3\| = o(1)$ and $\|f^4\| = o(1)$, we get

$$\begin{aligned} \rho_2\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon}\mathsf{f}_4\left|\lambda v^3\right|^2dx = k_2\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon}\mathsf{f}_4|v^3_x|^2dx + k_2\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon}\mathsf{f}_4'\overline{v^3_x}v^3dx \\ + k_1\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon}\mathsf{f}_4(\overline{v^1_x} + \overline{v^3} + l\overline{v^5})v^3dx + o(\lambda^{-2}). \end{aligned}$$

From the above estimation, the first estimation in (4.4.28) and the fact that $(v^1_x + v^3 + lv^5)$ is uniformly bounded in $L^2(0,L)$ and $\|v^3\| = O(|\lambda|^{-1})$, we obtain

$$\rho_2\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon}\mathsf{f}_4\left|\lambda v^3\right|^2dx = o(1).$$

Finally, from the above estimation and the definition of f_4, we obtain the second estimation desired. The proof is thus complete. $\square$

Lemma 4.4.5. If $\frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2}$ and $\ell = 4$. Then, the solution $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A})$ of (4.4.3)-(4.4.8) satisfies the following estimation

$$\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} |\lambda v^1|^2 dx = o(\lambda^{-2}). \tag{4.4.36}$$

Proof. For clarity, we divide the proof into five steps:
Step 1: In this step, we will prove that:

$$\begin{aligned} &l\rho_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 \left|\lambda v^1\right|^2 dx - lk_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 \left|v^1_x\right|^2 dx - \Re\left\{lk_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 v^3 \overline{v^1_x} dx\right\} \\ &-\Re\left\{l^2 k_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 v^5 \overline{v^1_x} dx\right\} = o(\lambda^{-2}). \end{aligned} \tag{4.4.37}$$

For this aim, take $\ell = 4$ in (4.4.9) and multiply it by $l\mathsf{f}_3\overline{v^1}$, integrating over $(\alpha + 2\varepsilon, \beta - 2\varepsilon)$, using the fact that $\|v^1\| = O(|\lambda|^{-1})$, $\|f^1\| = o(1)$ and $\|f^2\| = o(1)$, then taking the real part, we get

$$\begin{aligned} &l\rho_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 \left|\lambda v^1\right|^2 dx + \underbrace{\Re\left\{lk_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 (v^1_x + v^3 + lv^5)_x \overline{v^1} dx\right\}}_{:=\mathtt{I}_4} \\ &+\Re\left\{l^2 k_3 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 (v^5_x - lv^1)\overline{v^1} dx\right\} + \Re\left\{l^2 d_0 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 (v^6_x - lv^2)\overline{v^1} dx\right\} = o(\lambda^{-4}). \end{aligned} \tag{4.4.38}$$

Using integration by parts and the definition of f_3, we obtain

$$\begin{aligned} \mathtt{I}_4 &= -\Re\left\{lk_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3' (v^1_x + v^3 + lv^5)\overline{v^1} dx\right\} \\ &\quad -\Re\left\{lk_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 (v^1_x + v^3 + lv^5)\overline{v^1_x} dx\right\} \\ &= -\frac{lk_1}{2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3' \left(\left|v^1\right|^2\right)_x dx - \Re\left\{lk_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3' v^3 \overline{v^1} dx\right\} \\ &\quad - \Re\left\{l^2 k_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3' v^5 \overline{v^1} dx\right\} - lk_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 \left|v^1_x\right|^2 dx \\ &\quad -\Re\left\{lk_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 v^3 \overline{v^1_x} dx\right\} - \Re\left\{l^2 k_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 v^5 \overline{v^1_x} dx\right\}. \end{aligned} \tag{4.4.39}$$

Using integration by parts and the fact that $\mathsf{f}_3'(\alpha + 2\varepsilon) = \mathsf{f}_3'(\beta - 2\varepsilon) = 0$, then using Lemma 4.4.3, we obtain

$$-\frac{lk_1}{2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3' \left(\left|v^1\right|^2\right)_x dx = \frac{lk_1}{2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3'' \left|v^1\right|^2 dx = o(\lambda^{-2}). \tag{4.4.40}$$

Using the definition of f_3, Lemmas 4.4.1, 4.4.3 with $\ell = 4$ and the fact that $\|v^3\| = O(|\lambda|^{-1})$, $\|v^5\| = O(|\lambda|^{-1})$, we obtain

$$-\Re\left\{lk_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3' v^3 \overline{v^1} dx\right\} = o(\lambda^{-2}), \quad -\Re\left\{l^2 k_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3' v^5 \overline{v^1} dx\right\} = o(\lambda^{-2}). \tag{4.4.41}$$

Inserting (4.4.40) and (4.4.41) in (4.4.39), we obtain

$$\begin{aligned}\mathtt{I}_4 = &-lk_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3\left|v_x^1\right|^2dx - \Re\left\{lk_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3v^3\overline{v_x^1}dx\right\} - \Re\left\{l^2k_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3v^5\overline{v_x^1}dx\right\}\\ &+o(\lambda^{-2}).\end{aligned} \tag{4.4.42}$$

Moreover, from Lemmas 4.4.1, 4.4.3 and the fact that $\ell = 4$, we obtain

$$\begin{cases}\Re\left\{l^2k_3\displaystyle\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3(v_x^5-lv^1)\overline{v^1}dx\right\} = o(\lambda^{-4}),\\ \Re\left\{l^2d_0\displaystyle\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3(v_x^6-lv^2)\overline{v^1}dx\right\} = o(|\lambda|^{-3}).\end{cases} \tag{4.4.43}$$

Inserting (4.4.42) and (4.4.43) in (4.4.38), we obtain (4.4.37).

Step 2: In this step, we will prove that:

$$\begin{aligned}2l\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3\left|\lambda v^1\right|^2dx = \Re\left\{i\lambda\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3'v^6\overline{v^1}dx\right\}\\ -\Re\left\{d_0\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3(v_x^6-lv^2)\overline{v_{xx}^1}dx\right\}+o(\lambda^{-2}).\end{aligned} \tag{4.4.44}$$

For this aim, multiplying (4.4.8) by $\mathsf{f}_3\overline{v_x^1}$, integrating over $(\alpha+2\varepsilon,\beta-2\varepsilon)$, using the fact that v_x^1 is uniformly bounded in $L^2(0,L)$ and $\|f^6\| = o(1)$, then taking the real part, we get

$$\begin{aligned}&\underbrace{\Re\left\{i\lambda\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3v^6\overline{v_x^1}dx\right\}}_{:=\mathtt{I}_5}+\underbrace{\Re\left\{-\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3\left[k_3(v_x^5-lv^1)+d(x)(v_x^6-lv^2)\right]_x\overline{v_x^1}dx\right\}}_{:=\mathtt{I}_6}\\ &+lk_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3|v_x^1|^2dx+\Re\left\{lk_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3v^3\overline{v_x^1}dx\right\}+\Re\left\{l^2k_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3v^5\overline{v_x^1}dx\right\}\\ &= o(\lambda^{-4}).\end{aligned} \tag{4.4.45}$$

Adding (4.4.37) and (4.4.45), we obtain

$$l\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3\left|\lambda v^1\right|^2dx+\mathtt{I}_5+\mathtt{I}_6 = o(\lambda^{-2}). \tag{4.4.46}$$

Using integration by parts and the fact that $\mathsf{f}_3(\alpha+2\varepsilon) = \mathsf{f}_3(\beta-2\varepsilon) = 0$, we obtain

$$\mathtt{I}_5 = -\Re\left\{i\lambda\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3v_x^6\overline{v^1}dx\right\}-\Re\left\{i\lambda\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3'v^6\overline{v^1}dx\right\}. \tag{4.4.47}$$

Now, it is easy to see that

$$\begin{aligned}\Re\left\{-i\lambda\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_2v_x^6\overline{v^1}dx\right\} &= \Re\left\{-i\lambda\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3(v_x^6-lv^2+lv^2)\overline{v^1}dx\right\}\\ &= \Re\left\{-i\lambda\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3(v_x^6-lv^2)\overline{v^1}dx\right\}-\Re\left\{i\lambda\rho_1l\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3v^2\overline{v^1}dx\right\},\end{aligned}$$

using Lemma 4.4.1 and the fact that $\|v^1\| = O(|\lambda|^{-1})$, we get

$$\Re\left\{-i\lambda\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3 v_x^6\overline{v^1}dx\right\} = \Re\left\{-i\lambda\rho_1 l\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3 v^2\overline{v^1}dx\right\} + o(\lambda^{-2}).$$

Inserting $v^2 = i\lambda v^1 - \lambda^{-4}f^1$ in the above estimation, then using the fact that $\|v^1\| = O(|\lambda|^{-1})$ and $\|f^1\| = o(1)$, we get

$$\Re\left\{-i\lambda\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3 v_x^6\overline{v^1}dx\right\} = l\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3|\lambda v^1|^2dx + o(\lambda^{-2}),$$

Inserting the above estimation in (4.4.47), we obtain

$$\mathtt{I}_5 = l\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3|\lambda v^1|^2dx - \Re\left\{i\lambda\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3' v^6\overline{v^1}dx\right\} + o(\lambda^{-2}). \tag{4.4.48}$$

Now, Using integration by parts and the fact that $\mathsf{f}_3(\alpha+2\varepsilon) = \mathsf{f}_3(\beta-2\varepsilon) = 0$, then using the definition of $d(x)$, we obtain

$$\begin{aligned}
\mathtt{I}_6 &= \Re\left\{\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3'\left[k_3(v_x^5 - lv^1) + d(x)(v_x^6 - lv^2)\right]\overline{v_x^1}dx\right\}\\
&\quad+\Re\left\{\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3\left[k_3(v_x^5 - lv^1) + d(x)(v_x^6 - lv^2)\right]\overline{v_{xx}^1}dx\right\}\\
&= \Re\left\{k_3\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3'(v_x^5 - lv^1)\overline{v_x^1}dx\right\} - \Re\left\{d_0\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3'(v_x^6 - lv^2)\overline{v_x^1}dx\right\}\\
&\quad+\Re\left\{k_3\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3(v_x^5 - lv^1)\overline{v_{xx}^1}dx\right\} + \Re\left\{d_0\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3(v_x^6 - lv^2)\overline{v_{xx}^1}dx\right\},
\end{aligned}$$

consequently, by using Lemma 4.4.1 with $\ell = 4$ and the fact that v_x^1 is uniformly bounded in $L^2(0,L)$, $\|v_{xx}^1\| = O(|\lambda|)$, we get

$$\mathtt{I}_6 = \Re\left\{d_0\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3(v_x^6 - lv^2)\overline{v_{xx}^1}dx\right\} + o(\lambda^{-2}). \tag{4.4.49}$$

Thus, by inserting (4.4.48) and (4.4.49) in (4.4.46), we obtain (4.4.44).

Step 3: In this step, we will prove that:

$$\Re\left\{i\lambda\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3' v^6\overline{v^1}dx\right\} = o(\lambda^{-2}). \tag{4.4.50}$$

For this aim, take $\ell = 4$ in (4.4.8) and multiply it by $\mathsf{f}_3'\overline{v^1}$, integrating over $(\alpha+2\varepsilon, \beta-2\varepsilon)$, using the fact that $\|v^1\| = O(|\lambda|^{-1})$, $\|f^6\| = o(1)$, then taking the real part, we get

$$\begin{aligned}
&\Re\left\{i\lambda\rho_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3' v^6\overline{v^1}dx\right\} + \underbrace{\Re\left\{-\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3'\left[k_3(v_x^5 - lv^1) + d(x)(v_x^6 - lv^2)\right]_x\overline{v^1}dx\right\}}_{:=\mathtt{I}_7}\\
&+\frac{lk_1}{2}\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3'\left(|v^1|^2\right)_x dx + \Re\left\{lk_1\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}\mathsf{f}_3'(v^3 + lv^5)\overline{v^1}dx\right\} = o(|\lambda|^{-5}),
\end{aligned} \tag{4.4.51}$$

using (4.4.40), Lemma 4.4.3 and the fact that $\|v^3\| = O(|\lambda|^{-1})$, $\|v^5\| = O(|\lambda|^{-1})$, we obtain

$$\frac{lk_1}{2}\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathtt{f}_3' \left(\left|v^1\right|^2\right)_x dx = o(\lambda^{-2}) \quad \text{and} \quad \Re\left\{lk_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathtt{f}_3'(v^3 + lv^5)\overline{v^1} dx\right\} = o(\lambda^{-2}).$$

Consequently, (4.4.51) implies

$$\Re\left\{i\lambda\rho_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathtt{f}_3' v^6 \overline{v^1} dx\right\} + \mathtt{I}_7 = o(\lambda^{-2}). \tag{4.4.52}$$

Using integration by parts and the fact that $\mathtt{f}_3'(\alpha+2\varepsilon) = \mathtt{f}_3'(\beta-2\varepsilon) = 0$, then using Lemma 4.4.1 and the fact that v_x^1 is uniformly bounded in $L^2(0,L)$, $\|v^1\| = O(|\lambda|^{-1})$, we obtain

$$\begin{aligned}
\mathtt{I}_7 = &\Re\left\{\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathtt{f}_3'' \left[k_3(v_x^5 - lv^1) + d(x)(v_x^6 - lv^2)\right] \overline{v^1} dx\right\} \\
&+ \Re\left\{\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathtt{f}_3' \left[k_3(v_x^5 - lv^1) + d(x)(v_x^6 - lv^2)\right] \overline{v_x^1} dx\right\} \\
= &\; o(\lambda^{-2}).
\end{aligned}$$

Therefore, from the above estimation and (4.4.52), we obtain (4.4.50).

Step 4: In this step, we will prove that:

$$\Re\left\{d_0 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathtt{f}_3(v_x^6 - lv^2)\overline{v_{xx}^1} dx\right\} = -\Re\left\{\frac{d_0\rho_1}{k_1}\lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathtt{f}_3(v_x^6 - lv^2)\overline{v^1} dx\right\} + o(\lambda^{-2}). \tag{4.4.53}$$

For this aim, take $\ell = 4$ in (4.4.9) and multiply it by $\dfrac{d_0}{k_1}\mathtt{f}_3(\overline{v_x^6} - l\overline{v^2})$, integrating over $(\alpha + 2\varepsilon, \beta - 2\varepsilon)$ and taking the real part, then using Lemmas 4.4.1 and the fact that $\|f^1\| = o(1)$, $\|f^2\| = o(1)$, we get

$$\begin{cases}
\Re\left\{\dfrac{d_0\rho_1}{k_1}\lambda^2 \displaystyle\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathtt{f}_3 v^1 (\overline{v_x^6} - l\overline{v^2}) dx\right\} + \Re\left\{d_0 \displaystyle\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathtt{f}_3 v_{xx}^1 (\overline{v_x^6} - l\overline{v^2}) dx\right\} \\
+ \Re\left\{d_0 \displaystyle\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathtt{f}_3 v_x^3 (\overline{v_x^6} - l\overline{v^2}) dx\right\} + \Re\left\{d_0 l \displaystyle\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathtt{f}_3 v_x^5 (\overline{v_x^6} - l\overline{v^2}) dx\right\} \\
+ \Re\left\{\dfrac{d_0 lk_3}{k_1} \displaystyle\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathtt{f}_3 (v_x^5 - lv^1)(\overline{v_x^6} - l\overline{v^2}) dx\right\} + \dfrac{ld_0^2}{k_1} \displaystyle\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathtt{f}_3 |v_x^6 - lv^2|^2 dx \\
= o(|\lambda|^{-5}),
\end{cases}$$

consequently, by using Lemma 4.4.1 and the fact that v_x^3 is uniformly bounded in $L^2(0,L)$, we get

$$\Re\left\{d_0 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathtt{f}_3 v_{xx}^1 (\overline{v_x^6} - l\overline{v^2}) dx\right\} = -\Re\left\{\frac{d_0\rho_1}{k_1}\lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathtt{f}_3 v^1 (\overline{v_x^6} - l\overline{v^2}) dx\right\} + o(\lambda^{-2}).$$

Thus, from the above estimation, we obtain (4.4.53).

Step 5: In this step, we conclude the proof of (4.4.36). For this aim, inserting (4.4.50) and

(4.4.53) in (4.4.44), then using Young's inequality, Lemma 4.4.1 and the fact that $\ell = 4$, we get

$$\begin{aligned}
2l\rho_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 \left|\lambda v^1\right|^2 dx &= \Re\left\{\frac{d_0\rho_1}{k_1}\lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 (v_x^6 - lv^2)\overline{v^1} dx\right\} + o(\lambda^{-2}) \\
&\leq \frac{d_0\rho_1}{k_1}\lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 |v_x^6 - lv^2||v^1| dx + o(\lambda^{-2}) \\
&= \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \left(\frac{d_0\sqrt{\rho_1}}{k_1\sqrt{l}}|\lambda|\sqrt{\mathsf{f}_3}|v_x^6 - lv^2|\right)\left(\sqrt{l\rho_1}|\lambda|\sqrt{\mathsf{f}_3}|v^1|\right) dx + o(\lambda^{-2}) \\
&\leq \underbrace{\frac{\rho_1 d_0^2}{2k_1^2 l}\lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 |v_x^6 - lv^2|^2 dx}_{=o(\lambda^{-2})} + \frac{l\rho_1}{2}\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 |\lambda v^1|^2 dx + o(\lambda^{-2}),
\end{aligned}$$

consequently, we obtain

$$\frac{3l\rho_1}{2}\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \mathsf{f}_3 \left|\lambda v^1\right|^2 dx = o(\lambda^{-2}).$$

Finally, from the above estimation and the definition of f_3, we obtain (4.4.36). The proof is thus complete. □

Lemma 4.4.6. If $\frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2}$ and $\ell = 4$. Then, the solution $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A})$ of system (4.4.3)-(4.4.8) satisfies the following estimations

$$\int_{\alpha+4\varepsilon}^{\beta-4\varepsilon} \left|v_x^3\right|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha+5\varepsilon}^{\beta-5\varepsilon} \left|\lambda v^3\right|^2 dx = o(1). \tag{4.4.54}$$

Proof. First, take $\ell = 4$ in (4.4.9) and multiply it by $k_1^{-1}\mathsf{f}_4\overline{v_x^3}$, integrating over $(\alpha+3\varepsilon, \beta-3\varepsilon)$, using the definition of $d(x)$ and the fact that v_x^3 is uniformly bounded in $L^2(0,L)$, $\|f^1\| = o(1)$, $\|f^2\| = o(1)$, then taking the real part, we obtain

$$\begin{aligned}
\Re\Bigg\{\frac{\lambda^2\rho_1}{k_1}\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \mathsf{f}_4 v^1\overline{v_x^3} dx + \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \mathsf{f}_4 v_{xx}^1\overline{v_x^3} dx + \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \mathsf{f}_4 |v_x^3|^2 dx + l\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \mathsf{f}_4 v_x^5\overline{v_x^3} dx \\
+ \frac{lk_3}{k_1}\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \mathsf{f}_4 (v_x^5 - lv^1)\overline{v_x^3} dx + \frac{ld_0}{k_1}\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \mathsf{f}_4 (v_x^6 - lv^2)\overline{v_x^3} dx\Bigg\} = o(|\lambda|^{-3}),
\end{aligned}$$

consequently, from Lemmas 4.4.1, 4.4.5 with $\ell = 4$ and the fact that v_x^3 is uniformly bounded in $L^2(0,L)$, we obtain

$$\Re\left\{\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \mathsf{f}_4 v_{xx}^1\overline{v_x^3} dx\right\} + \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \mathsf{f}_4 |v_x^3|^2 dx = o(1). \tag{4.4.55}$$

Now, take $\ell = 4$ in (4.4.10) and multiply it by $k_2^{-1}\mathsf{f}_4\overline{v_x^1}$, integrating over $(\alpha+3\varepsilon, \beta-3\varepsilon)$ and integrating by parts, using the fact that v_x^1 is uniformly bounded in $L^2(0,L)$ and $\|f^3\| = o(1)$, $\|f^4\| = o(1)$, then taking the real part, we obtain

$$\begin{aligned}
\Re\Bigg\{-\frac{\lambda^2\rho_2}{k_2}\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \mathsf{f}_4 v_x^3\overline{v^1} dx - \frac{\lambda^2\rho_2}{k_2}\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \mathsf{f}_4' v^3\overline{v^1} dx - \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \mathsf{f}_4 v_x^3\overline{v_{xx}^1} dx \\
- \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \mathsf{f}_4' v_x^3\overline{v_x^1} dx - \frac{k_1}{k_2}\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \mathsf{f}_4 (v_x^1 + v^3 + lv^5)\overline{v_x^1} dx\Bigg\} = o(|\lambda|^{-3}),
\end{aligned}$$

consequently, from Lemmas 4.4.3, 4.4.5 with $\ell = 4$ and the fact that v_x^3, $(v_x^1 + v^3 + lv^5)$ are uniformly bounded in $L^2(0,L)$ and $\|v^3\| = O(|\lambda|^{-1})$, we obtain

$$\Re\left\{-\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \mathsf{f}_4 v_x^3 \overline{v_{xx}^1} dx\right\} = o(1). \tag{4.4.56}$$

Adding (4.4.55) and (4.4.56), then using the definition of f_4, we obtain the first estimation in (4.4.54). Next, take $\ell = 4$ in (4.4.10) and multiply it by $\mathsf{f}_5\overline{v^3}$, integrating over $(\alpha + 4\varepsilon, \beta - 4\varepsilon)$ and integrating by parts, then using the fact that $\|v^3\| = O(|\lambda|^{-1})$, $\|f^3\| = o(1)$, $\|f^4\| = o(1)$, we obtain

$$\begin{aligned}\rho_2 \int_{\alpha+4\varepsilon}^{\beta-4\varepsilon} \mathsf{f}_5 |\lambda v^3|^2 dx = k_2 \int_{\alpha+4\varepsilon}^{\beta-4\varepsilon} \mathsf{f}_5 |v_x^3|^2 dx + k_2 \int_{\alpha+4\varepsilon}^{\beta-4\varepsilon} \mathsf{f}_5' v_x^3 \overline{v^3} dx \\ + k_1 \int_{\alpha+4\varepsilon}^{\beta-4\varepsilon} \mathsf{f}_5 (v_x^1 + v^3 + lv^5)\overline{v^3} dx + o(\lambda^{-4}).\end{aligned}$$

From the above estimation, the first estimation in (4.4.54) and the fact that $(v_x^1 + v^3 + lv^5)$ is uniformly bounded in $L^2(0,L)$ and $\|v^3\| = O(|\lambda|^{-1})$, we obtain

$$\rho_2 \int_{\alpha+4\varepsilon}^{\beta-4\varepsilon} \mathsf{f}_5 |\lambda v^3|^2 dx = o(1).$$

Finally, from the above estimation and the definition of f_5, we obtain the second estimation in (4.4.54). The proof is thus complete. $\square$

Lemma 4.4.7. Let $\mathsf{h} \in C^1([0,L])$ such that $\mathsf{h}(0) = \mathsf{h}(L) = 0$. If $\left(\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \text{ and } \ell = 2\right)$ or $\left(\frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2} \text{ and } \ell = 4\right)$, then the solution $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A})$ of system (4.4.3)-(4.4.8) satisfies the following estimation

$$\begin{aligned}\int_0^L \mathsf{h}' \Big(\rho_1 \left|\lambda v^1\right|^2 + k_1 \left|v_x^1\right|^2 + \rho_2 \left|\lambda v^3\right|^2 + k_2 \left|v_x^3\right|^2 + \rho_1 \left|\lambda v^5\right|^2 \\ + k_3^{-1} \left|k_3 v_x^5 + d(x)(v_x^6 - lv^2)\right|^2\Big) dx = o(1).\end{aligned}$$

Proof. First, multiplying (4.4.9) by $2\mathsf{h}\overline{v_x^1}$, integrating over $(0,L)$, taking the real part, then using Lemma 4.4.1, the fact that v_x^1 is uniformly bounded in $L^2(0,L)$, $\|v^1\| = O(|\lambda|^{-1})$, $\|f^1\| = o(1)$ and $\|f^2\| = o(1)$, we obtain

$$\begin{aligned}&\int_0^L \mathsf{h}\left(\rho_1 \left|\lambda v^1\right|^2 + k_1 \left|v_x^1\right|^2\right)_x dx + \Re\left\{2k_1 \int_0^L \mathsf{h} v_x^3 \overline{v_x^1} dx\right\} \\ &+ \Re\left\{2l(k_1+k_3)\int_0^L \mathsf{h} v_x^5 \overline{v_x^1} dx\right\} \underbrace{-\Re\left\{2l^2 k_3 \int_0^L \mathsf{h} v^1 \overline{v_x^1} dx\right\}}_{=o(1)} \\ &+ \underbrace{\Re\left\{2ld_0 \int_\alpha^\beta \mathsf{h}(v_x^6 - lv^2)\overline{v_x^1} dx\right\}}_{=o\left(|\lambda|^{-\frac{\ell}{2}}\right)} = o(|\lambda|^{-\ell+1}).\end{aligned} \tag{4.4.57}$$

Now, multiplying (4.4.10) by $2\mathsf{h}\overline{v_x^3}$, integrating over $(0,L)$, taking the real part, then using the fact that v_x^3 is uniformly bounded in $L^2(0,L)$, $\|v^3\|=O(|\lambda|^{-1})$, $\|v^5\|=O(|\lambda|^{-1})$, $\|f^3\|=o(1)$ and $\|f^4\|=o(1)$, we obtain

$$\begin{aligned}&\int_0^L \mathsf{h}\left(\left|\rho_2\lambda v^3\right|^2+k_2\left|v_x^3\right|^2\right)_x dx-\Re\left\{2k_1\int_0^L \mathsf{h}v_x^1\overline{v_x^3}dx\right\}\\ &\underbrace{-\Re\left\{2k_1\int_0^L \mathsf{h}(v^3+lv^5)\overline{v_x^3}dx\right\}}_{=o(1)}=o(|\lambda|^{1-\ell}).\end{aligned}\tag{4.4.58}$$

Let $\mathsf{S}:=k_3v_x^5+d(x)(v_x^6-lv^2)$, from Lemma 4.4.1, the definition of $d(x)$ and the fact that v_x^5 is uniformly bounded in $L^2(0,L)$, we get S is uniformly bounded in $L^2(0,L)$. Now, multiplying (4.4.11) by $2k_3^{-1}\mathsf{h}\overline{\mathsf{S}}$, integrating over $(0,L)$, taking the real part, then using the fact that $\|v^3\|=O(|\lambda|^{-1})$, $\|v^5\|=O(|\lambda|^{-1})$, $\|f^5\|=o(1)$ and $\|f^6\|=o(1)$, we obtain

$$\begin{aligned}&\Re\left\{\frac{2\lambda^2\rho_1}{k_3}\int_0^L \mathsf{h}v^5\overline{\mathsf{S}}dx\right\}+k_3^{-1}\int_0^L \mathsf{h}\left(|\mathsf{S}|^2\right)_x dx-\Re\left\{\frac{2l(k_1+k_3)}{k_3}\int_0^L \mathsf{h}v_x^1\overline{\mathsf{S}}dx\right\}\\ &\underbrace{-\Re\left\{\frac{2lk_1}{k_3}\int_0^L \mathsf{h}(v^3+lv^5)\overline{\mathsf{S}}dx\right\}}_{=o(1)}=\underbrace{\Re\left\{2k_3^{-1}\int_0^L \mathsf{h}\left(-\rho_1\lambda^{-\ell}f^6-i\lambda^{1-\ell}\rho_1f^5\right)\overline{\mathsf{S}}dx\right\}}_{=o\left(|\lambda|^{-\ell+1}\right)}.\end{aligned}\tag{4.4.59}$$

Moreover, from the definition of S and $d(x)$, Lemma 4.4.1 and the fact that v_x^1 is uniformly bounded in $L^2(0,L)$, $\|v^5\|=O(|\lambda|^{-1})$, we obtain

$$\left\{\begin{aligned}&\Re\left\{\frac{2\lambda^2\rho_1}{k_3}\int_0^L \mathsf{h}v^5\overline{\mathsf{S}}dx\right\}=\lambda^2\rho_1\int_0^L \mathsf{h}\left(\left|v^5\right|^2\right)_x dx+\underbrace{\Re\left\{\frac{2\lambda^2\rho_1d_0}{k_3}\int_\alpha^\beta \mathsf{h}v^5(\overline{v_x^6}-l\overline{v^2})dx\right\}}_{=o\left(|\lambda|^{-\frac{\ell}{2}+1}\right)},\\ &-\Re\left\{\frac{2l(k_1+k_3)}{k_3}\int_0^L \mathsf{h}v_x^1\overline{\mathsf{S}}dx\right\}=-\Re\left\{2l(k_1+k_3)\int_0^L \mathsf{h}v_x^1\overline{v_x^5}dx\right\}\\ &\qquad\qquad\underbrace{-\Re\left\{\frac{2l(k_1+k_3)d_0}{k_3}\int_\alpha^\beta \mathsf{h}v_x^1(\overline{v_x^6}-l\overline{v^2})dx\right\}}_{=o\left(|\lambda|^{-\frac{\ell}{2}}\right)}.\end{aligned}\right.$$

Inserting the above estimations in (4.4.59) and using the fact that $\ell\in\{2,4\}$, we obtain

$$\int_0^L \mathsf{h}\left(\rho_1\left|\lambda v^5\right|^2+k_3^{-1}|\mathsf{S}|^2\right)_x dx-\Re\left\{2l(k_1+k_3)\int_0^L \mathsf{h}v_x^1\overline{v_x^5}dx\right\}=o(1).\tag{4.4.60}$$

Adding (4.4.57), (4.4.58), (4.4.60) and using the fact that $\ell\in\{2,4\}$, then using integration by parts, we obtain (4.4.7). The proof is thus complete. $\square$

Lemma 4.4.8. The solution $U=(v^1,v^2,v^3,v^4,v^5,v^6)^\top\in D(\mathcal{A})$ of system (4.4.3)-(4.4.8) satisfies the following estimations

$$\mathsf{J}(\alpha+4\varepsilon,\beta-4\varepsilon)=o(1)\quad\text{if}\quad\frac{k_1}{\rho_1}=\frac{k_2}{\rho_2}\ \text{and}\ \ell=2,\tag{4.4.61}$$

$$\mathsf{J}(\alpha+5\varepsilon,\beta-5\varepsilon)=o(1) \qquad \text{if} \qquad \frac{k_1}{\rho_1}\neq\frac{k_2}{\rho_2} \ \text{ and } \ \ell=4, \tag{4.4.62}$$

where

$$\begin{aligned}\mathsf{J}(\gamma_1,\gamma_2):=&\int_0^{\gamma_1}\left(\rho_1\left|\lambda v^1\right|^2+k_1\left|v_x^1\right|^2+\rho_2\left|\lambda v^3\right|^2+k_2\left|v_x^3\right|^2+\rho_1|\lambda v^5|^2\right)dx+k_3\int_0^{\alpha}|v_x^5|^2dx\\&+\int_{\gamma_2}^{L}\left(\rho_1\left|\lambda v^1\right|^2+k_1\left|v_x^1\right|^2+\rho_2\left|\lambda v^3\right|^2+k_2\left|v_x^3\right|^2+\rho_1\left|\lambda v^5\right|^2\right)dx+k_3\int_{\beta}^{L}|v_x^5|^2dx,\end{aligned}$$

for all $0<\alpha<\gamma_1<\gamma_2<\beta<L$.

Proof. First, take $\mathsf{h}=x\mathsf{q}_1+(x-L)\mathsf{q}_2$ in (4.4.7), then using the definition of $d(x)$ and the fact that $0<\alpha<\gamma_1<\gamma_2<\beta<L$, we obtain

$$\begin{aligned}&\int_0^{\gamma_1}\left(\rho_1\left|\lambda v^1\right|^2+k_1\left|v_x^1\right|^2+\rho_2\left|\lambda v^3\right|^2+k_2\left|v_x^3\right|^2+\rho_1|\lambda v^5|^2\right)dx+k_3\int_0^{\alpha}|v_x^5|^2dx\\&+\int_{\gamma_2}^{L}\left(\rho_1\left|\lambda v^1\right|^2+k_1\left|v_x^1\right|^2+\rho_2\left|\lambda v^3\right|^2+k_2\left|v_x^3\right|^2+\rho_1\left|\lambda v^5\right|^2\right)dx+k_3\int_{\beta}^{L}|v_x^5|^2dx\\&=-\int_{\gamma_1}^{\gamma_2}(\mathsf{q}_1+x\mathsf{q}_1')\left(\rho_1\left|\lambda v^1\right|^2+k_1\left|v_x^1\right|^2+\rho_2\left|\lambda v^3\right|^2+k_2\left|v_x^3\right|^2+\rho_1\left|\lambda v^5\right|^2\right.\\&\qquad\qquad\left.+k_3^{-1}\left|k_3v_x^5+d_0(v_x^6-lv^2)\right|^2\right)dx\\&\quad-\int_{\gamma_1}^{\gamma_2}(\mathsf{q}_2+(x-L)\mathsf{q}_2')\left(\rho_1\left|\lambda v^1\right|^2+k_1\left|v_x^1\right|^2+\rho_2\left|\lambda v^3\right|^2+k_2\left|v_x^3\right|^2+\rho_1\left|\lambda v^5\right|^2\right.\\&\qquad\qquad\left.+k_3^{-1}\left|k_3v_x^5+d_0(v_x^6-lv^2)\right|^2\right)dx\\&\quad+k_3^{-1}\int_{\alpha}^{\gamma_2}\mathsf{q}_1|k_3v_x^5+d_0(v_x^6-lv^2)|^2dx+k_3^{-1}\int_{\gamma_1}^{\beta}\mathsf{q}_2|k_3v_x^5+d_0(v_x^6-lv^2)|^2dx.\end{aligned}$$

Now, take $\gamma_1=\alpha+4\varepsilon$ and $\gamma_2=\beta-4\varepsilon$ in the above equation, then using Lemmas 4.4.1-4.4.4 in case of $\frac{k_1}{\rho_1}=\frac{k_2}{\rho_2}$ and $\ell=2$, we obtain (4.4.61). Finally, take $\gamma_1=\alpha+5\varepsilon$ and $\gamma_2=\beta-5\varepsilon$ in the above equation, then using Lemmas 4.4.1-4.4.3, 4.4.6 in case of $\frac{k_1}{\rho_1}\neq\frac{k_2}{\rho_2}$ and $\ell=4$, we obtain (4.4.62). The proof is thus complete. □

Proof of Theorem 4.4.1. First, from Lemmas 4.4.1-4.4.4 and the fact that $\ell=2$, we obtain

$$\left\{\begin{array}{l}\displaystyle\int_{\alpha}^{\beta}|v_x^5|^2dx=O(\lambda^{-2})=o(1),\ \int_{\alpha+\varepsilon}^{\beta-\varepsilon}|v^6|^2dx=o(1),\ \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}|v_x^1|^2dx=o(1)\\ \displaystyle\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon}|\lambda v^1|^2dx=o(1),\ \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon}|v_x^3|^2dx=o(1)\quad\text{and}\quad\int_{\alpha+4\varepsilon}^{\beta-4\varepsilon}|\lambda v^3|^2dx=o(1).\end{array}\right. \tag{4.4.63}$$

Now, from (4.4.61), (4.4.63) and the fact that $0<\varepsilon<\dfrac{\beta-\alpha}{10}$, we deduce that $\|U\|_{\mathcal{H}}=o(1)$, which contradicts (H). This implies that

$$\limsup_{\lambda\in\mathbb{R},\ |\lambda|\to\infty}\frac{1}{\lambda^2}\|(i\lambda I-\mathcal{A})^{-1}\|_{\mathcal{H}}<\infty.$$

Finally, according to Theorem 1.3.7, we obtain the desired result. The proof is thus complete. $\square$

Proof of Theorem 4.4.2. First, from Lemmas 4.4.1, 4.4.2, 4.4.3, 4.4.6 and the fact that $\ell = 4$, we obtain

$$\begin{cases} \displaystyle\int_\alpha^\beta |v_x^5|^2 dx = O(\lambda^{-2}) = o(1), \ \int_{\alpha+\varepsilon}^{\beta-\varepsilon} |v^6|^2 dx = o(1), \ \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} |v_x^1|^2 dx = o(\lambda^{-2}) \\ \displaystyle\int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} |\lambda v^1|^2 dx = o(1), \ \int_{\alpha+4\varepsilon}^{\beta-4\varepsilon} |v_x^3|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha+5\varepsilon}^{\beta-5\varepsilon} |\lambda v^3|^2 dx = o(1). \end{cases} \tag{4.4.64}$$

Now, from (4.4.62), (4.4.64) and the fact that $0 < \varepsilon < \dfrac{\beta - \alpha}{10}$, we deduce that $\|U\|_{\mathcal{H}} = o(1)$, which contradicts (H). This implies that

$$\limsup_{\lambda\in\mathbb{R},\ |\lambda|\to\infty} \frac{1}{\lambda^4} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} < \infty.$$

Finally, according to Theorem 1.3.7, we obtain the desired result. The proof is thus complete. $\square$

4.5 Conclusion

We have studied the stabilization of a Bresse system with one discontinuous local internal viscoelastic damping of Kelvin-Voigt type acting on the axial force under fully Dirichlet boundary conditions. We proved the strong stability of the system by using Arendt-Batty criteria. We proved that the energy of our system decays polynomially with the rates:

$$\begin{cases} t^{-1} \quad \text{if} \quad \dfrac{k_1}{\rho_1} = \dfrac{k_2}{\rho_2}, \\ t^{-\frac{1}{2}} \quad \text{if} \quad \dfrac{k_1}{\rho_1} \neq \dfrac{k_2}{\rho_2}. \end{cases}$$

Chapter 5

Stability and instability results of the Kirchhoff plate equation with delay terms on the boundary or dynamical boundary controls

In this chapter, we consider two models of the Kirchhoff plate equation, the first one with delay terms on the dynamical boundary controls (see system (5.1.1) below), and the second one where delay terms on the boundary control are added (see system (5.1.2) below). For the first system, we prove its well-posedness, strong stability, non-exponential stability, and polynomial stability under a multiplier geometric control condition. For the second one, we prove its well-posedness, strong stability, and exponential stability under the same multiplier geometric control condition. Finally, we give some instability examples of system (5.1.2) for some choices of delays.

5.1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with boundary Γ of class C^4 consisting of a clamped part $\Gamma_0 \neq \emptyset$ and a rimmed part $\Gamma_1 \neq \emptyset$ such that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. In the first part of this chapter, we study the stability of a Kirchhoff plate equation with delay terms on the dynamical boundary controls, namely we consider

$$
\begin{cases}
u_{tt}(x,t) + \Delta^2 u(x,t) = 0 \ \text{ in } \ \Omega \times (0,\infty), \\
u(x,t) = \partial_\nu u(x,t) = 0 \ \text{ on } \ \Gamma_0 \times (0,\infty), \\
\mathcal{B}_1 u(x,t) + \eta(x,t) = 0 \ \text{ on } \ \Gamma_1 \times (0,\infty), \\
\mathcal{B}_2 u(x,t) - \xi(x,t) = 0 \ \text{ on } \ \Gamma_1 \times (0,\infty), \\
\eta_t(x,t) - \partial_\nu u_t(x,t) + \beta_1 \eta(x,t) + \beta_2 \eta(x,t-\tau_1) = 0 \ \text{ on } \ \Gamma_1 \times (0,\infty), \\
\xi_t(x,t) - u_t(x,t) + \gamma_1 \xi(x,t) + \gamma_2 \xi(x,t-\tau_2) = 0 \ \text{ on } \ \Gamma_1 \times (0,\infty), \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \ \text{ in } \ \Omega, \\
\eta(x,0) = \eta_0(x), \quad \xi(x,0) = \xi_0(x) \ \text{ on } \ \Gamma_1, \\
\eta(x,t) = f_0(x,t) \ \text{ on } \ \Gamma_1 \times (-\tau_1, 0), \\
\xi(x,t) = g_0(x,t) \ \text{ on } \ \Gamma_1 \times (-\tau_2, 0).
\end{cases}
\tag{5.1.1}
$$

In the second part of this chapter, we study the stability of the Kirchhoff plate equation with delay terms on the boundary controls, by considering:

$$\begin{cases} u_{tt}(x,t)+\Delta^2 u(x,t)=0 \;\text{ in }\; \Omega\times(0,\infty), \\ u(x,t)=\partial_\nu u(x,t)=0 \;\text{ on }\; \Gamma_0\times(0,\infty), \\ \mathcal{B}_1 u(x,t)=-\beta_1\partial_\nu u_t(x,t)-\beta_2\partial_\nu u_t(x,t-\tau_1) \;\text{ on }\; \Gamma_1\times(0,\infty), \\ \mathcal{B}_2 u(x,t)=\gamma_1 u_t(x,t)+\gamma_2 u_t(x,t-\tau_2) \;\text{ on }\; \Gamma_1\times(0,\infty), \\ u(x,0)=u_0(x), \;\; u_t(x,0)=u_1(x) \;\text{ in }\; \Omega, \\ u_t(x,t)=f_0(x,t) \;\text{ on }\; \Gamma_1\times(-\tau_1,0), \\ \partial_\nu u_t(x,t)=g_0(x,t) \;\text{ on }\; \Gamma_1\times(-\tau_2,0). \end{cases} \tag{5.1.2}$$

Here and below, β_1, γ_1, τ_1 and τ_2 are positive real numbers, β_2 and γ_2 are non-zero real numbers, $\nu=(\nu_1,\nu_2)$ is the unit outward normal vector along Γ, and $\tau=(-\nu_2,\nu_1)$ is the unit tangent vector along Γ. The constant $0<\mu<\frac{1}{2}$ is the Poisson coefficient and the boundary operators $\mathcal{B}_1$ and $\mathcal{B}_2$ are defined by

$$\mathcal{B}_1\mathsf{f}=\Delta\mathsf{f}+(1-\mu)\mathcal{C}_1\mathsf{f}$$

and

$$\mathcal{B}_2\mathsf{f}=\partial_\nu\Delta\mathsf{f}+(1-\mu)\partial_\tau\mathcal{C}_2\mathsf{f},$$

where

$$\mathcal{C}_1\mathsf{f}=2\nu_1\nu_2\mathsf{f}_{x_1x_2}-\nu_1^2\mathsf{f}_{x_2x_2}-\nu_2^2\mathsf{f}_{x_1x_1} \;\text{ and }\; \mathcal{C}_2\mathsf{f}=(\nu_1^2-\nu_2^2)\mathsf{f}_{x_1x_2}-\nu_1\nu_2\left(\mathsf{f}_{x_1x_1}-\mathsf{f}_{x_2x_2}\right).$$

Moreover, easy computations show that

$$\mathcal{C}_1\mathsf{f}=-\partial_\tau^2\mathsf{f}-\partial_\tau\nu_2\mathsf{f}_{x_1}+\partial_\tau\nu_1\mathsf{f}_{x_2} \;\text{ and }\; \mathcal{C}_2\mathsf{f}=\partial_{\nu\tau}\mathsf{f}-\partial_\tau\nu_1\mathsf{f}_{x_1}-\partial_\tau\nu_2\mathsf{f}_{x_2}. \tag{5.1.3}$$

In 1993, Rao in [95] studied the stabilization of the Kirchhoff plate equation with non-linear boundary controls (in the linear case, it corresponds to system (5.1.2) with $\beta_2=\gamma_2=0$), under a multiplier geometric control condition he established an exponential energy decay rate. Furthermore, in 2005, Rao and Wehbe in [96] studied the stabilization of the Kirchhoff plate equation with dynamical boundary controls (corresponding to system (5.1.1) with $\beta_2=\gamma_2=0$), under the same mulitplier geometric control condition they established a polynomial energy decay rate of order t^{-1}.

Time delays appear in several applications such as in physics, chemistry, biology, thermal phenomena not only depending on the present state but also on some past occurrences (see [44, 72]). In the last years, the control of partial differential equations with time delays have become popular among scientists, since in many cases time delays induce some instabilities see [36, 38, 39, 42].

In 2006, Nicaise and Pignotti in [88] studied the multidimensional wave equation with boundary feedback and a delay term at the boundary, by considering the following system:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \text{ in } \Omega \times (0,\infty), \\ u(x,t) = 0 \text{ on } \Gamma_D \times (0,\infty), \\ \frac{\partial u}{\partial \nu}(x,t) = -\mu_1 u_t(x,t) - \mu_2 u_t(x,t-\tau) \text{ on } \Gamma_N \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \text{ in } \Omega, \\ u_t(x,t) = f_0(x,t) \text{ on } \Gamma_N \times (-\tau,0), \end{cases} \tag{5.1.4}$$

where μ_1 and μ_2 are positive real numbers, and Ω is an open bounded domain of $\mathbb{R}^n$ with a boundary Γ of class C^2 and $\Gamma = \Gamma_D \cup \Gamma_N$, such that $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$. Under the assumption $\mu_2 < \mu_1$, an exponential decay is achieved. If this assumption does not hold, they found a sequences of delays $\{\tau_k\}_k$, $\tau_k \to 0$, for which the corresponding solutions have increasing energy. In 2020, Bayili *et al.* in [28] studied the multidimensional wave equation with a delay term in the dynamical control, by considering the following system:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \text{ in } \Omega \times (0,\infty), \\ u(x,t) = 0 \text{ on } \Gamma_D \times (0,\infty), \\ \frac{\partial u}{\partial \nu}(x,t) + \eta(x,t) = 0 \text{ on } \Gamma_N \times (0,\infty), \\ \eta_t(x,t) - u_t(x,t) + \beta_1 \eta(x,t) + \beta_2 \eta(x,t-\tau) = 0 \text{ on } \Gamma_N \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \text{ in } \Omega, \\ \eta(x,0) = \eta_0(x) \text{ on } \Gamma_N, \\ \eta(x,t-\tau) = f_0(x,t-\tau) \text{ on } \Gamma_N \times (0,\tau), \end{cases} \tag{5.1.5}$$

where β_1 and β_2 are positive real numbers, and Ω is an open bounded domain in $\mathbb{R}^n$ with a lipschitz boundary $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ with $meas(\Gamma_D) \neq 0$ and $meas(\Gamma_N \neq 0)$. Under the assumption $\beta_2 < \beta_1$, they showed that the system is not exponentially stable, but they proved that the system has the same decay rate than the one without delay.

But to the best of our knowledge, it seems that there is no result in the existing literature concerning the case of the Kirchhoff plate equation with dynamical boundary controls and time delays (or with boundary controls and time delay). The goal of the present chapter is to fill this gap by studying the stability of systems (5.1.1) and (5.1.2).

In the first part of this chapter, we study the stability of system (5.1.1). In Subsection 5.2.1, we prove the well-posedness of our system by using semigroup approach. In Subsection 5.2.2, following a general criteria of Arendt and Batty, we show the strong stability of our system in the absence of the compactness of the resolvent. In subsection 5.2.3, we prove that the system (5.1.1) is not exponentially stable. Next, in Subsection 5.2.4, by combining the frequency domain approach with a specific multiplier method, we prove under the multiplier geometric control condition **(MGC)** that the energy of our system decays polynomially with the rate t^{-1}.

In the second part of this chapter, we study both stability and instability of system (5.1.2). In subsection 5.3.1, we prove the well-posedness and the strong stability of our system. In subsection 5.3.2, we prove under the same **(MGC)** condition that system (5.1.2) is exponentially stable. Finally, in subsection 5.3.3, if $|\beta_2| \geq \beta_1$ and $|\gamma_2| \geq \gamma_1$, we give some instability examples of system (5.1.2) for some particular choices of delays

5.2 Kirchhoff plate equation with delay terms on the dynamical boundary control

5.2.1 Well-posedness of the system

In this section, we will establish the well-posedness of system (5.1.1) by using semigroup approach. To this aim, as in [88], we introduce the following auxiliary variables

$$\begin{aligned} z^1(x,\rho,t) &:= \eta(x,t-\rho\tau_1), \quad x\in\Gamma_1,\ \rho\in(0,1),\ t>0,\\ z^2(x,\rho,t) &:= \xi(x,t-\rho\tau_2), \quad x\in\Gamma_1,\ \rho\in(0,1),\ t>0. \end{aligned} \tag{5.2.1}$$

Then, system (5.1.1) becomes

$$\begin{aligned}
u_{tt}+\Delta^2 u &= 0 \ \text{ in } \ \Omega\times(0,\infty), && (5.2.2)\\
u=\partial_\nu u &= 0 \ \text{ on } \ \Gamma_0\times(0,\infty), && (5.2.3)\\
\mathcal{B}_1 u+\eta &= 0 \ \text{ on } \ \Gamma_1\times(0,\infty), && (5.2.4)\\
\mathcal{B}_2 u-\xi &= 0 \ \text{ on } \ \Gamma_1\times(0,\infty), && (5.2.5)\\
\eta_t-\partial_\nu u_t+\beta_1\eta+\beta_2 z^1(\cdot,1,t) &= 0 \ \text{ on } \ \Gamma_1\times(0,\infty), && (5.2.6)\\
\xi_t-u_t+\gamma_1\xi+\gamma_2 z^2(\cdot,1,t) &= 0 \ \text{ on } \ \Gamma_1\times(0,\infty), && (5.2.7)\\
\tau_1 z^1_t(\cdot,\rho,t)+z^1_\rho(\cdot,\rho,t) &= 0 \ \text{ on } \ \Gamma_1\times(0,1)\times(0,\infty), && (5.2.8)\\
\tau_2 z^2_t(\cdot,\rho,t)+z^2_\rho(\cdot,\rho,t) &= 0 \ \text{ on } \ \Gamma_1\times(0,1)\times(0,\infty), && (5.2.9)
\end{aligned}$$

with the following initial conditions

$$\begin{cases} u(\cdot,0)=u_0(\cdot), \quad u_t(\cdot,0)=u_1(\cdot) \ \text{ in } \ \Omega,\\ \eta(\cdot,0)=\eta_0(\cdot), \quad \xi(\cdot,0)=\xi_0(\cdot) \ \text{ on } \ \Gamma_1,\\ z^1(\cdot,\rho,0)=f_0(\cdot,-\rho\tau_1) \ \text{ on } \ \Gamma_1\times(0,1),\\ z^2(\cdot,\rho,0)=g_0(\cdot,-\rho\tau_2) \ \text{ on } \ \Gamma_1\times(0,1). \end{cases} \tag{5.2.10}$$

The energy of system (5.2.2)-(5.2.10) is given by

$$\begin{aligned} E(t)=\frac{1}{2}\Bigg\{& a(u,u)+\int_\Omega |u_t|^2dx+\int_{\Gamma_1}|\eta|^2d\Gamma+\int_{\Gamma_1}|\xi|^2d\Gamma\\ &+\ \tau_1|\beta_2|\int_{\Gamma_1}\int_0^1\left|z^1(\cdot,\rho,t)\right|^2d\rho d\Gamma+\tau_2|\gamma_2|\int_{\Gamma_1}\int_0^1\left|z^2(\cdot,\rho,t)\right|^2d\rho d\Gamma\Bigg\}, \end{aligned}$$

where the sequilinear form $a: H^2(\Omega)\times H^2(\Omega)\longmapsto\mathbb{C}$ is defined by

$$\begin{aligned} a(\mathsf{f},\mathsf{g})=\int_\Omega & \left[\mathsf{f}_{x_1x_1}\overline{\mathsf{g}}_{x_1x_1}+\mathsf{f}_{x_2x_2}\overline{\mathsf{g}}_{x_2x_2}+\mu\left(\mathsf{f}_{x_1x_1}\overline{\mathsf{g}}_{x_2x_2}+\mathsf{f}_{x_2x_2}\overline{\mathsf{g}}_{x_1x_1}\right)\right.\\ &\left.+2(1-\mu)\mathsf{f}_{x_1x_2}\overline{\mathsf{g}}_{x_1x_2}\right]dx. \end{aligned} \tag{5.2.11}$$

We first recall the following Green's formula (see [73]):

$$a(\mathsf{f},\mathsf{g})=\int_\Omega \Delta^2\mathsf{f}\overline{\mathsf{g}}dx+\int_\Gamma\left(\mathcal{B}_1\mathsf{f}\partial_\nu\overline{\mathsf{g}}-\mathcal{B}_2\mathsf{f}\overline{\mathsf{g}}\right)d\Gamma, \quad \forall \mathsf{f}\in H^4(\Omega),\ \mathsf{g}\in H^2(\Omega). \tag{5.2.12}$$

For further purposes, we need a weaker version of it. Indeed as $\mathcal{D}(\overline{\Omega})$ is dense in $E(\Delta^2, L^2(\Omega)) :=$
$\{\mathsf{f} \in H^2(\Omega) \mid \Delta^2\mathsf{f} \in L^2(\Omega)\}$ equipped with its natural norm, we deduce that $\mathsf{f} \in E(\Delta^2, L^2(\Omega))$ (see Theorem 5.6 in [87]) satisfies $\mathcal{B}_1\mathsf{f} \in H^{-\frac{1}{2}}(\Gamma)$ and $\mathcal{B}_2\mathsf{f} \in H^{-\frac{3}{2}}(\Gamma)$ with

$$a(\mathsf{f},\mathsf{g}) = \int_\Omega \Delta^2\mathsf{f}\overline{\mathsf{g}}dx + \langle \mathcal{B}_1\mathsf{f}, \partial_\nu \mathsf{g}\rangle_{H^{-\frac{1}{2}}(\Gamma),H^{\frac{1}{2}}(\Gamma)} - \langle \mathcal{B}_2\mathsf{f}, \mathsf{g}\rangle_{H^{-\frac{3}{2}}(\Gamma),H^{\frac{3}{2}}(\Gamma)}, \quad \forall \mathsf{g} \in H^2(\Omega). \tag{5.2.13}$$

Lemma 5.2.1. Let $U = (u, u_t, \eta, \xi, z^1, z^2)$ be a regular solution of system (5.2.2)-(5.2.10). Then, the energy $E(t)$ satisfies the following estimation

$$\frac{d}{dt}E(t) \leq -(\beta_1 - |\beta_2|)\int_{\Gamma_1} |\eta|^2 d\Gamma - (\gamma_1 - |\gamma_2|)\int_{\Gamma_1} |\xi|^2 d\Gamma.$$

Proof. First, multiplying (5.2.2) by $\overline{u_t}$, integrating over Ω, using (5.2.12) and (5.2.3), then taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2 + \frac{1}{2}\frac{d}{dt}a(u,u) - \Re\left\{\int_{\Gamma_1}\left(\mathcal{B}_1 u \partial_\nu \overline{u_t} - \mathcal{B}_2 u \overline{u_t}\right)d\Gamma\right\} = 0. \tag{5.2.14}$$

Now, from (5.2.4)-(5.2.7), we get

$$\begin{aligned}
-\Re\left\{\int_{\Gamma_1}\left(\mathcal{B}_1 u \partial_\nu \overline{u_t} - \mathcal{B}_2 u \overline{u_t}\right)d\Gamma\right\} &= \Re\left\{\int_{\Gamma_1} \eta\overline{(\eta_t + \beta_1\eta + \beta_2 z^1(\cdot,1,t))}d\Gamma\right.\\
&\quad \left.+\int_{\Gamma_1} \xi\overline{(\xi_t + \gamma_1\xi + \gamma_2 z^2(\cdot,1,t))}d\Gamma\right\}\\
&= \frac{1}{2}\frac{d}{dt}\int_{\Gamma_1}|\eta|^2 d\Gamma + \beta_1\int_{\Gamma_1}|\eta|^2 d\Gamma + \Re\left\{\beta_2\int_{\Gamma_1}\eta\overline{z^1}(\cdot,1,t)d\Gamma\right\}\\
&\quad +\frac{1}{2}\frac{d}{dt}\int_{\Gamma_1}|\xi|^2 d\Gamma + \gamma_1\int_{\Gamma_1}|\xi|^2 d\Gamma + \Re\left\{\gamma_2\int_{\Gamma_1}\xi\overline{z^2}(\cdot,1,t)d\Gamma\right\}.
\end{aligned}$$

Inserting the above equation in (5.2.14), then using Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2 + \frac{1}{2}\frac{d}{dt}a(u,u) + \frac{1}{2}\frac{d}{dt}\int_{\Gamma_1}|\eta|^2 d\Gamma + \frac{1}{2}\frac{d}{dt}\int_{\Gamma_1}|\xi|^2 d\Gamma\\
&= -\beta_1\int_{\Gamma_1}|\eta|^2 d\Gamma - \Re\left\{\beta_2\int_{\Gamma_1}\eta z^1(\cdot,1,t)d\Gamma\right\} - \gamma_1\int_{\Gamma_1}|\xi|^2 d\Gamma\\
&\quad -\Re\left\{\gamma_2\int_{\Gamma_1}\xi z^2(\cdot,1,t)d\Gamma\right\} \qquad (5.2.15)\\
&\leq -\beta_1\int_{\Gamma_1}|\eta|^2 d\Gamma + \frac{|\beta_2|}{2}\int_{\Gamma_1}|\eta|^2 d\Gamma + \frac{|\beta_2|}{2}\int_{\Gamma_1}|z^1(\cdot,1,t)|^2 d\Gamma\\
&\quad -\gamma_1\int_{\Gamma_1}|\xi|^2 d\Gamma + \frac{|\gamma_2|}{2}\int_{\Gamma_1}|\xi|^2 d\Gamma + \frac{|\gamma_2|}{2}\int_{\Gamma_1}|z^2(\cdot,1,t)|^2 d\Gamma.
\end{aligned}$$

Multiplying (5.2.8) and (5.2.9) by $|\beta_2|\overline{z^1}(\cdot,\rho,t)$ and $|\gamma_2|\overline{z^2}(\cdot,\rho,t)$ respectively, integrating over $\Gamma_1 \times (0,1)$, using the fact that $z^1(\cdot,0,t) = \eta$ and $z^2(\cdot,0,t) = \xi$, then taking the real part, we obtain

$$\frac{\tau_1|\beta_2|}{2}\frac{d}{dt}\int_{\Gamma_1}\int_0^1 |z^1(\cdot,\rho,t)|^2 d\rho d\Gamma = -\frac{|\beta_2|}{2}\int_{\Gamma_1}|z^1(\cdot,1,t)|^2 d\Gamma + \frac{|\beta_2|}{2}\int_{\Gamma_1}|\eta|^2 d\Gamma \tag{5.2.16}$$

and

$$\frac{\tau_2|\gamma_2|}{2}\frac{d}{dt}\int_{\Gamma_1}\int_0^1 |z^2(\cdot,\rho,t)|^2 d\rho d\Gamma = -\frac{|\gamma_2|}{2}\int_{\Gamma_1}|z^2(\cdot,1,t)|^2 d\Gamma + \frac{|\gamma_2|}{2}\int_{\Gamma_1}|\xi|^2 d\Gamma. \tag{5.2.17}$$

Finally, by adding (5.2.15), (5.2.16) and (5.2.17), we obtain the desired result. The proof is thus complete. □

In the sequel, we make the following assumptions

$$\beta_1,\gamma_1>0,\quad \beta_2,\gamma_2\in\mathbb{R}^*,\quad |\beta_2|<\beta_1 \ \text{ and } \ |\gamma_2|<\gamma_1. \tag{H}$$

Under the hypothesis (H) and from Lemma 5.2.1, system (5.2.2)-(5.2.10) is dissipative in the sense that its energy is non-increasing with respect to time (i.e. $E'(t)\leq 0$). Let us define the Hilbert space $\mathcal{H}$ by

$$\mathcal{H} = H^2_{\Gamma_0}(\Omega)\times L^2(\Omega)\times\left(L^2(\Gamma_1)\right)^2\times\left(L^2(\Gamma_1\times(0,1))\right)^2,$$

where

$$H^2_{\Gamma_0}(\Omega) = \left\{\mathsf{f}\in H^2(\Omega)\mid \mathsf{f}=\partial_\nu\mathsf{f}=0 \text{ on } \Gamma_0\right\}.$$

The Hilbert space $\mathcal{H}$ is equipped with the following inner product

$$\begin{aligned}(U,U_1)_{\mathcal{H}} &= a(u,u_1)+\int_\Omega v\overline{v_1}dx+\int_{\Gamma_1}\eta\overline{\eta_1}d\Gamma+\int_{\Gamma_1}\xi\overline{\xi_1}d\Gamma\\ &\quad+\tau_1|\beta_2|\int_{\Gamma_1}\int_0^1 z^1\overline{z_1^1}d\rho d\Gamma+\tau_2|\gamma_2|\int_{\Gamma_1}\int_0^1 z^2\overline{z_1^2}d\rho d\Gamma,\end{aligned} \tag{5.2.18}$$

where $U=(u,v,\eta,\xi,z^1,z^2)^\top$, $U^1=(u_1,v_1,\eta_1,\xi_1,z_1^1,z_1^2)^\top\in\mathcal{H}$. Now, we define the linear unbounded operator $\mathcal{A}: D(\mathcal{A})\subset\mathcal{H}\longmapsto\mathcal{H}$ by:

$$D(\mathcal{A}) = \left\{\begin{array}{l} U=(u,v,\eta,\xi,z^1,z^2)^\top\in \mathsf{D}_{\Gamma_0}(\Delta^2)\times H^2_{\Gamma_0}(\Omega)\times(L^2(\Gamma_1))^2\times(L^2(\Gamma_1;H^1(0,1)))^2 \mid \\ \mathcal{B}_1u=-\eta,\quad \mathcal{B}_2u=\xi,\quad z^1(\cdot,0)=\eta,\quad z^2(\cdot,0)=\xi \quad\text{on}\quad \Gamma_1\end{array}\right\}$$

where

$$\mathsf{D}_{\Gamma_0}(\Delta^2) = \left\{\mathsf{f}\in H^2_{\Gamma_0}(\Omega)\mid \Delta^2\mathsf{f}\in L^2(\Omega),\ \mathcal{B}_1\mathsf{f}\in L^2(\Gamma_1),\ \text{and } \mathcal{B}_2\mathsf{f}\in L^2(\Gamma_1)\right\}$$

and

$$\mathcal{A}\begin{pmatrix}u\\v\\\eta\\\xi\\z^1\\z^2\end{pmatrix} = \begin{pmatrix}v\\-\Delta^2u\\\partial_\nu v-\beta_1\eta-\beta_2z^1(\cdot,1)\\v-\gamma_1\xi-\gamma_2z^2(\cdot,1)\\-\dfrac{1}{\tau_1}z^1_\rho\\-\dfrac{1}{\tau_2}z^2_\rho\end{pmatrix},\forall U=(u,v,\eta,\xi,z^1,z^2)^\top\in D(\mathcal{A}). \tag{5.2.19}$$

Remark 5.2.1. From the fact that $2\Re\left(u_{x_1x_1}\overline{u}_{x_2x_2}\right) = |u_{x_1x_1}+u_{x_2x_2}|^2 - |u_{x_1x_1}|^2 - |u_{x_2x_2}|^2$, we remark that

$$\begin{aligned} &|u_{x_1x_1}|^2 + |u_{x_2x_2}|^2 + 2\mu\Re\left(u_{x_1x_1}\overline{u}_{x_2x_2}\right) + 2(1-\mu)|u_{x_1x_2}|^2 \\ &\quad = (1-\mu)|u_{x_1x_1}|^2 + (1-\mu)|u_{x_2x_2}|^2 + \mu|u_{x_1x_1}+u_{x_2x_2}|^2 + 2(1-\mu)|u_{x_1x_2}|^2 \geq 0, \end{aligned} \tag{5.2.20}$$

consequently, from (5.2.11), we get

$$a(u,u) \geq (1-\mu)|u|_{H^2(\Omega)}.$$

Hence the sesquilinear form a is coercive on $H^2_{\Gamma_0}(\Omega)$, since Γ_0 is non empty. On the other hand, from (5.2.13) (see also Lemma 3.1 and Remark 3.1 in [95]), we remark that

$$a(\mathsf{f},\mathsf{g}) = \int_\Omega \Delta^2 \mathsf{f}\overline{\mathsf{g}}dx + \int_{\Gamma_1} (\mathcal{B}_1\mathsf{f}\partial_\nu\overline{\mathsf{g}} - \mathcal{B}_2\mathsf{f}\overline{\mathsf{g}})d\Gamma, \quad \forall \mathsf{f} \in \mathsf{D}_{\Gamma_0}(\Delta^2),\ \mathsf{g} \in H^2_{\Gamma_0}(\Omega). \tag{5.2.21}$$

□

Now, if $U = (u, u_t, \eta, \xi, z^1, z^2)^\top$ is regular enough, then system (5.2.2)-(5.2.10) can be written as the following first order evolution equation

$$U_t = \mathcal{A}U, \quad U(0) = U_0, \tag{5.2.22}$$

where $U_0 = (u_0, u_1, \eta_0, \xi_0, f_0(\cdot, -\rho\tau_1), g_0(\cdot, -\rho\tau_2))^\top \in \mathcal{H}$.

Proposition 5.2.1. Under the hypothesis (H), the unbounded linear operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$.

Proof. For all $U = (u, v, \eta, \xi, z^1, z^2)^\top \in D(\mathcal{A})$, from (5.2.18) and (5.2.19), we have

$$\begin{aligned} \Re\left(\mathcal{A}U, U\right)_{\mathcal{H}} = \Re\Big\{ & a(v,u) - \int_\Omega \Delta^2 u\overline{v}dx + \int_{\Gamma_1} \left(\partial_\nu v - \beta_1\eta - \beta_2 z^1(\cdot,1)\right)\overline{\eta}d\Gamma \\ & + \int_{\Gamma_1} \left(v - \gamma_1\xi - \gamma_2 z^2(\cdot,1)\right)\overline{\xi}d\Gamma \Big\} \\ & - \frac{|\beta_2|}{2}\int_{\Gamma_1}\left[\left|z^1_\rho(\cdot,\rho)\right|^2\right]_0^1 d\Gamma - \frac{|\gamma_2|}{2}\int_{\Gamma_1}\left[\left|z^2_\rho(\cdot,\rho)\right|^2\right]_0^1 d\Gamma. \end{aligned}$$

Using (5.2.21) and the fact that $U \in D(\mathcal{A})$, we obtain

$$\begin{aligned} \Re\left(\mathcal{A}U, U\right)_{\mathcal{H}} = & -\beta_1\int_{\Gamma_1}|\eta|^2 d\Gamma - \Re\left\{\beta_2\int_{\Gamma_1} z^1(\cdot,1)\overline{\eta}d\Gamma\right\} \\ & -\gamma_1\int_{\Gamma_1}|\xi|^2 d\Gamma - \Re\left\{\gamma_2\int_{\Gamma_1} z^2(\cdot,1)\overline{\xi}d\Gamma\right\} - \frac{|\beta_2|}{2}\int_{\Gamma_1}|z^1(\cdot,1)|^2 d\Gamma \\ & + \frac{|\beta_2|}{2}\int_{\Gamma_1}|\eta|^2 d\Gamma - \frac{|\gamma_2|}{2}\int_{\Gamma_1}|z^2(\cdot,1)|^2 d\Gamma + \frac{|\gamma_2|}{2}\int_{\Gamma_1}|\xi|^2 d\Gamma. \end{aligned} \tag{5.2.23}$$

Now, by using Young's inequality, we get

$$\left\{\begin{aligned} -\Re\left\{\beta_2\int_{\Gamma_1} z^1(\cdot,1)\overline{\eta}d\Gamma\right\} &\leq \frac{|\beta_2|}{2}\int_{\Gamma_1}|z^1(\cdot,1)|^2 d\Gamma + \frac{|\beta_2|}{2}\int_{\Gamma_1}|\eta|^2 d\Gamma, \\ -\Re\left\{\gamma_2\int_{\Gamma_1} z^2(\cdot,1)\overline{\xi}d\Gamma\right\} &\leq \frac{|\gamma_2|}{2}\int_{\Gamma_1}|z^2(\cdot,1)|^2 d\Gamma + \frac{|\gamma_2|}{2}\int_{\Gamma_1}|\xi|^2 d\Gamma. \end{aligned}\right. \tag{5.2.24}$$

Inserting (5.2.24) in (5.2.23) and using the hypothesis (H), we obtain

$$\Re(\mathcal{A}U,U)_{\mathcal{H}} \leq -(\beta_1 - |\beta_2|)\int_{\Gamma_1}|\eta|^2 d\Gamma - (\gamma_1 - |\gamma_2|)\int_{\Gamma_1}|\xi|^2 d\Gamma \leq 0, \tag{5.2.25}$$

which implies that $\mathcal{A}$ is dissipative. Now, let us prove that $\mathcal{A}$ is maximal. To this aim, if $F = (f_1, f_2, f_3, f_4, g_1, g_2)^\top \in \mathcal{H}$, we look for $U = (u, v, \eta, \xi, z^1, z^2)^\top \in D(\mathcal{A})$ unique solution of

$$-\mathcal{A}U = F. \tag{5.2.26}$$

Equivalently, we have the following system

$$\begin{aligned}
-v &= f_1, && (5.2.27)\\
\Delta^2 u &= f_2, && (5.2.28)\\
-\partial_\nu v + \beta_1\eta + \beta_2 z^1(\cdot,1) &= f_3, && (5.2.29)\\
-v + \gamma_1\xi + \gamma_2 z^2(\cdot,1) &= f_4, && (5.2.30)\\
\frac{1}{\tau_1} z^1_\rho &= g_1, && (5.2.31)\\
\frac{1}{\tau_2} z^2_\rho &= g_2, && (5.2.32)
\end{aligned}$$

with the following boundary conditions

$$u = \partial_\nu u = 0 \ \text{ on } \ \Gamma_0 \ \text{ and } \ \mathcal{B}_1 u = -\eta,\ \mathcal{B}_2 u = \xi,\ z^1(\cdot,0) = \eta,\ z^2(\cdot,0) = \xi \ \text{ on } \ \Gamma_1. \tag{5.2.33}$$

From (5.2.27) and the fact that $F \in \mathcal{H}$, we get

$$v = -f_1 \in H^2_{\Gamma_0}(\Omega). \tag{5.2.34}$$

From (5.2.31), (5.2.32), (5.2.33) and the fact that $F \in \mathcal{H}$, we obtain

$$z^1_\rho \in L^2(\Gamma_1 \times (0,1)) \ \text{ and } \ z^1(\cdot,\rho) = \tau_1 \int_0^\rho g_1(\cdot,s)ds + \eta \tag{5.2.35}$$

and

$$z^2_\rho \in L^2(\Gamma_1 \times (0,1)) \ \text{ and } \ z^2(\cdot,\rho) = \tau_2 \int_0^\rho g_2(\cdot,s)ds + \xi. \tag{5.2.36}$$

Consequently, from (5.2.34), (5.2.29), (5.2.30), (5.2.35), (5.2.36) and the fact that $F \in \mathcal{H}$, we get

$$\eta = \frac{1}{\beta_1 + \beta_2}\left(-\partial_\nu f_1 - \tau_1\beta_2 \int_0^1 g_1(\cdot,s)ds + f_3\right) \in L^2(\Gamma_1) \tag{5.2.37}$$

and

$$\xi = \frac{1}{\gamma_1 + \gamma_2}\left(-f_1 - \tau_2\gamma_2 \int_0^1 g_2(\cdot,s)ds + f_4\right) \in L^2(\Gamma_1). \tag{5.2.38}$$

Now, from (5.2.35)-(5.2.38) and the fact that $g_1, g_2 \in L^2(\Gamma_1 \times (0,1))$, we deduce that

$$z^1, z^2 \in L^2(\Gamma_1; H^1(0,1)).$$

It follows from (5.2.28), (5.2.33), (5.2.37) and (5.2.38) that

$$\begin{cases} \Delta^2 u = f_2 \quad \text{in} \quad \Omega, \\ u = \partial_\nu u = 0 \quad \text{on} \quad \Gamma_0, \\ \mathcal{B}_1 u = -\dfrac{1}{\beta_1+\beta_2}\left(-\partial_\nu f_1 - \tau_1\beta_2 \displaystyle\int_0^1 g_1(\cdot,s)ds + f_3\right) \quad \text{on} \quad \Gamma_1, \\ \mathcal{B}_2 u = \dfrac{1}{\gamma_1+\gamma_2}\left(-f_1 - \tau_2\gamma_2 \displaystyle\int_0^1 g_2(\cdot,s)ds + f_4\right) \quad \text{on} \quad \Gamma_1. \end{cases} \tag{5.2.39}$$

Let $\varphi \in H^2_{\Gamma_0}(\Omega)$. Multiplying the first equation in (5.2.39) by $\overline{\varphi}$ and integrating over Ω, then using Green's formula, we obtain

$$a(u,\varphi) = l(\varphi), \quad \forall \varphi \in H^2_{\Gamma_0}(\Omega), \tag{5.2.40}$$

where

$$l(\varphi) = \int_\Omega f_2\overline{\varphi}dx - \frac{1}{\beta_1+\beta_2}\int_{\Gamma_1}\left(-\partial_\nu f_1 - \tau_1\beta_2\int_0^1 g_1(\cdot,s)ds + f_3\right)\partial_\nu\overline{\varphi}d\Gamma$$
$$-\frac{1}{\gamma_1+\gamma_2}\int_{\Gamma_1}\left(-f_1 - \tau_2\gamma_2\int_0^1 g_2(\cdot,s)ds + f_4\right)\overline{\varphi}d\Gamma.$$

It is easy to see that, a is a sesquilinear, continuous and coercive form on $H^2_{\Gamma_0}(\Omega)\times H^2_{\Gamma_0}(\Omega)$ and l is an antilinear and continuous form on $H^2_{\Gamma_0}(\Omega)$. Then, it follows by Lax-Milgram theorem that (5.2.40) admits a unique solution $u \in H^2_{\Gamma_0}(\Omega)$. By taking the test function $\varphi \in \mathcal{D}(\Omega)$, we see that the first identity of (5.2.39) holds in the distributional sense, hence $\Delta^2 u \in L^2(\Omega)$. Coming back to (5.2.40), and again applying Greens's formula (5.2.13), we find that

$$\mathcal{B}_1 u = -\frac{1}{\beta_1+\beta_2}\left(-\partial_\nu f_1 - \tau_1\beta_2\int_0^1 g_1(\cdot,s)ds + f_3\right) \quad \text{on} \quad \Gamma_1$$

and

$$\mathcal{B}_2 u = \frac{1}{\gamma_1+\gamma_2}\left(-f_1 - \tau_2\gamma_2\int_0^1 g_2(\cdot,s)ds + f_4\right) \quad \text{on} \quad \Gamma_1.$$

Further since $F \in \mathcal{H}$, we deduce that $u \in \mathsf{D}_{\Gamma_0}(\Delta^2)$. Consequently, if we define $U = (u,v,\eta,\xi,z^1,z^2)^\top$ with $u \in H^2_{\Gamma_0}(\Omega)$ the unique solution of (5.2.40), $v = -f_1$, ξ (resp. η) defined by (5.2.37) (resp. (5.2.38)) and z^1 (resp. z^2) defined by (5.2.35) (resp. (5.2.36)), U belongs to $D(\mathcal{A})$ is the unique solution of (5.2.26). Then, $\mathcal{A}$ is an isomorphism and since $\rho(\mathcal{A})$ is open set of $\mathbb{C}$ (see Theorem 1.1.13), we easily get $\mathcal{R}(\lambda I - \mathcal{A}) = \mathcal{H}$ for a sufficiently small $\lambda > 0$. This, together with the dissipativeness of $\mathcal{A}$, imply that $D(\mathcal{A})$ is dense in $\mathcal{H}$ and that $\mathcal{A}$ is m-dissipative in $\mathcal{H}$ (see Theorems 1.2.6, 1.2.9). The proof is thus complete. □

According to Lumer-Phillips theorem (see Theorem 1.2.8), Proposition 5.2.1 implies that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in $\mathcal{H}$ which gives the well-posedness of (5.2.22). Then, we have the following result:

Theorem 5.2.1. For all $U_0 \in \mathcal{H}$, system (5.2.22) admits a unique weak solution

$$U(t) = e^{t\mathcal{A}}U_0 \in C^0(\mathbb{R}_+,\mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, then the system (5.2.22) admits a unique strong solution

$$U(t) = e^{t\mathcal{A}}U_0 \in C^0(\mathbb{R}_+,D(\mathcal{A})) \cap C^1(\mathbb{R}_+,\mathcal{H}).$$

5.2.2 Strong Stability

In this section, we will prove the strong stability of system (5.2.2)-(5.2.10). The main result of this section is the following theorem.

Theorem 5.2.2. Under the hypothesis (H), the C_0-semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is strongly stable in $\mathcal{H}$; i.e., for all $U_0 \in \mathcal{H}$, the solution of (5.2.22) satisfies

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

According to Theorem 1.3.3, to prove Theorem 5.2.2, we need to prove that the operator $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable. The proof of these results is not reduced to the analysis of the point spectrum of $\mathcal{A}$ on the imaginary axis since its resolvent is not compact. Hence the proof of Theorem 5.2.2 has been divided into the following two Lemmas.

Lemma 5.2.2. For all $\lambda \in \mathbb{R}$, $i\lambda I - \mathcal{A}$ is injective i.e.,

$$\ker(i\lambda I - \mathcal{A}) = \{0\}.$$

Proof. From Proposition 5.2.1, we have $0 \in \rho(\mathcal{A})$. We still need to show the result for $\lambda \in \mathbb{R}^*$. To this aim, suppose that $\lambda \neq 0$ and let $U = (u, v, \eta, \xi, z^1, z^2)^\top \in D(\mathcal{A})$ be such that

$$\mathcal{A}U = i\lambda U. \tag{5.2.41}$$

Equivalently, we have the following system

$$\begin{aligned}
v &= i\lambda u, && (5.2.42)\\
-\Delta^2 u &= i\lambda v, && (5.2.43)\\
\partial_\nu v - \beta_1\eta - \beta_2 z^1(\cdot,1) &= i\lambda\eta, && (5.2.44)\\
v - \gamma_1\xi - \gamma_2 z^2(\cdot,1) &= i\lambda\xi, && (5.2.45)\\
-\frac{1}{\tau_1} z^1_\rho &= i\lambda z^1, && (5.2.46)\\
-\frac{1}{\tau_2} z^2_\rho &= i\lambda z^2. && (5.2.47)
\end{aligned}$$

From (5.2.23), (5.2.41) and (H), we get

$$0 = \Re\left(i\lambda\|U\|^2_{\mathcal{H}}\right) = \Re\left(\mathcal{A}U, U\right)_{\mathcal{H}} \leq -(\beta_1 - |\beta_2|)\int_{\Gamma_1}|\eta|^2 d\Gamma - (\gamma_1 - |\gamma_2|)\int_{\Gamma_1}|\xi|^2 d\Gamma \leq 0.$$

Thus, we have

$$\eta = \xi = 0 \quad \text{on} \quad \Gamma_1. \tag{5.2.48}$$

Using (5.2.46), (5.2.47) and the fact that $z^1(\cdot,0) = \eta$, $z^2(\cdot,0) = \xi$ on Γ_1, then using (5.2.48), we obtain

$$z^1(\cdot,\rho) = \eta e^{-i\lambda\tau_1\rho} = 0 \quad \text{on} \quad \Gamma_1 \times (0,1), \tag{5.2.49}$$

$$z^2(\cdot,\rho) = \xi e^{-i\lambda\tau_2\rho} = 0 \quad \text{on} \quad \Gamma_1 \times (0,1). \tag{5.2.50}$$

From (5.2.44), (5.2.45), (5.2.48), (5.2.49) and (5.2.50), we get

$$v = \partial_\nu v = 0 \quad \text{on} \quad \Gamma_1, \tag{5.2.51}$$

consequently, from (5.2.42) and the fact that $\lambda \neq 0$, we obtain

$$u = \partial_\nu u = 0 \quad \text{on} \quad \Gamma_1. \tag{5.2.52}$$

Now, from (5.2.48) and the fact that $U \in D(\mathcal{A})$, we get

$$\mathcal{B}_1 u = \Delta u + (1-\mu)\mathcal{C}_1 u = 0 \quad \text{on} \quad \Gamma_1, \tag{5.2.53}$$

$$\mathcal{B}_2 u = \partial_\nu \Delta u + (1-\mu)\partial_\tau \mathcal{C}_2 u = 0 \quad \text{on} \quad \Gamma_1. \tag{5.2.54}$$

Using (5.2.52) and the fact that $\nabla u = \partial_\tau u \tau + \partial_\nu u \nu$ on Γ_1, we obtain

$$u_{x_1} = u_{x_2} = 0 \quad \text{on} \quad \Gamma_1. \tag{5.2.55}$$

Now, from (5.1.3), (5.2.52) and (5.2.55), we get

$$\mathcal{C}_1 u = \mathcal{C}_2 u = 0 \quad \text{on} \quad \Gamma_1, \tag{5.2.56}$$

consequently, from (5.2.53) and (5.2.54), we get

$$\Delta u = \partial_\nu \Delta u = 0 \quad \text{on} \quad \Gamma_1. \tag{5.2.57}$$

Inserting (5.2.42) in (5.2.43), we obtain

$$\begin{cases} \lambda^2 u - \Delta^2 u = 0 \quad \text{in} \quad \Omega, \\ u = \partial_\nu u = 0 \quad \text{on} \quad \Gamma_0, \\ u = \partial_\nu u = \Delta u = \partial_\nu \Delta u = 0 \quad \text{on} \quad \Gamma_1. \end{cases} \tag{5.2.58}$$

Holmgren uniqueness theorem (see [75]) yields

$$u = 0 \quad \text{in} \quad \Omega. \tag{5.2.59}$$

Finally, from (5.2.42), (5.2.48), (5.2.49), (5.2.50), and (5.2.59), we get

$$U = 0.$$

The proof is thus complete. □

Lemma 5.2.3. Under the hypothesis (H), for all $\lambda \in \mathbb{R}$, we have

$$\mathcal{R}(i\lambda I - \mathcal{A}) = \mathcal{H}.$$

Proof. From Proposition 5.2.1, we have $0 \in \rho(\mathcal{A})$. We still need to show the result for $\lambda \in \mathbb{R}^\star$. For this aim, for $F = (f_1, f_2, f_3, f_4, g_1, g_2)^\top \in \mathcal{H}$, we look for $U = (u, v, \eta, \xi, z^1, z^2)^\top \in D(\mathcal{A})$ solution of

$$(i\lambda I - \mathcal{A})U = F. \tag{5.2.60}$$

Equivalently, we have the following system

$$\begin{aligned} i\lambda u - v &= f_1, && (5.2.61)\\ i\lambda v + \Delta^2 u &= f_2, && (5.2.62)\\ i\lambda \eta - \partial_\nu v + \beta_1 \eta + \beta_2 z^1(\cdot, 1) &= f_3, && (5.2.63)\\ i\lambda \xi - v + \gamma_1 \xi + \gamma_2 z^2(\cdot, 1) &= f_4, && (5.2.64)\\ i\lambda z^1 + \frac{1}{\tau_1} z^1_\rho &= g_1, && (5.2.65)\\ i\lambda z^2 + \frac{1}{\tau_2} z^2_\rho &= g_2, && (5.2.66) \end{aligned}$$

with the following boundary conditions

$$u = \partial_\nu u = 0 \text{ on } \Gamma_0 \text{ and } \mathcal{B}_1 u = -\eta, \ \mathcal{B}_2 u = \xi, \ z^1(\cdot,0) = \eta, \ z^2(\cdot,0) = \xi \text{ on } \Gamma_1. \tag{5.2.67}$$

From (5.2.65), (5.2.66) and (5.2.67), we deduce that

$$z^1(\cdot,\rho) = \eta e^{-i\lambda\tau_1\rho} + \tau_1 \int_0^\rho g_1(x,s) e^{i\lambda\tau_1(s-\rho)} ds \text{ on } \Gamma_1 \times (0,1), \tag{5.2.68}$$

$$z^2(\cdot,\rho) = \xi e^{-i\lambda\tau_2\rho} + \tau_2 \int_0^\rho g_2(x,s) e^{i\lambda\tau_2(s-\rho)} ds \text{ on } \Gamma_1 \times (0,1). \tag{5.2.69}$$

Eliminating v, $z^1(\cdot,1)$ and $z^2(\cdot,1)$ in (5.2.63) and (5.2.64), we get

$$\eta = C_{i\lambda}\left(i\lambda\partial_\nu u + \mathsf{F}_{i\lambda}\right) \text{ on } \Gamma_1 \text{ and } \xi = K_{i\lambda}\left(i\lambda u + \mathsf{G}_{i\lambda}\right) \text{ on } \Gamma_1, \tag{5.2.70}$$

where

$$\begin{cases} C_{i\lambda} = \dfrac{1}{i\lambda + \beta_1 + \beta_2 e^{-i\lambda\tau_1}}, & \mathsf{F}_{i\lambda} = -\partial_\nu f_1 - \beta_2\tau_2 \displaystyle\int_0^1 g_1(\cdot,s) e^{i\lambda\tau_1(s-1)} ds + f_3, \\ K_{i\lambda} = \dfrac{1}{i\lambda + \gamma_1 + \gamma_2 e^{-i\lambda\tau_2}}, & \mathsf{G}_{i\lambda} = -f_1 - \beta_2\tau_2 \displaystyle\int_0^1 g_2(\cdot,s) e^{i\lambda\tau_2(s-1)} ds + f_4. \end{cases} \tag{5.2.71}$$

It follows from (5.2.61), (5.2.62), (5.2.67) and (5.2.70) that

$$\begin{cases} -\lambda^2 u + \Delta^2 u = i\lambda f_1 + f_2 \text{ in } \Omega, \\ u = \partial_\nu u = 0 \text{ on } \Gamma_0, \\ \mathcal{B}_1 u = -C_{i\lambda}(i\lambda\partial_\nu u + \mathsf{F}_{i\lambda}) \text{ on } \Gamma_1, \\ \mathcal{B}_2 u = K_{i\lambda}(i\lambda u + \mathsf{G}_{i\lambda}) \text{ on } \Gamma_1. \end{cases} \tag{5.2.72}$$

Let $\varphi \in H^2_{\Gamma_0}(\Omega)$. Multiplying the first equation in (5.2.72) by $\overline{\varphi}$, integrating over Ω, then using Green's formula, we obtain

$$b(u,\varphi) = l(\varphi), \quad \forall \varphi \in \mathbb{V} := H^2_{\Gamma_0}(\Omega), \tag{5.2.73}$$

where

$$b(u,\varphi) = b_1(u,\varphi) + b_2(u,\varphi),$$

with

$$\begin{cases} b_1(u,\varphi) = a(u,\varphi), \\ b_2(u,\varphi) = -\lambda^2 \displaystyle\int_\Omega u\overline{\varphi} dx + i\lambda C_{i\lambda} \int_{\Gamma_1} \partial_\nu u \partial_\nu \overline{\varphi} d\Gamma + i\lambda K_{i\lambda} \int_{\Gamma_1} u\overline{\varphi} d\Gamma \end{cases} \tag{5.2.74}$$

and

$$l(\varphi) = \int_\Omega (i\lambda f_1 + f_2)\overline{\varphi} dx - i\lambda C_{i\lambda} \int_{\Gamma_1} \mathsf{F}_{i\lambda} \partial_\nu \overline{\varphi} d\Gamma - i\lambda K_{i\lambda} \int_{\Gamma_1} \mathsf{G}_{i\lambda} \overline{\varphi} d\Gamma. \tag{5.2.75}$$

Let $\mathbb{V}'$ be the dual space of $\mathbb{V}$. Let us define the following operators

$$\begin{array}{rcl} \mathbb{B} : \mathbb{V} & \longmapsto & \mathbb{V}' \\ u & \longmapsto & \mathbb{B}u \end{array} \text{ and } \begin{array}{rcl} \mathbb{B}_i : \mathbb{V} & \longmapsto & \mathbb{V}' \\ u & \longmapsto & \mathbb{B}_i u \end{array}, \quad i \in \{1,2\}, \tag{5.2.76}$$

such that

$$\begin{cases} (\mathbb{B}u)(\varphi) = b(u,\varphi), \quad \forall \varphi \in \mathbb{V}, \\ (\mathbb{B}_i u)(\varphi) = b_i(u,\varphi), \quad \forall \varphi \in \mathbb{V}, \ i \in \{1,2\}. \end{cases} \tag{5.2.77}$$

We need to prove that the operator $\mathbb{B}$ is an isomorphism. For this aim, we divide the proof into two steps:

Step 1. In this step, we prove that the operator $\mathbb{B}_2$ is compact. For this aim, let us define the following Hilbert space

$$H^s_{\Gamma_0}(\Omega) := \{\varphi \in H^s(\Omega) \mid \varphi = \partial_\nu \varphi = 0 \ \text{ on } \ \Gamma_0\} \ \text{ with } \ s \in \left(\frac{3}{2}, 2\right).$$

Now, from (5.2.74) and a trace theorem, we get

$$\begin{aligned} |b_2(u,\varphi)| &\lesssim \|u\|_{L^2(\Omega)}\|\varphi\|_{H^2(\Omega)} + \|\partial_\nu u\|_{L^2(\Gamma_1)}\|\partial_\nu \varphi\|_{L^2(\Gamma_1)} + \|u\|_{L^2(\Gamma_1)}\|\varphi\|_{L^2(\Gamma_1)} \\ &\lesssim \|u\|_{H^s(\Omega)}\|\varphi\|_{H^2(\Omega)}, \end{aligned}$$

for all $s \in \left(\frac{3}{2}, 2\right)$. As $\mathbb{V}$ is compactly embedded into $H^s_{\Gamma_0}(\Omega)$ for any $s \in \left(\frac{3}{2}, 2\right)$, $\mathbb{B}_2$ is indeed a compact operator.

This compactness property and the fact that $\mathbb{B}_1$ is an isomorphism imply that the operator $\mathbb{B} = \mathbb{B}_1 + \mathbb{B}_2$ is a Fredholm operator of index zero. Now, following Fredholm alternative, we simply need to prove that the operator $\mathbb{B}$ is injective to obtain that it is an isomorphism.

Step 2. In this step, we prove that the operator $\mathbb{B}$ is injective (i.e. $\ker(\mathbb{B}) = \{0\}$). For this aim, let $\mathsf{u} \in \ker(\mathbb{B})$ which gives

$$b(\mathsf{u},\varphi) = 0, \quad \forall \varphi \in \mathbb{V}.$$

Equivalently, we have

$$a(\mathsf{u},\varphi) - \lambda^2 \int_\Omega \mathsf{u}\overline{\varphi}dx + i\lambda C_{i\lambda} \int_{\Gamma_1} \partial_\nu \mathsf{u} \partial_\nu \overline{\varphi} d\Gamma + i\lambda K_{i\lambda} \int_{\Gamma_1} \mathsf{u}\overline{\varphi} d\Gamma = 0, \ \forall \varphi \in \mathbb{V}.$$

Thus, we find that

$$\begin{cases} -\lambda^2 \mathsf{u} + \Delta^2 \mathsf{u} = 0 \ \text{ in } \ \mathcal{D}'(\Omega), \\ \mathsf{u} = \partial_\nu \mathsf{u} = 0 \ \text{ on } \ \Gamma_0 \\ \mathcal{B}_1 \mathsf{u} = -i\lambda C_{i\lambda} \partial_\nu \mathsf{u} \ \text{ on } \ \Gamma_1, \\ \mathcal{B}_2 \mathsf{u} = i\lambda K_{i\lambda} \mathsf{u} \ \text{ on } \ \Gamma_1. \end{cases}$$

Therefore, the vector U defined by

$$\mathsf{U} = (\mathsf{u}, i\lambda \mathsf{u}, i\lambda C_{i\lambda}\partial_\nu \mathsf{u}, i\lambda K_{i\lambda}\mathsf{u}, i\lambda C_{i\lambda}\partial_\nu \mathsf{u} e^{-i\lambda\tau_1\rho}, i\lambda K_{i\lambda}\mathsf{u} e^{-i\lambda\tau_2\rho})^\top$$

belongs to $D(\mathcal{A})$ and satisfies

$$i\lambda \mathsf{U} - \mathcal{A}\mathsf{U} = 0,$$

and consequently $\mathsf{U} \in \ker(i\lambda I - \mathcal{A})$. Hence Lemma 5.2.2 yields $\mathsf{U} = 0$ and consequently $\mathsf{u} = 0$ and $\ker(\mathbb{B}) = \{0\}$.

Steps 1 and 2 guarantee that the operator $\mathbb{B}$ is isomorphism. Furthermore it is easy to see that the operator l is an antilinear and continuous form on $\mathbb{V}$. Consequently, (5.2.73) admits a unique solution $u \in \mathbb{V}$. In (5.2.73), by taking test functions $\varphi \in \mathcal{D}(\Omega)$, we see that the first identity of (5.2.72) holds in the distributional sense, hence $\Delta^2 u \in L^2(\Omega)$. Coming back to (5.2.73), and again applying Green's formula (5.2.13), we find that

$$\mathcal{B}_1 u = -C_{i\lambda}(i\lambda\partial_\nu u + \mathsf{F}_{i\lambda}) \quad \text{on} \quad \Gamma_1$$

and

$$\mathcal{B}_2 u = K_{i\lambda}(i\lambda u + \mathsf{G}_{i\lambda}) \quad \text{on} \quad \Gamma_1.$$

Further since u, $\partial_\nu u$, $\mathsf{F}_{i\lambda}$ and $\mathsf{G}_{i\lambda}$ belong to $L^2(\Gamma_1)$, we deduce that $u \in \mathsf{D}_{\Gamma_0}(\Delta^2)$. Consequently, if $u \in \mathbb{V}$ is the unique solution of (5.2.73) and if we define η and ξ by (5.2.70) and z^1 (resp. z^2) by (5.2.68) (resp. (5.2.69)), we deduce that

$$U = (u, i\lambda u - f_1, \eta, \xi, z^1, z^2)^\top$$

belongs to $D(\mathcal{A})$ and is the unique solution of (5.2.60). The proof is thus complete. □

Proof of Theorem 5.2.2. From Lemma 5.2.2, the operator $\mathcal{A}$ has no pure imaginary eigenvalues (i.e. $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset$). Moreover, from Lemma 5.2.2 and Lemma 5.2.3, $i\lambda I - \mathcal{A}$ is bijective for all $\lambda \in \mathbb{R}$ and since $\mathcal{A}$ is closed, we conclude with the help of the closed graph theorem that $i\lambda I - \mathcal{A}$ is an isomorphism for all $\lambda \in \mathbb{R}$, hence that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Therefore, according to Theorem 1.3.3, we get that the C$_0$-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable. The proof is thus complete. □

5.2.3 Lack of exponential stability

In this section, we will prove that the system (5.2.2)-(5.2.10) is not exponential stable. Let us start with a technical result.

Lemma 5.2.4. Define the linear unbounded operator $T : D(T) \longmapsto L^2(\Omega)$ by

$$D(T) = \left\{\mathsf{f} \in E(\Delta^2, L^2(\Omega)) \cap H^2_{\Gamma_0}(\Omega) \mid \mathcal{B}_1\mathsf{f} + \partial_\nu \mathsf{f} = 0 \text{ on } \Gamma_1, \ \ \mathcal{B}_2\mathsf{f} - \mathsf{f} = 0 \text{ on } \Gamma_1\right\} \tag{5.2.78}$$

and

$$T\mathsf{f} = \Delta^2 \mathsf{f}, \quad \forall \mathsf{f} \in D(T). \tag{5.2.79}$$

Then, T is a positive self-adjoint operator with a compact resolvent.

Proof. We check that T is the Friedrichs extension of the sesquilinear, symmetric and coercive form

$$\widetilde{a}(\mathsf{f}, \mathsf{g}) = a(\mathsf{f}, \mathsf{g}) + \int_{\Gamma_1} (\partial_\nu f \partial_\nu \overline{g} + f\overline{g}) d\Gamma,$$

defined in $H^2_{\Gamma_0}(\Omega) \times H^2_{\Gamma_0}(\Omega)$. Indeed, by Friedrichs extension Theorem, we can write

$$D(T) = \left\{\mathsf{f} \in H^2_{\Gamma_0}(\Omega) \ : \ \exists ! \mathsf{F}_\mathsf{f} \in L^2(\Omega) \ \text{ such that } \ \widetilde{a}(\mathsf{f}, \mathsf{g}) = (\mathsf{F}_\mathsf{f}, \mathsf{g}), \ \ \forall \mathsf{g} \in H^2_{\Gamma_0}(\Omega)\right\}$$

and

$$T\mathsf{f} = \mathsf{F}_\mathsf{f}, \quad \forall \mathsf{f} \in D(T).$$

We now need to show that this operator T coincides with the one defined by (5.2.78)-(5.2.79). For that purpose, let us denote by $\widetilde{D}(T)$ the right-hand side of (5.2.78). By Green's formula (5.2.13), we directly see that $\widetilde{D}(T) \subseteq D(T)$ and that for $\mathsf{f} \in \widetilde{D}(T)$, $T\mathsf{f}$ is indeed given by (5.2.79). Let us then prove the converse inclusion. For this aim, let $\mathsf{f} \in D(T)$, then we have

$$\Delta^2 \mathsf{f} = \mathsf{F}_{\mathsf{f}} \;\text{ in }\; \mathcal{D}'(\Omega).$$

Hence f belongs to $E(\Delta^2, L^2(\Omega))$ and using Green's formula (5.2.13), we obtain

$$\mathcal{B}_1 \mathsf{f} = -\partial_\nu \mathsf{f} \;\text{ and }\; \mathcal{B}_2 \mathsf{f} = \mathsf{f}.$$

This proves that $D(T) = \widetilde{D}(T)$. Finally as $H^2_{\Gamma_0}(\Omega)$ is compactly embedded in $L^2(\Omega)$, T has clearly a compact resolvent. The proof is thus complete. $\square$

The main result of this section is the following theorem.

Theorem 5.2.3. The C$_0$-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is not uniformly stable in the energy space $\mathcal{H}$.

Proof. According to Theorem 1.3.6 due to Huang [67] and Prüss [94], it is sufficient to show that the resolvent of $\mathcal{A}$ is not uniformly bounded on the imaginary axis. In other words, it is enough to show the existence of a positive real number M and some sequences $\lambda_n \in i\mathbb{R}$, $U_n = (u_n, v_n, \eta_n, \xi_n, z_n^1, z_n^2)^\top \in D(\mathcal{A})$ and $F_n = (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, g_{1,n}, g_{2,n})^\top \in \mathcal{H}$, where $n \in \mathbb{N}$ such that

$$(\lambda_n I - \mathcal{A})U_n = F_n, \forall n \in \mathbb{N}, \tag{5.2.80}$$

$$\|U_n\|_{\mathcal{H}} \geq M, \forall n \in \mathbb{N}, \tag{5.2.81}$$

$$\lim_{n\to\infty} \|F_n\|_{\mathcal{H}} = 0. \tag{5.2.82}$$

From Lemma 5.2.4, we can consider the sequence of eigenfunctions $(\varphi_n)_{n\in\mathbb{N}}$ (that form an orthonormal basis of $L^2(\Omega)$) of the operator T corresponding to the eigenvalues $(\mu_n^4)_{n\in\mathbb{N}}$ such that μ_n^4 tends to infinity as n goes to infinity. Consequently for all $n \in \mathbb{N}$, they satisfy

$$\begin{cases} \Delta^2 \varphi_n = \mu_n^4 \varphi_n \;\text{ in }\; \Omega, \\ \varphi_n = \partial_\nu \varphi_n = 0 \;\text{ on }\; \Gamma_0, \\ \mathcal{B}_1 \varphi_n + \partial_\nu \varphi_n = 0 \;\text{ on }\; \Gamma_1, \\ \mathcal{B}_2 \varphi_n - \varphi_n = 0 \;\text{ on }\; \Gamma_1, \end{cases} \tag{5.2.83}$$

with

$$\|\varphi_n\|_{L^2(\Omega)} = 1. \tag{5.2.84}$$

Now, let us choose

$$\begin{cases} \lambda_n = i\mu_n^2, \quad u_n = \dfrac{\varphi_n}{\lambda_n}, \quad v_n = \varphi_n, \quad \eta_n = \dfrac{1}{\lambda_n}\partial_\nu \varphi_n, \quad \xi_n = \dfrac{\varphi_n}{\lambda_n} \\ \qquad z_n^1(\cdot, \rho) = \eta_n e^{-i\mu_n^2 \tau_1 \rho} \;\text{ and }\; z_n^2(\cdot, \rho) = \xi_n e^{-i\mu_n^2 \tau_2 \rho}. \end{cases} \tag{5.2.85}$$

It is easy to see that

$$z_n^1(\cdot, 0) = \eta_n = -\mathcal{B}_1 u_n \;\text{ on }\; \Gamma_1, \tag{5.2.86}$$

$$z_n^2(\cdot, 0) = \xi_n = -\mathcal{B}_2 u_n \;\text{ on }\; \Gamma_1. \tag{5.2.87}$$

Thus

$$U_n = \left(\frac{\varphi_n}{\lambda_n}, \varphi_n, \frac{1}{\lambda_n}\partial_\nu\varphi_n, \frac{\varphi_n}{\lambda_n}, \eta_n e^{-i\mu_n^2\tau_1\rho}, \xi_n e^{-i\mu_n^2\tau_2\rho}\right)^\top \tag{5.2.88}$$

belongs to $D(\mathcal{A})$ and is a solution of (5.2.80) with

$$F_n = (0,0,f_{3,n},f_{4,n},0,0)^\top, \quad f_{3,n} = \frac{\beta_1+\beta_2 e^{-i\mu_n^2\tau_1}}{i\mu_n^2}\partial_\nu\varphi_n \text{ and } f_{4,n} = \frac{\gamma_1+\gamma_2 e^{-i\mu_n^2\tau_2}}{i\mu_n^2}\varphi_n. \tag{5.2.89}$$

Now, we have

$$\|U_n\|_{\mathcal{H}}^2 \geq \|\varphi_n\|_{L^2(\Omega)}^2 = 1, \quad \forall n\in\mathbb{N},$$

which means that (5.2.81) holds with $M=1$. Moreover, we have

$$\begin{aligned}\|F_n\|_{\mathcal{H}}^2 &= \|f_{3,n}\|_{L^2(\Gamma_1)}^2 + \|f_{4,n}\|_{L^2(\Gamma_1)}^2\\ &\leq \frac{(\beta_1+|\beta_2|)^2}{\mu_n^4}\|\partial_\nu\varphi\|_{L^2(\Gamma_1)}^2 + \frac{(\gamma_1+|\gamma_2|)^2}{\mu_n^4}\|\varphi\|_{L^2(\Gamma_1)}^2\\ &\lesssim \frac{1}{\mu_n^4}\left(\|\partial_\nu\varphi_n\|_{L^2(\Gamma_1)}^2 + \|\varphi_n\|_{L^2(\Gamma_1)}^2\right).\end{aligned} \tag{5.2.90}$$

By using the trace theorem of interpolation type (see Theorem 1.4.4 in [82] and Theorem 1.5.1.10 in [53]), we obtain

$$\|\partial_\nu\varphi_n\|_{L^2(\Gamma_1)}^2 \lesssim \|\varphi_n\|_{H^2(\Omega)}\|\varphi_n\|_{H^1(\Omega)}, \tag{5.2.91}$$

$$\|\varphi_n\|_{L^2(\Gamma_1)}^2 \lesssim \|\varphi_n\|_{H^1(\Omega)}\|\varphi_n\|_{L^2(\Omega)}. \tag{5.2.92}$$

Now, it follows from Theorem 4.17 in [3] that

$$\|\varphi_n\|_{H^1(\Omega)} \lesssim \|\varphi_n\|_{H^2(\Omega)}^{\frac{1}{2}}\|\varphi_n\|_{L^2(\Omega)}^{\frac{1}{2}}.$$

Inserting the above inequality in (5.2.91) and (5.2.92), we get

$$\|\partial_\nu\varphi_n\|_{L^2(\Gamma_1)}^2 \lesssim \|\varphi_n\|_{H^2(\Omega)}^{\frac{3}{2}}\|\varphi_n\|_{L^2(\Omega)}^{\frac{1}{2}}, \tag{5.2.93}$$

$$\|\varphi_n\|_{L^2(\Gamma_1)}^2 \lesssim \|\varphi_n\|_{H^2(\Omega)}^{\frac{1}{2}}\|\varphi_n\|_{L^2(\Omega)}^{\frac{3}{2}}, \tag{5.2.94}$$

Now, we notice that

$$\widetilde{a}(\varphi_n,\varphi_n)^{\frac{1}{2}} = \left(a(\varphi_n,\varphi_n) + \int_{\Gamma_1}|\partial_\nu\varphi_n|^2 d\Gamma + \int_{\Gamma_1}|\varphi_n|^2 d\Gamma\right)^{\frac{1}{2}} = \mu_n^2\|\varphi_n\|_{L^2(\Omega)} = \mu_n^2.$$

Since the norm defined on the left-hand side of the above equation is equivalent to the usual norm of $H^2(\Omega)$, then we get

$$\|\varphi_n\|_{H^2(\Omega)} \lesssim \mu_n^2.$$

Inserting the above inequality and (5.2.84) in (5.2.93) and (5.2.94), we obtain

$$\|\partial_\nu\varphi_n\|_{L^2(\Gamma_1)}^2 \lesssim |\mu_n|^3, \tag{5.2.95}$$

$$\|\varphi_n\|_{L^2(\Gamma_1)}^2 \lesssim |\mu_n|. \tag{5.2.96}$$

Finally, from the above inequalities and (5.2.90), we obtain

$$\|F_n\|_{\mathcal{H}}^2 \lesssim \frac{|\mu_n|^3+|\mu_n|}{\mu_n^4} \to 0 \text{ as } n\to\infty.$$

The proof is thus complete. □

5.2.4 Polynomial stability

In this section, we will prove the polynomial stability of system (5.2.2)-(5.2.10). The main result of this section is the following theorem.

Theorem 5.2.4. Under the hypothesis (H) and the multiplier geometric control condition **MGC** (see Definition 1.4.1), for all $U_0 \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of U_0 such that the energy of system (5.2.2)-(5.2.10) satisfies the following estimation

$$E(t) \leq \frac{C}{t}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall\, t > 0.$$

According to Theorem 1.3.7, to prove Theorem 5.2.4, we need to prove the following two conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}) \tag{5.2.97}$$

and

$$\limsup_{\lambda\in\mathbb{R},\ |\lambda|\to\infty} \frac{1}{\lambda^2}\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{5.2.98}$$

As condition (5.2.97) was checked in Subsection 5.2.2, we only need to prove the second condition. This condition (5.2.98) is proved by a contradiction argument. For this purpose, suppose that (5.2.98) is false, then there exists $\{(\lambda_n, U_n := (u_n, v_n, \eta_n, \xi_n, z^1_n, z^2_n)^\top)\}_{n\geq 1} \subset \mathbb{R}^* \times D(\mathcal{A})$ with

$$|\lambda_n| \to \infty \quad \text{as } n\to\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}} = \left\|(u_n, v_n, \eta_n, \xi_n, z^1_n, z^2_n)^\top\right\|_{\mathcal{H}} = 1, \forall n \geq 1, \tag{5.2.99}$$

such that

$$(\lambda_n)^2(i\lambda_n I - \mathcal{A})U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, g_{1,n}, g_{2,n})^\top \to 0 \quad \text{in} \quad \mathcal{H} \quad \text{as } n\to\infty. \tag{5.2.100}$$

For simplicity, we now drop the index n. Equivalently, from (5.2.100), we have

$$\begin{aligned}
i\lambda u - v &= \lambda^{-2} f_1, \quad f_1 \to 0 \ \text{ in } H^2_{\Gamma_0}(\Omega), && (5.2.101)\\
i\lambda v + \Delta^2 u &= \lambda^{-2} f_2, \quad f_2 \to 0 \ \text{ in } L^2(\Omega), && (5.2.102)\\
i\lambda\eta - \partial_\nu v + \beta_1\eta + \beta_2 z^1(\cdot,1) &= \lambda^{-2} f_3, \quad f_3 \to 0 \ \text{ in } L^2(\Gamma_1), && (5.2.103)\\
i\lambda\xi - v + \gamma_1\xi + \gamma_2 z^2(\cdot,1) &= \lambda^{-2} f_4, \quad f_4 \to 0 \ \text{ in } L^2(\Gamma_1), && (5.2.104)\\
i\lambda z^1 + \frac{1}{\tau_1} z^1_\rho &= \lambda^{-2} g_1, \quad g_1 \to 0 \ \text{ in } L^2(\Gamma_1\times(0,1)), && (5.2.105)\\
i\lambda z^2 + \frac{1}{\tau_2} z^2_\rho &= \lambda^{-2} g_2, \quad g_2 \to 0 \ \text{ in } L^2(\Gamma_1\times(0,1)). && (5.2.106)
\end{aligned}$$

Here we will check the condition (5.2.98) by finding a contradiction with (5.2.99) by showing $\|U\|_{\mathcal{H}} = o(1)$. For clarity, we divide the proof into several Lemmas.

Lemma 5.2.5. Under the hypothesis (H), the solution $U = (u, v, \eta, \xi, z^1, z^2)^\top \in D(\mathcal{A})$ of (5.2.101)-(5.2.106) satisfies the following estimations

$$\int_{\Gamma_1} |\eta|^2 d\Gamma = o(\lambda^{-2}), \quad \int_{\Gamma_1} |\xi|^2 d\Gamma = o(\lambda^{-2}), \quad \int_{\Gamma_1} |\mathcal{B}_1 u|^2 d\Gamma = o(\lambda^{-2}) \ \text{ and } \ \int_{\Gamma_1} |\mathcal{B}_2 u|^2 d\Gamma = o(\lambda^{-2}). \tag{5.2.107}$$

Proof. First, taking the inner product of (5.2.100) with U in $\mathcal{H}$ and using (5.2.25), we get

$$(\beta_1 - |\beta_2|)\int_{\Gamma_1} |\eta|^2 d\Gamma + (\gamma_1 - |\gamma_2|)\int_{\Gamma_1} |\xi|^2 d\Gamma \leq -\Re(\mathcal{A}U, U)_{\mathcal{H}} = \frac{1}{\lambda^2}\Re(F, U)_{\mathcal{H}} \leq \frac{1}{\lambda^2}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

from the hypothesis (H), we notice that

$$\beta_1 - |\beta_2| > 0 \ \text{ and } \ \gamma_1 - |\gamma_2| > 0,$$

using the fact that $\|F\|_{\mathcal{H}} = o(1)$ and $\|U\|_{\mathcal{H}} = 1$, we obtain the first two estimations in (5.2.107). The last two estimations in (5.2.107) directly follows from the first two estimations in (5.2.107) and the fact that $\mathcal{B}_1 u = -\eta$, $\mathcal{B}_2 u = \xi$ on Γ_1. □

Lemma 5.2.6. Under the hypothesis (H), the solution $U = (u, v, \eta, \xi, z^1, z^2)^\top \in D(\mathcal{A})$ of (5.2.101)-(5.2.106) satisfies the following estimations

$$\int_{\Gamma_1}\int_0^1 |z^1|^2 d\rho d\Gamma = o(\lambda^{-2}) \ \text{ and } \ \int_{\Gamma_1} |z^1(\cdot, 1)|^2 d\Gamma = o(\lambda^{-2}), \tag{5.2.108}$$

$$\int_{\Gamma_1}\int_0^1 |z^2|^2 d\rho d\Gamma = o(\lambda^{-2}) \ \text{ and } \ \int_{\Gamma_1} |z^2(\cdot, 1)|^2 d\Gamma = o(\lambda^{-2}). \tag{5.2.109}$$

Proof. First, from (5.2.105) and the fact that $z^1(\cdot, 0) = \eta(\cdot)$ on Γ_1, we obtain

$$z^1(\cdot, \rho) = \eta e^{-i\lambda\tau_1\rho} + \frac{\tau_1}{\lambda^2}\int_0^\rho g_1(\cdot, s)e^{i\lambda\tau_1(s-\rho)}ds \ \text{ on } \ \Gamma_1 \times (0, 1). \tag{5.2.110}$$

From (5.2.110), Cauchy-Schwarz inequality and the fact that $\rho \in (0, 1)$, we get

$$\begin{aligned}
\int_{\Gamma_1}\int_0^1 |z^1|^2 d\rho d\Gamma &\leq 2\int_{\Gamma_1} |\eta|^2 d\Gamma + \frac{2(\tau_1)^2}{\lambda^4}\int_{\Gamma_1}\int_0^1 \left(\int_0^\rho |g_1(\cdot, s)|ds\right)^2 d\rho d\Gamma \\
&\leq 2\int_{\Gamma_1} |\eta|^2 d\Gamma + \frac{2(\tau_1)^2}{\lambda^4}\int_{\Gamma_1}\int_0^1 \rho \int_0^\rho |g_1(\cdot, s)|^2 ds d\rho d\Gamma \\
&\leq 2\int_{\Gamma_1} |\eta|^2 d\Gamma + \frac{2(\tau_1)^2}{\lambda^4}\left(\int_0^1 \rho d\rho\right)\int_{\Gamma_1}\int_0^1 |g_1(\cdot, s)|^2 ds d\Gamma \\
&= 2\int_{\Gamma_1} |\eta|^2 d\Gamma + \frac{(\tau_1)^2}{\lambda^4}\int_{\Gamma_1}\int_0^1 |g_1(\cdot, s)|^2 ds d\Gamma.
\end{aligned}$$

The above inequality, (5.2.107) and the fact that $g_1 \to 0$ in $L^2(\Gamma_1 \times (0, 1))$ lead to the first estimation in (5.2.108). Now, from (5.2.110), we deduce that

$$z^1(\cdot, 1) = \eta e^{-i\lambda\tau_1} + \frac{\tau_1}{\lambda^2}\int_0^1 g_1(\cdot, s)e^{i\lambda\tau_1}(s - 1)ds \ \text{ on } \ \Gamma_1,$$

consequently, by using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\int_{\Gamma_1} |z^1(\cdot, 1)|^2 d\Gamma &\leq 2\int_{\Gamma_1} |\eta|^2 d\Gamma + \frac{2(\tau_1)^2}{\lambda^4}\int_{\Gamma_1}\left(\int_0^1 |g_1(\cdot, s)|ds\right)^2 d\Gamma \\
&\leq 2\int_{\Gamma_1} |\eta|^2 d\Gamma + \frac{2(\tau_1)^2}{\lambda^4}\int_{\Gamma_1}\int_0^1 |g_1(\cdot, s)|^2 ds d\Gamma.
\end{aligned}$$

Therefore, from the above inequality, (5.2.107) and the fact that $g_1 \to 0$ in $L^2(\Gamma_1 \times (0, 1))$, we get the second estimation in (5.2.108). The same argument as before yielding (5.2.109), the proof is complete. □

Lemma 5.2.7. Under the hypothesis (H), the solution $U = (u, v, \eta, \xi, z^1, z^2)^\top \in D(\mathcal{A})$ of (5.2.101)-(5.2.106) satisfies the following estimations

$$\int_{\Gamma_1} |\partial_\nu u|^2 d\Gamma = o(\lambda^{-2}) \quad \text{and} \quad \int_{\Gamma_1} |u|^2 d\Gamma = o(\lambda^{-2}). \tag{5.2.111}$$

Proof. First, inserting (5.2.101) in (5.2.103), we obtain

$$i\lambda\partial_\nu u = \frac{1}{\lambda^2}(\partial_\nu f_1 - f_3) + (i\lambda + \beta_1)\eta + \beta_2 z^1(\cdot, 1) \quad \text{on} \quad \Gamma_1.$$

From the above equation, we get

$$\begin{aligned}\int_{\Gamma_1} |\lambda\partial_\nu u|^2 d\Gamma \lesssim \frac{1}{\lambda^4}\int_{\Gamma_1} \left(|\partial_\nu f_1|^2 + |f_3|^2\right) d\Gamma \\ + (\lambda^2 + \beta_1^2)\int_{\Gamma_1} |\eta|^2 d\Gamma + \beta_2^2 \int_{\Gamma_1} |z^1(\cdot, 1)|^2 d\Gamma.\end{aligned} \tag{5.2.112}$$

Using a trace theorem and the fact that $a(f_1, f_1) = o(1)$, we get

$$\int_{\Gamma_1} |\partial_\nu f_1|^2 d\Gamma \lesssim \|f_1\|^2_{H^2(\Omega)} \lesssim a(f_1, f_1) = o(1).$$

Thus, from the above estimation, (5.2.107), (5.2.108), (5.2.112), and the fact that $f_3 \to 0$ in $L^2(\Gamma_1)$, we get the first estimation in (5.2.111). Now, inserting (5.2.101) in (5.2.104), we obtain

$$i\lambda u = \frac{1}{\lambda^2}(f_1 - f_4) + (i\lambda + \gamma_1)\xi + \gamma_2 z^2(\cdot, 1) \quad \text{on} \quad \Gamma_1.$$

From the above equation, we deduce that

$$\int_{\Gamma_1} |\lambda u|^2 d\Gamma \lesssim \frac{1}{\lambda^4}\int_{\Gamma_1} (|f_1|^2 + |f_4|^2) d\Gamma + (\lambda^2 + \gamma_1^2)\int_{\Gamma_1} |\xi|^2 d\Gamma + \gamma_2^2 \int_{\Gamma_1} |z^2(\cdot, 1)|^2 d\Gamma \tag{5.2.113}$$

Again by a trace theorem and the fact that $a(f_1, f_1) = o(1)$, we get

$$\int_{\Gamma_1} |f_1|^2 d\Gamma \lesssim \|f_1\|^2_{H^2(\Omega)} \lesssim a(f_1, f_1) = o(1).$$

Finally, from the above estimation, (5.2.107), (5.2.109), (5.2.113) and the fact that $f_4 \to 0$ in $L^2(\Gamma_1)$, we obtain the second estimation in (5.2.111). The proof is thus complete. □

Lemma 5.2.8. Under the hypotheses (H) and (1.4.1), for all $u \in \mathsf{D}_{\Gamma_0}(\Delta^2)$, we have

$$\begin{aligned}-\Re\left\{\int_\Omega \Delta^2 u(h \cdot \nabla \overline{u}) dx\right\} \leq -\frac{1}{2}a(u, u) + \frac{\varepsilon_1 R^2}{2}\int_{\Gamma_1} |\mathcal{B}_2 u|^2 d\Gamma \\ + \left(\int_{\Gamma_1} |\mathcal{B}_1 u|^2 d\Gamma\right)^{\frac{1}{2}} \left(\int_{\Gamma_1} |\partial_\nu u|^2 d\Gamma\right)^{\frac{1}{2}} + \frac{R^2\varepsilon_2}{2}\int_{\Gamma_1} |\mathcal{B}_1 u|^2 d\Gamma,\end{aligned} \tag{5.2.114}$$

where $R = \|h\|_{L^\infty(\Omega)}$ and ε_1, ε_2 are positive constants explicitly given below.

Proof. In this proof, we follow the arguments of the proof of Lemma 3.1 in [95] and Lemma 3.1 in [96]. First, we assume that $\mathcal{B}_1 u = \Delta u+(1-\mu)\mathcal{C}_1 u \in H^{\frac{3}{2}}(\Gamma_1)$ and $\mathcal{B}_2 u = \partial_\nu \Delta u+(1-\mu)\partial_\tau \mathcal{C}_2 u \in H^{\frac{1}{2}}(\Gamma_1)$, then as $u \in \mathsf{D}_{\Gamma_0}(\Delta^2)$ we get $u \in H^4(\Omega)$. Now, by the identity (3.5) in [95] (see also [73], [96] and [20]), we get

$$\begin{aligned}
-\Re\left\{\int_\Omega \Delta^2 u(h\cdot\nabla\overline{u})dx\right\} &= -a(u,u)\\
&-\Re\left\{\int_\Gamma \left(\partial_\nu\Delta u+(1-\mu)\partial_\tau\mathcal{C}_2 u\right)(h\cdot\nabla\overline{u})d\Gamma\right\}\\
&+\Re\left\{\int_\Gamma \left(\Delta u+(1-\mu)\mathcal{C}_1 u\right)\partial_\nu(h\cdot\nabla\overline{u})d\Gamma\right\}-\frac{1}{2}\int_\Gamma (h\cdot\nu)c(u,u)d\Gamma,
\end{aligned}\tag{5.2.115}$$

where

$$c(u,u) = |u_{x_1x_1}|^2+|u_{x_2x_2}|^2+2\mu\Re\left(u_{x_1x_1}\overline{u}_{x_2x_2}\right)+2(1-\mu)|u_{x_1x_2}|^2.$$

From (5.2.20), we deduce that

$$c(u,u)\geq (1-\mu)d(u,u)\geq 0,\tag{5.2.116}$$

where

$$d(u,u) = |u_{x_1x_1}|^2+|u_{x_2x_2}|^2+2|u_{x_1x_2}|^2.$$

Now, since $u=\partial_\nu u = 0$ on Γ_0, then using the identities (3.5) and (3.6) in [96], we have

$$\nabla u = 0,\quad \mathcal{C}_1 u = 0,\quad \partial_\nu(h\cdot\nabla u) = (h\cdot\nu)\Delta u,\quad c(u,u) = |\Delta u|^2 \quad \text{on} \quad \Gamma_0,\tag{5.2.117}$$

where $\mathcal{C}_1$ is defined in (5.1.3). Consequently, we get

$$\left\{\begin{aligned}
&\int_{\Gamma_0}\left(\partial_\nu\Delta u+(1-\mu)\partial_\tau\mathcal{C}_2 u\right)(h\cdot\nabla\overline{u})d\Gamma = 0,\\
&\int_{\Gamma_0}\left(\Delta u+(1-\mu)\mathcal{C}_1 u\right)\partial_\nu(h\cdot\nabla\overline{u})d\Gamma = \int_{\Gamma_0}(h\cdot\nu)|\Delta u|^2 d\Gamma,\\
&\frac{1}{2}\int_{\Gamma_0}(h\cdot\nu)c(u,u)d\Gamma = \frac{1}{2}\int_{\Gamma_0}(h\cdot\nu)|\Delta u|^2 d\Gamma.
\end{aligned}\right.\tag{5.2.118}$$

Now, by using Young's inequality, we get

$$-\Re\left\{\int_{\Gamma_1}\mathcal{B}_2 u(h\cdot\nabla\overline{u})d\Gamma\right\}\leq \frac{\varepsilon_1 R^2}{2}\int_{\Gamma_1}|\mathcal{B}_2 u|^2 d\Gamma+\frac{1}{2\varepsilon_1}\int_{\Gamma_1}|\nabla u|^2 d\Gamma,\tag{5.2.119}$$

where $R = \|h\|_{L^\infty(\Omega)}$ and ε_1 is an arbitrary positive constant fixed below. Now, according to the identity (3.9) in [96] (see also (3.7) in [95]), we notice that

$$|\partial_\nu(h\cdot\nabla u)|\leq |\partial_\nu u|+R\sqrt{d(u,u)}\quad \text{on } \Gamma_1.\tag{5.2.120}$$

Using (5.2.120), Cauchy-Schwarz and Young's inequalities, (5.2.107), and (5.2.111), we get

$$\begin{aligned}
\Re\left\{\int_{\Gamma_1}\mathcal{B}_1 u\partial_\nu(h\cdot\nabla\overline{u})d\Gamma\right\} &\leq \int_{\Gamma_1}|\mathcal{B}_1 u|\left(|\partial_\nu u|+R\sqrt{d(u,u)}\right)d\Gamma\\
&\leq \left(\int_{\Gamma_1}|\mathcal{B}_1 u|^2 d\Gamma\right)^{\frac{1}{2}}\left(\int_{\Gamma_1}|\partial_\nu u|^2 d\Gamma\right)^{\frac{1}{2}}\\
&\quad +\frac{R^2\varepsilon_2}{2}\int_{\Gamma_1}|\mathcal{B}_1 u|^2 d\Gamma+\frac{1}{2\varepsilon_2}\int_{\Gamma_1}d(u,u)d\Gamma,
\end{aligned}\tag{5.2.121}$$

for all $\varepsilon_2 > 0$. Now, from (5.2.116) and (1.4.1), we get

$$\frac{1}{2}\int_{\Gamma_1}(h\cdot\nu)c(u,u)d\Gamma \geq \frac{1-\mu}{2\delta}\int_{\Gamma_1} d(u,u)d\Gamma. \tag{5.2.122}$$

Inserting (5.2.118), (5.2.119), (5.2.121) and (5.2.122) in (5.2.115), we obtain

$$\begin{aligned}
-\Re\left\{\int_\Omega \Delta^2 u(h\cdot\nabla\overline{u})dx\right\} &\leq -a(u,u) + \frac{1}{2}\int_{\Gamma_0}(h\cdot\nu)|\Delta u|^2 d\Gamma + \frac{\varepsilon_1 R^2}{2}\int_{\Gamma_1}|\mathcal{B}_2 u|^2 d\Gamma + \frac{1}{2\varepsilon_1}\int_{\Gamma_1}|\nabla u|^2 d\Gamma \\
&+ \left(\frac{1}{2\varepsilon_2} - \frac{1-\mu}{2\delta}\right)\int_{\Gamma_1} d(u,u)d\Gamma + \left(\int_{\Gamma_1}|\mathcal{B}_1 u|^2 d\Gamma\right)^{\frac{1}{2}}\left(\int_{\Gamma_1}|\partial_\nu u|^2 d\Gamma\right)^{\frac{1}{2}} + \frac{R^2\varepsilon_2}{2}\int_{\Gamma_1}|\mathcal{B}_1 u|^2 d\Gamma,
\end{aligned}$$

using (1.4.1) and taking $\varepsilon_2 \geq \delta(1-\mu)^{-1}$, we obtain

$$\begin{aligned}
-\Re\left\{\int_\Omega \Delta^2 u(h\cdot\nabla\overline{u})dx\right\} &\leq -a(u,u) + \frac{\varepsilon_1 R^2}{2}\int_{\Gamma_1}|\mathcal{B}_2 u|^2 d\Gamma + \frac{1}{2\varepsilon_1}\int_{\Gamma_1}|\nabla u|^2 d\Gamma \\
&+ \left(\int_{\Gamma_1}|\mathcal{B}_1 u|^2 d\Gamma\right)^{\frac{1}{2}}\left(\int_{\Gamma_1}|\partial_\nu u|^2 d\Gamma\right)^{\frac{1}{2}} + \frac{R^2\varepsilon_2}{2}\int_{\Gamma_1}|\mathcal{B}_1 u|^2 d\Gamma.
\end{aligned} \tag{5.2.123}$$

Now, by using a trace theorem, there exists a positive constant C_{tr} such that

$$\int_{\Gamma_1}|\nabla u|^2 d\Gamma \leq C_{tr}\|u\|^2_{H^2(\Omega)}.$$

From the equivalence between the norm $\sqrt{a(u,u)}$ and the usual norm of $H^2(\Omega)$, there then exists a positive constant C_{eq} such that

$$\int_{\Gamma_1}|\nabla u|^2 d\Gamma \leq C_{tr}\|u\|^2_{H^2(\Omega)} \leq C_{tr}C_{eq}a(u,u).$$

Inserting the above inequality in (5.2.123) and taking $\varepsilon_1 = C_{tr}C_{eq}$, we obtain (5.2.114). Finally, the case when $\mathcal{B}_1 u, \mathcal{B}_2 u \in L^2(\Gamma_1)$ can be easily obtained by the standard density arguments as in Lemma 3.1 in [95]. The proof is thus complete.

□

Lemma 5.2.9. Under the hypotheses (H) and (1.4.1), the solution $U = (u, v, \eta, \xi, z^1, z^2)^\top \in D(\mathcal{A})$ of (5.2.101)-(5.2.106) satisfies the following estimations

$$\int_\Omega |\lambda u|^2 dx = o(1) \quad \text{and} \quad a(u,u) = o(1). \tag{5.2.124}$$

Proof. First, inserting (5.2.101) in (5.2.102), we get

$$-\lambda^2 u + \Delta^2 u = \frac{if_1}{\lambda} + \frac{f_2}{\lambda^2} \quad \text{in} \quad \Omega.$$

Multiplying the above equation by $(h\cdot\nabla\overline{u})$, integrating over Ω, then taking the real part, we obtain

$$\begin{aligned}
&\Re\left\{-\lambda^2\int_\Omega u(h\cdot\nabla\overline{u})dx + \int_\Omega \Delta^2 u(h\cdot\nabla\overline{u})dx\right\} \\
&= \Re\left\{\frac{i}{\lambda}\int_\Omega f_1(h\cdot\nabla\overline{u})dx + \frac{1}{\lambda^2}\int_\Omega f_2(h\cdot\nabla\overline{u})dx\right\}
\end{aligned} \tag{5.2.125}$$

Now, by using Green's formula and the fact that $u = 0$ on Γ_0, then using (5.2.111), we get

$$\Re\left\{-\lambda^2 \int_\Omega u(h\cdot\nabla\overline{u})dx\right\} = \int_\Omega |\lambda u|^2 dx - \frac{1}{2}\int_{\Gamma_1}(h\cdot\nu)|\lambda u|^2 d\Gamma = \int_\Omega |\lambda u|^2 dx + o(1). \quad (5.2.126)$$

Using the fact that $a(u,u) = O(1)$ and $a(f_1, f_1) = o(1)$, we obtain

$$\begin{cases} \|\nabla u\|_{L^2(\Omega)} \leq \|u\|_{H^2(\Omega)} \lesssim \sqrt{a(u,u)} = O(1), \\ \|f_1\|_{L^2(\Omega)} \leq \|f_1\|_{H^2(\Omega)} \lesssim \sqrt{a(f_1,f_1)} = o(1). \end{cases}$$

Thus, from the above estimations and the fact that $f_2 \to 0$ in $L^2(\Omega)$, we obtain

$$\Re\left\{\frac{i}{\lambda}\int_\Omega f_1(h\cdot\nabla\overline{u})dx + \frac{1}{\lambda^2}\int_\Omega f_2(h\cdot\nabla\overline{u})dx\right\} = o(|\lambda|^{-1}). \quad (5.2.127)$$

Inserting (5.2.126) in (5.2.125) and using (5.2.127), we obtain

$$\int_\Omega |\lambda u|^2 dx = -\Re\left\{\int_\Omega \Delta^2 u(h\cdot\nabla\overline{u})dx\right\} + o(1). \quad (5.2.128)$$

As (5.2.107), (5.2.111) and (5.2.114) yield

$$-\Re\left\{\int_\Omega \Delta^2 u(h\cdot\nabla\overline{u})dx\right\} \leq -\frac{1}{2}a(u,u) + o(\lambda^{-2}),$$

inserting the above estimation in (5.2.128), we get

$$\int_\Omega |\lambda u|^2 dx + \frac{1}{2}a(u,u) = o(1).$$

The proof is thus complete. □

Proof of Theorem 5.2.4 From Lemmas 5.2.5, 5.2.6 and 5.2.9, we deduce that

$$\|U\|_{\mathcal{H}} = o(1),$$

which contradicts (5.2.99). □

5.3 Kirchhoff plate equation with delay terms on the boundary controls

5.3.1 Wellposedness and strong stability

In this section, we will establish the well-posedness and the strong stability of system (5.1.2). For this aim, as in [88], we introduce the following auxiliary variables

$$\begin{aligned} &z^1(x,\rho,t) := \partial_\nu u_t(x, t-\rho\tau_1), \quad x\in\Gamma_1,\ \rho\in(0,1),\ t>0,\\ &z^2(x,\rho,t) := u_t(x,t-\rho\tau_2), \quad x\in\Gamma_1,\ \rho\in(0,1),\ t>0. \end{aligned} \quad (5.3.1)$$

Then, system (5.1.2) becomes

$$
\begin{aligned}
u_{tt}+\Delta^2 u &= 0 \text{ in } \Omega\times(0,\infty), && (5.3.2)\\
u=\partial_\nu u &= 0 \text{ on } \Gamma_0\times(0,\infty), && (5.3.3)\\
\mathcal{B}_1 u+\beta_1\partial_\nu u_t+\beta_2 z^1(\cdot,1,t) &= 0 \text{ on } \Gamma_1\times(0,\infty), && (5.3.4)\\
\mathcal{B}_2 u-\gamma_1 u_t-\gamma_2 z^2(\cdot,1,t) &= 0 \text{ on } \Gamma_1\times(0,\infty), && (5.3.5)\\
\tau_1 z_t^1(\cdot,\rho,t)+z_\rho^1(\cdot,\rho,t) &= 0 \text{ on } \Gamma_1\times(0,1)\times(0,\infty), && (5.3.6)\\
\tau_2 z_t^2(\cdot,\rho,t)+z_\rho^2(\cdot,\rho,t) &= 0 \text{ on } \Gamma_1\times(0,1)\times(0,\infty), && (5.3.7)
\end{aligned}
$$

with the following initial conditions

$$
\begin{cases}
u(\cdot,0)=u_0(\cdot), \quad u_t(\cdot,0)=u_1(\cdot) \text{ in } \Omega,\\
z^1(\cdot,\rho,0)=f_0(\cdot,-\rho\tau_1) \text{ on } \Gamma_1\times(0,1),\\
z^2(\cdot,\rho,0)=g_0(\cdot,-\rho\tau_2) \text{ on } \Gamma_1\times(0,1).
\end{cases} \tag{5.3.8}
$$

The energy of system (5.3.2)-(5.3.8) is given by

$$
\begin{aligned}
E^0(t)=\frac{1}{2}\Bigg\{a(u,u)+\int_\Omega |u_t|^2dx+\tau_1|\beta_2|\int_{\Gamma_1}\int_0^1\left|z^1(\cdot,\rho,t)\right|^2d\rho d\Gamma\\
+\tau_2|\gamma_2|\int_{\Gamma_1}\int_0^1\left|z^2(\cdot,\rho,t)\right|^2d\rho d\Gamma\Bigg\},
\end{aligned} \tag{5.3.9}
$$

where a is defined in (5.2.11). If (u,u_t,z^1,z^2) is a regular solution of (5.3.2)-(5.3.8), then similarly to the proof of Lemma 5.2.1, we obtain

$$
\frac{d}{dt}E^0(t)\le-(\beta_1-|\beta_2|)\int_{\Gamma_1}|\partial_\nu u_t|^2d\Gamma-(\gamma_1-|\gamma_2|)\int_{\Gamma_1}|u_t|^2d\Gamma. \tag{5.3.10}
$$

Hence under the hypothesis (H), system (5.3.2)-(5.3.8) is dissipative in the sense that its energy is non-increasing with respect to time . Let us define the Hilbert space $\mathcal{H}^0$ by

$$
\mathcal{H}^0=H^2_{\Gamma_0}(\Omega)\times L^2(\Omega)\times\left(L^2(\Gamma_1\times(0,1))\right)^2,
$$

equipped with the following inner product

$$
\begin{aligned}
(U,U_1)_{\mathcal{H}^0}=a(u,u_1)+\int_\Omega v\overline{v_1}dx+\tau_1|\beta_2|\int_{\Gamma_1}\int_0^1 z^1\overline{z_1^1}d\rho d\Gamma\\
+\tau_2|\gamma_2|\int_{\Gamma_1}\int_0^1 z^2\overline{z_1^2}d\rho d\Gamma,
\end{aligned} \tag{5.3.11}
$$

where $U=(u,v,z^1,z^2)^\top$, $U^1=(u_1,v_1,z_1^1,z_1^2)^\top\in\mathcal{H}^0$. Now, we define the linear unbounded operator $\mathcal{A}^0:D(\mathcal{A}^0)\subset\mathcal{H}^0\longmapsto\mathcal{H}^0$ by:

$$
D(\mathcal{A}^0)=\left\{\begin{array}{l}
U=(u,v,z^1,z^2)^\top\in \mathsf{D}_{\Gamma_0}(\Delta^2)\times H^2_{\Gamma_0}(\Omega)\times(L^2(\Gamma_1;H^1(0,1)))^2 \mid\\
\mathcal{B}_1u=-\beta_1\partial_\nu v-\beta_2 z^1(\cdot,1),\quad \mathcal{B}_2 u=\gamma_1 v+\gamma_2 z^2(\cdot,1) \text{ on } \Gamma_1\\
z^1(\cdot,0)=\partial_\nu v, \quad z^2(\cdot,0)=v \text{ on } \Gamma_1
\end{array}\right\}
$$

and

$$\mathcal{A}^0 \begin{pmatrix} u \\ v \\ z^1 \\ z^2 \end{pmatrix} = \begin{pmatrix} v \\ -\Delta^2 u \\ -\dfrac{1}{\tau_1} z^1_\rho \\ -\dfrac{1}{\tau_2} z^2_\rho \end{pmatrix}, \forall U = (u, v, z^1, z^2)^\top \in D(\mathcal{A}^0). \tag{5.3.12}$$

Now, if $U = (u, u_t, z^1, z^2)^\top$ is solution of (5.3.2)-(5.3.8) and is sufficiently regular, then system (5.3.2)-(5.3.8) can be written as the following first order evolution equation

$$U_t = \mathcal{A}^0 U, \quad U(0) = U_0, \tag{5.3.13}$$

where $U_0 = (u_0, u_1, f_0(\cdot, -\rho\tau_1), g_0(\cdot, -\rho\tau_2))^\top \in \mathcal{H}^0$.

Proposition 5.3.1. Under the hypothesis (H), the unbounded linear operator $\mathcal{A}^0$ is m-dissipative in the energy space $\mathcal{H}^0$.

Proof. Similarly to the proof of Lemma 5.2.1, we show that

$$\begin{aligned} \Re(\mathcal{A}^0 U, U)_{\mathcal{H}^0} \leq & -(\beta_1 - |\beta_2|) \int_{\Gamma_1} |\partial_\nu v|^2 d\Gamma \\ & -(\gamma_1 - |\gamma_2|) \int_{\Gamma_1} |v|^2 d\Gamma \leq 0, \quad \forall U \in D(\mathcal{A}^0) \end{aligned} \tag{5.3.14}$$

and that $0 \in \rho(\mathcal{A}^0)$. □

According to Lumer-Phillips theorem (see Theorem 1.2.8), Proposition 5.3.1 implies that the operator $\mathcal{A}^0$ generates a C_0-semigroup of contractions $e^{t\mathcal{A}^0}$ in $\mathcal{H}^0$ which gives the well-posedness of (5.3.13). Then, we have the following result:

Theorem 5.3.1. For all $U_0 \in \mathcal{H}^0$, system (5.3.13) admits a unique weak solution $U(t) = e^{t\mathcal{A}^0} U_0 \in C^0(\mathbb{R}_+, \mathcal{H}^0)$. Moreover, if $U_0 \in D(\mathcal{A}^0)$, then the system (5.3.13) admits a unique strong solution $U(t) = e^{t\mathcal{A}^0} U_0 \in C^0(\mathbb{R}_+, D(\mathcal{A}^0)) \cap C^1(\mathbb{R}_+, \mathcal{H}^0)$.

Theorem 5.3.2. Under the hypotheses (H) and (1.4.1), the C_0-semigroup of contractions $\left(e^{t\mathcal{A}^0}\right)_{t\geq 0}$ is strongly stable in $\mathcal{H}^0$; i.e., for all $U_0 \in \mathcal{H}^0$, the solution of (5.3.13) satisfies

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}^0} U_0\|_{\mathcal{H}^0} = 0.$$

Proof. Similarly to the proof of Theorem 5.2.2, we can show that

$$\begin{cases} \ker(i\lambda I - \mathcal{A}^0) = \{0\}, & \forall \lambda \in \mathbb{R}, \\ \mathcal{R}(i\lambda I - \mathcal{A}^0) = \mathcal{H}^0, & \forall \lambda \in \mathbb{R}, \end{cases} \tag{5.3.15}$$

consequently $\mathcal{A}^0$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}^0) \cap i\mathbb{R} = \emptyset$, and we conclude by Theorem 1.3.3. □

5.3.2 Exponential stability

Theorem 5.3.1. Under the hypotheses (H) and (1.4.1), the C_0-semigroup $e^{t\mathcal{A}^0}$ is exponentially stable; i.e., for all $U_0 \in \mathcal{H}^0$, there exist constants $M \geq 1$ and $\epsilon > 0$ independent of U_0 such that

$$\|e^{t\mathcal{A}^0}U_0\|_{\mathcal{H}^0} \leq Me^{-\epsilon t}\|U_0\|_{\mathcal{H}^0},\ t > 0.$$

Proof. Since $i\mathbb{R} \subset \rho(\mathcal{A}^0)$ (see Section 5.3.1), according to Theorem 1.3.6, to prove Theorem 5.3.1, it remains to prove that

$$\limsup_{\lambda\in\mathbb{R},\ |\lambda|\to\infty} \| \left(i\lambda I - \mathcal{A}^0\right)^{-1} \|_{\mathcal{L}(\mathcal{H}^0)} < \infty. \tag{5.3.16}$$

We will prove condition (5.3.16) by a contradiction argument. For this purpose, suppose that (5.3.16) is false, then there exists $\left\{(\lambda_n, U_n := (u_n, v_n, z_n^1, z_n^2)^\top)\right\}_{n\geq 1} \subset \mathbb{R}^* \times D(\mathcal{A}^0)$ with

$$|\lambda_n| \to \infty \quad \text{as } n \to \infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}^0} = 1, \forall n \geq 1, \tag{5.3.17}$$

such that

$$(i\lambda_n I - \mathcal{A}^0)U_n = F_n := (f_{1,n}, f_{2,n}, g_{1,n}, g_{2,n})^\top \to 0 \quad \text{in} \quad \mathcal{H}^0, \quad \text{as } n \to \infty. \tag{5.3.18}$$

For simplicity, we drop the index n. Equivalently, from (5.3.18), we have

$$\begin{aligned} i\lambda u - v &= f_1 \to 0 \ \text{ in } \ H^2_{\Gamma_0}(\Omega), && (5.3.19)\\ i\lambda v + \Delta^2 u &= f_2 \to 0 \ \text{ in } \ L^2(\Omega), && (5.3.20)\\ i\lambda z^1 + \frac{1}{\tau_1} z^1_\rho &= g_1 \to 0 \ \text{ in } \ L^2(\Gamma_1 \times (0,1)), && (5.3.21)\\ i\lambda z^2 + \frac{1}{\tau_2} z^2_\rho &= g_2 \to 0 \ \text{ in } \ L^2(\Gamma_1 \times (0,1)). && (5.3.22) \end{aligned}$$

Taking the inner product of (5.3.18) with U in $\mathcal{H}^0$ and using (5.3.14), we get

$$(\beta_1 - |\beta_2|)\int_{\Gamma_1} |\partial_\nu v|^2 d\Gamma + (\gamma_1 - |\gamma_2|)\int_{\Gamma_1} |v|^2 d\Gamma \leq -\Re(\mathcal{A}^0 U, U)_{\mathcal{H}^0} = \Re(F, U)_{\mathcal{H}^0} \leq \|F\|_{\mathcal{H}^0}\|U\|_{\mathcal{H}^0},$$

From the above estimation, (H) and the fact that $\|F\|_{\mathcal{H}^0} = o(1)$ and $\|U\|_{\mathcal{H}^0} = 1$, we obtain

$$\int_{\Gamma_1} |\partial_\nu v|^2 d\Gamma = o(1) \quad \text{and} \quad \int_{\Gamma_1} |v|^2 d\Gamma = o(1), \tag{5.3.23}$$

consequently, from (5.3.19), a trace theorem and the fact that $\|F\|_{\mathcal{H}^0} = o(1)$, we get

$$\int_{\Gamma_1} |\partial_\nu u|^2 d\Gamma = o(\lambda^{-2}) \quad \text{and} \quad \int_{\Gamma_1} |u|^2 d\Gamma = o(\lambda^{-2}). \tag{5.3.24}$$

Now, from (5.3.21), (5.3.22) and the fact that $z^1(\cdot,0) = \partial_\nu v(\cdot)$, $z^2(\cdot,0) = v(\cdot)$ on Γ_1, we may write

$$z^1(\cdot,\rho) = \partial_\nu v e^{-i\lambda\tau_1\rho} + \tau_1 \int_0^\rho g_1(\cdot,s)e^{i\lambda\tau_1(s-\rho)}ds \quad \text{on} \quad \Gamma_1 \times (0,1),$$

$$z^2(\cdot,\rho) = v e^{-i\lambda\tau_2\rho} + \tau_2 \int_0^\rho g_2(\cdot,s)e^{i\lambda\tau_2(s-\rho)}ds \quad \text{on} \quad \Gamma_1 \times (0,1).$$

From the above equations, (5.3.23) and the fact that $\|F\|_{\mathcal{H}^0} = o(1)$, we obtain

$$\int_{\Gamma_1}\int_0^1 |z^1|^2 d\rho\Gamma = o(1) \quad \text{and} \quad \int_{\Gamma_1}\int_0^1 |z^2|^2 d\rho\Gamma = o(1), \tag{5.3.25}$$

and

$$\int_{\Gamma_1} |z^1(\cdot,1)|^2 d\Gamma = o(1) \quad \text{and} \quad \int_{\Gamma_1} |z^2(\cdot,1)|^2 d\Gamma = o(1). \tag{5.3.26}$$

Next, from the above estimations, (5.3.23) and the fact that $U \in D(\mathcal{A}^0)$, we get

$$\int_{\Gamma_1} |\mathcal{B}_1 u|^2 d\Gamma = o(1) \quad \text{and} \quad \int_{\Gamma_1} |\mathcal{B}_2 u|^2 d\Gamma = o(1). \tag{5.3.27}$$

Moreover, from (5.3.24), (5.3.27) and Lemma 5.2.8, we obtain

$$-\Re\left\{\int_\Omega \Delta^2 u(h\cdot\nabla\overline{u})dx\right\} \leq -\frac{1}{2}a(u,u) + o(1). \tag{5.3.28}$$

On the other hand, inserting (5.3.19) in (5.3.20), then multiplying the resulting equation by $(h\cdot\nabla\overline{u})$ and continue with the same argument as in the proof of Lemma 5.2.9, we obtain

$$\int_\Omega |\lambda u|^2 dx = -\Re\left\{\int_\Omega \Delta^2 u(h\cdot\nabla\overline{u})dx\right\} + o(1), \tag{5.3.29}$$

and consequently, from (5.3.28), we deduce that

$$\int_\Omega |\lambda u|^2 dx = o(1) \quad \text{and} \quad a(u,u) = o(1). \tag{5.3.30}$$

Finally, from (5.3.25) and (5.3.30), we obtain

$$\|U\|_{\mathcal{H}^0} = o(1),$$

which contradicts (5.3.17). The proof is thus complete. □

5.3.3 Some instability results

In this subsection, we will give some instability examples of system (5.1.2) in the cases $|\beta_2| \geq \beta_1$ and $|\gamma_2| \geq \gamma_1$. This is achieved by distinguishing between the following cases:

$$|\beta_2| = \beta_1 \quad \text{and} \quad |\gamma_2| = \gamma_1, \tag{IS$_1$}$$

$$|\beta_2| \geq \beta_1 \quad \text{and} \quad |\gamma_2| \geq \gamma_1 \quad \text{and} \quad |\beta_2| - \beta_1 + |\gamma_2| - \gamma_1 > 0. \tag{IS$_2$}$$

Theorem 5.3.3. If (IS$_1$) or (IS$_2$) hold, then there exist sequences of delays and solutions of (5.1.2) corresponding to these delays such that their standard energy is constant.

Proof. We seek for a solution of system (5.1.2) in the form

$$u(x,t) = e^{i\lambda t}\varphi(x), \quad \text{with } \lambda \neq 0. \tag{5.3.31}$$

Inserting (5.3.31) in (5.1.2), we get

$$\begin{cases} -\lambda^2\varphi + \Delta^2\varphi = 0 \ \text{ in } \ \Omega, \\ \varphi = \partial_\nu\varphi = 0 \ \text{ on } \ \Gamma_0, \\ \mathcal{B}_1\varphi = -i\lambda(\beta_1 + \beta_2 e^{-i\lambda\tau_1})\partial_\nu\varphi \ \text{ on } \ \Gamma_1, \\ \mathcal{B}_2\varphi = i\lambda(\gamma_1 + \gamma_2 e^{-i\lambda\tau_2})\varphi \ \text{ on } \ \Gamma_1. \end{cases} \tag{5.3.32}$$

Let $\theta \in H^2_{\Gamma_0}(\Omega)$. Multiplying the first equation in (5.3.32) by $\overline{\theta}$, then using Green's formula, we get

$$-\lambda^2 \int_\Omega \varphi\overline{\theta}dx + a(\varphi,\theta) + i\lambda(\beta_1+\beta_2 e^{-i\lambda\tau_1}) \int_{\Gamma_1} \partial_\nu\varphi\partial_\nu\overline{\theta}d\Gamma + i\lambda(\gamma_1+\gamma_2 e^{-i\lambda\tau_2}) \int_{\Gamma_1} \varphi\overline{\theta}d\Gamma = 0, \tag{5.3.33}$$

for all $\theta \in H^2_{\Gamma_0}(\Omega)$. Now, since $|\beta_2| \geq \beta_1$ and $|\gamma_2| \geq \gamma_1$, then we assume that

$$\cos(\lambda\tau_1) = -\frac{\beta_1}{\beta_2} \quad \text{and} \quad \cos(\lambda\tau_2) = -\frac{\gamma_1}{\gamma_2}. \tag{5.3.34}$$

Thus, we choose

$$\beta_2 \sin(\lambda\tau_1) = \sqrt{\beta_2^2 - \beta_1^2} \ \text{ and } \ \gamma_2 \sin(\lambda\tau_2) = \sqrt{\gamma_2^2 - \gamma_1^2}. \tag{5.3.35}$$

Inserting (5.3.34) and (5.3.35) in (5.3.33), we obtain

$$-\lambda^2 \int_\Omega \varphi\overline{\theta}dx + a(\varphi,\theta) + \lambda\sqrt{\beta_2^2 - \beta_1^2} \int_{\Gamma_1} \partial_\nu\varphi\partial_\nu\overline{\theta}d\Gamma + \lambda\sqrt{\gamma_2^2 - \gamma_1^2} \int_{\Gamma_1} \varphi\overline{\theta}d\Gamma = 0, \tag{5.3.36}$$

for all $\theta \in H^2_{\Gamma_0}(\Omega)$. Now, taking $\theta = \varphi$ in (5.3.36), we obtain

$$-\lambda^2 \int_\Omega |\varphi|^2 dx + a(\varphi,\varphi) + \lambda\sqrt{\beta_2^2 - \beta_1^2} \int_{\Gamma_1} |\partial_\nu\varphi|^2 d\Gamma + \lambda\sqrt{\gamma_2^2 - \gamma_1^2} \int_{\Gamma_1} |\varphi|^2 d\Gamma = 0. \tag{5.3.37}$$

Without loss of generality, we can assume that

$$\|\varphi\|_{L^2(\Omega)} = 1. \tag{5.3.38}$$

Thus, from (5.3.37) and (5.3.38), we get

$$\lambda^2 - a(\varphi,\varphi) - \lambda\sqrt{\beta_2^2 - \beta_1^2}\, q_\nu(\varphi) - \lambda\sqrt{\gamma_2^2 - \gamma_1^2}\, q(\varphi) = 0, \tag{5.3.39}$$

where

$$q(\varphi) = \int_{\Gamma_1} |\varphi|^2 d\Gamma \ \text{ and } \ q_\nu(\varphi) = \int_{\Gamma_1} |\partial_\nu\varphi|^2 d\Gamma. \tag{5.3.40}$$

We define

$$W := \left\{ w \in H^2_{\Gamma_0}(\Omega) \mid \|w\|_{L^2(\Omega)} = 1 \right\}.$$

Now, we distinguish two cases.
Case 1: If (IS_1) holds, then from (5.3.39), we have

$$a(\varphi,\varphi) = \lambda^2. \tag{5.3.41}$$

Let us define

$$\lambda^2 := \min_{w\in W} a(w,w). \tag{5.3.42}$$

Now, if φ verifies

$$a(\varphi,\varphi) = \min_{w\in W} a(w,w),$$

then it easy to see that φ is a solution of (5.3.33) and consequently (5.3.31) is a solution of (5.1.2). Moreover, from (5.3.31) and (5.3.9), we get

$$E^0(t) = E^0(0) \geq a(\varphi,\varphi) + \lambda^2 \int_\Omega |\varphi|^2 dx = 2\lambda^2 > 0, \quad \forall t \geq 0.$$

Thus, the energy of (5.1.2) is constant and positive. Further from our assumptions

$$\cos(\lambda\tau_1) = -1, \quad \sin(\lambda\tau_1) = 0, \quad \cos(\lambda\tau_2) = -1, \quad \sin(\lambda\tau_2) = 0,$$

system (5.3.32) becomes

$$\begin{cases} -\lambda^2\varphi + \Delta^2\varphi = 0 \ \text{ in } \ \Omega, \\ \varphi = \partial_\nu\varphi = 0 \ \text{ on } \ \Gamma_0, \\ \mathcal{B}_1\varphi = 0 \ \text{ on } \ \Gamma_1, \\ \mathcal{B}_2\varphi = 0 \ \text{ on } \ \Gamma_1. \end{cases} \tag{5.3.43}$$

So, we can take a sequence $(\lambda_n)_n$ of positive real numbers defined by

$$\lambda_n^2 = \Lambda_n^2, \quad n \in \mathbb{N},$$

where Λ_n^2, $n \in \mathbb{N}$, are the eigenvalues for the bi-Laplacian operator with the boundary conditions $(5.3.43)_2$-$(5.3.43)_4$. Then, setting

$$\lambda_n\tau_1 = (2k+1)\pi, \quad k \in \mathbb{N} \ \text{ and } \ \lambda_n\tau_2 = (2l+1)\pi, \quad l \in \mathbb{N},$$

we get the following sequences of delays

$$\tau_{1,n,k} = \frac{(2k+1)\pi}{\lambda_n}, \quad k,n \in \mathbb{N} \ \text{ and } \ \tau_{2,n,l} = \frac{(2l+1)\pi}{\lambda_n}, \quad l,n \in \mathbb{N},$$

which becomes arbitrarily small (or large) for suitable choices of the indices $n,k,l \in \mathbb{N}$. Therefore, we have found sets of time delays for which system (5.1.2) is not asymptotically stable.

Case 2: If (IS_2) holds, then from (5.3.39), we have

$$\lambda = \frac{1}{2}\left[\sqrt{\beta_2^2 - \beta_1^2 q_\nu(\varphi)} + \sqrt{\gamma_2^2 - \gamma_1^2 q(\varphi)} \right. \\ \left. \pm\sqrt{\left(\sqrt{\beta_2^2 - \beta_1^2 q_\nu(\varphi)} + \sqrt{\gamma_2^2 - \gamma_1^2 q(\varphi)}\right)^2 + 4a(\varphi,\varphi)}\right]. \tag{5.3.44}$$

Let us define

$$\lambda := \frac{1}{2}\min_{w\in W}\left\{\sqrt{\beta_2^2 - \beta_1^2 q_\nu(w)} + \sqrt{\gamma_2^2 - \gamma_1^2 q(w)} \right. \\ \left. +\sqrt{\left(\sqrt{\beta_2^2 - \beta_1^2 q_\nu(w)} + \sqrt{\gamma_2^2 - \gamma_1^2 q(w)}\right)^2 + 4a(w,w)}\right\}. \tag{5.3.45}$$

Let us prove that if the minimum in the right-hand side of (5.3.45) is attained at φ, that is

$$\begin{aligned}
&\sqrt{\beta_2^2-\beta_1^2}q_\nu(\varphi)+\sqrt{\gamma_2^2-\gamma_1^2}q(\varphi)\\
&\quad+\sqrt{\left(\sqrt{\beta_2^2-\beta_1^2}q_\nu(\varphi)+\sqrt{\gamma_2^2-\gamma_1^2}q(\varphi)\right)^2+4a(\varphi,\varphi)}\\
&:=\min_{w\in W}\left\{\sqrt{\beta_2^2-\beta_1^2}q_\nu(w)\right.\\
&\quad\left.+\sqrt{\gamma_2^2-\gamma_1^2}q(w)+\sqrt{\left(\sqrt{\beta_2^2-\beta_1^2}q_\nu(w)+\sqrt{\gamma_2^2-\gamma_1^2}q(w)\right)^2+4a(w,w)}\right\},
\end{aligned}\tag{5.3.46}$$

then φ is a solution of (5.3.36). For this aim, take for $\varepsilon\in\mathbb{R}$

$$w=\varphi+\varepsilon\theta \quad\text{with}\quad \theta\in H^2_{\Gamma_0}(\Omega)\quad\text{such that}\quad \int_\Omega \varphi\overline{\theta}dx=0. \tag{5.3.47}$$

Thus, we have

$$\|w\|^2_{L^2(\Omega)}=\|\varphi\|^2_{L^2(\Omega)}+\varepsilon^2\|\theta\|^2_{L^2(\Omega)}=1+\varepsilon^2\|\theta\|^2_{L^2(\Omega)}. \tag{5.3.48}$$

Now, if we define

$$\begin{aligned}
f(\varepsilon):=&\frac{1}{1+\varepsilon^2\|\theta\|^2_{L^2(\Omega)}}\left(\sqrt{\beta_2^2-\beta_1^2}q_\nu(\varphi+\varepsilon\theta)+\sqrt{\gamma_2^2-\gamma_1^2}q(\varphi+\varepsilon\theta)\right.\\
&\left.+\sqrt{\left(\sqrt{\beta_2^2-\beta_1^2}q_\nu(\varphi+\varepsilon\theta)+\sqrt{\gamma_2^2-\gamma_1^2}q(\varphi+\varepsilon\theta)\right)^2+4a(\varphi+\varepsilon\theta,\varphi+\varepsilon\theta)}\right),
\end{aligned}\tag{5.3.49}$$

thus, from (5.3.46), we get

$$\begin{aligned}
f(\varepsilon)\geq f(0)=&\sqrt{\beta_2^2-\beta_1^2}q_\nu(\varphi)+\sqrt{\gamma_2^2-\gamma_1^2}q(\varphi)\\
&+\sqrt{\left(\sqrt{\beta_2^2-\beta_1^2}q_\nu(\varphi)+\sqrt{\gamma_2^2-\gamma_1^2}q(\varphi)\right)^2+4a(\varphi,\varphi)},
\end{aligned}$$

which gives

$$f'(0)=0.$$

Consequently, after an easy computation, we obtain

$$a(\varphi,\theta)+\lambda\sqrt{\beta_2^2-\beta_1^2}\int_{\Gamma_1}\partial_\nu\varphi\partial_\nu\overline{\theta}d\Gamma+\lambda\sqrt{\gamma_2^2-\gamma_1^2}\int_{\Gamma_1}\varphi\overline{\theta}d\Gamma=0. \tag{5.3.50}$$

Since any function $\tilde{\theta}\in H^2_{\Gamma_0}(\Omega)$ can be decomposed as

$$\tilde{\theta}=\alpha\varphi+\theta\quad\text{with}\quad\alpha\in\mathbb{R}\quad\text{and}\quad\theta\in H^2_{\Gamma_0}(\Omega)\quad\text{such that}\quad\int_\Omega\varphi\overline{\theta}dx=0,$$

from (5.3.50) and (5.3.37), we obtain that φ satisfies (5.3.36). Thus, for such $\lambda>0$

$$\lambda\tau_1=\arccos\left(-\frac{\beta_1}{\beta_2}\right)+2k\pi,\quad k\in\mathbb{N}\quad\text{and}\quad\lambda\tau_2=\arccos\left(-\frac{\gamma_1}{\gamma_2}\right)+2l\pi,\quad l\in\mathbb{N},$$

define a sequences of time delays for which (5.1.2) is not asymptotically stable. The proof is thus complete. ∎

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