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Quelques problèmes de contrôle optimal d'équations aux dérivées partielles et applications au procédé de fusion sélective par laser (SLM)

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Abstract

In this thesis, we study optimal control problems of partial differential equations with an application to the selective laser melting process. The thesis consists of four chapters. In chapter 1, we introduce the topic of the thesis and we present the main results. Chapter 2 deals with an optimal control problem of the heat equation with non-convex constraints on the control. The problem corresponds to laser path optimization in the selective laser melting process. First, we introduce this technology and the control problem. The laser path is the control. Then we show existence of optimal controls and we deduce a first order necessary optimality condition. The difficulty in discretizing the non-convex constraints leads us to introduce another constraint on the control, being non differentiable we treat it by penalization. In chapter 3, we study the existence of local solutions for the heat-Maxwell coupled system, the permittivity dependent on the temperature. The model describes the diffusion of heat with volumic heat source induced by electromagnetic waves. The model being nonlinear, we first show that Maxwell's equations are well posed using the theory of evolution systems (theory of T. Kato) in the hyperbolic case. Then, we show the existence of local solutions for the coupled problem using the Schauder's fixed point theorem. Finally in chapter 4, we study an optimal control problem related to external electromagnetic source. The state equation is the heat-Maxwell system presented in chapter 3 but with a permittivity independent of the temperature. The control is the external electromagnetic source and could be the electric field of lasers. We show the well posedness of the state equation then we prove existence of optimal controls. At last, a first-order necessary condition for a control to be optimal is then derived in the form of a variational inequality.

Key words: Heat Equation, Maxwell's equations, heat-Maxwell coupled system, laser path optimization, optimal control, non convex constraints, first-order necessary optimality condition.

Résumé. Dans cette thèse, nous étudions quelques problèmes de contrôle optimal des équations aux dérivées partielles avec appliqués au procédé de fusion sélective par laser. La thèse se compose de quatre chapitres. Dans le chapitre 1, nous introduisons le sujet de la thèse et nous présentons les principaux résultats obtenus. Le chapitre 2 traite un problème de contrôle optimal de l'équation de la chaleur avec des contraintes non convexes sur le contrôle. Le problème correspond à l'optimisation de la trajectoire du laser dans le procédé de fusion sélective par laser. Tout d'abord, nous présentons cette technologie et le problème de contrôle optimal étudié. Le contrôle est la trajectoire du laser. Ensuite, nous montrons l'existence d'un contrôle optimal et nous en déduisons une condition d'optimalité nécessaire du premier ordre. La difficulté à discrétiser les contraintes non convexes nous amènent à introduire une autre contrainte géométrique sur la trajectoire du laser, étant non différentiable nous la traitons par pénalisation. Dans le chapitre 3, nous étudions l'existence de solutions locales pour le système couplé chaleur-Maxwell, la permittivité dépendante de la température. Le modèle décrit la diffusion de la chaleur avec une source de chaleur volumique induite par des ondes électromagnétiques. Le modèle étant non linéaire, nous montrons d'abord que les équations de Maxwell sont bien posées en utilisant la théorie des systèmes d'évolution (théorie de T. Kato) dans le cas hyperbolique. Puis, nous montrons l'existence de solutions locales pour le problème couplé en utilisant le théorème du point fixe de Schauder. Enfin au chapitre 4, nous étudions un problème de contrôle optimal lié à la source électromagnétique externe. L'équation d'état est le système chaleur-Maxwell présenté au chapitre 3 mais avec une permittivité indépendant de la température. Le contrôle est la source électromagnétique externe et pourrait être le champ électrique des lasers. Nous montrons que l'équation d'état est bien posé puis nous prouvons l'existence d'un contrôle optimal. Enfin, une condition nécessaire du premier ordre pour qu'un contrôle soit optimal est alors dérivée sous la forme d'une inégalité variationnelle.

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Chapter 1

Introduction

1.1 Aim and motivation

Additive manufacturing technologies (AMT), including 3D printing, have undergone considerable development during the last decade. It differs from traditional material removal processes like milling, turning, and spark erosion, because this technology family creates metal components by incremental addition of material instead of removal of chips. Starting from a three-dimensional CAD (computer aided design) representation of an object, the object is virtually "sliced" into a set of two-dimensional layers. These layers are then successively fused and consolidated on top of each other to recreate the three-dimensional object. AM technologies using laser avoid the need for toolings such as jigs, molds, and fixtures. Also, fabrication of optimized and complex patterns that are especially applicable in automotive, aerospace, and biomedical products becomes much easier. Specifically, AM can manufacture complex geometries which are hard or impossible to fabricate by conventional cutting tools and technologies. Moreover, it is economical to use AM for ranges of low to large batches in a short period with low material waste and low residual stress.

Laser sintering is a process in which a high energy laser beam scans the surface of a powder bed (the powder can be metal, polymer or ceramics) and the melted powder solidifies to form the bulk part. Selective Laser Melting (SLM) is the most commonly used terminology to describe laser sintering of metals. SLM makes it possible to create fully functional parts directly by melting metallic powder by a layer-by-layer technology without using any intermediate binders or any additional processing steps after the laser melting operation. Laser sintering is very complicated because of its fast laser scan rates and material transformations (solidification and liquefaction) in a very short time frame. The temperature field was found to be inhomogeneous by many previous researchers [57][30][61]. Meanwhile, the temperature evolution history in laser sintering has significant effects on the quality of the final parts, such as density, dimensions, mechanical properties, microstructure, etc. For metals, rapid repeated heating and cooling cycles of successive layers of the powder feedstock during SLM build process is responsible for large temperature gradients, as a consequence high residual stresses and deformation appear, and may even lead to crack formation in the fabricated part. Thermal distortion of the fabricated part is one serious problem in SLM [16].

Our aim in this thesis is to study two main problems in SLM. The first one is presented in chapter 2, where a laser path optimization problem in SLM is proposed and studied. In

fact, laser scanning strategies in SLM influence temperature distribution. Indeed highly localized heat inputs may result in large temperature gradient and thus lead to deformations during SLM process. Our aim in Chapter 2 is to propose a mathematical model to find an optimal trajectory minimizing thermal gradients within the produced part using optimal control theory of PDE's. The second problem studied in this thesis is the interaction between the laser and the heat in the heated medium. This is a heat-diffusion system that involves a laser beam source irradiating locally a three dimensional medium. This technique is used in laser melting process for additive metallic manufacturing. To model accurately the laser interaction with the medium, we consider the coupling between the heat diffusion equation and Maxwell's equations. We also take into account, the temperature dependence of the electric permittivity of the medium inside the domain. In chapter 3 we prove existence of local solutions for the non-linear coupled Heat-Maxwell system. In chapter 4 we study an optimal control problem of the external electromagnetic source.

1.2 Chapters content and manuscrit organization

1.2.1 Laser path optimization

Chapter 2 concerns a problem arising with selective laser melting. First, we explain the physical problem in section 2.1. In section 2.2 we give the system that we will study. The problem depends on the function γ, that represents the displacement of the laser beam on the top layer Γ_1, and the goal is to find γ satisfying constraints related to the problem: the laser beam has to cover in some way the surface Γ_1, and additionnally the gradient of the temperature inside the layer Ω is the smallest possible. We translate this in the following optimal control problem:

The state equation is given by

$$\left\{\begin{array}{lll} \rho\, c\, \partial_t y - \kappa\, \Delta y = 0 & \text{in} & Q = \Omega \times]0,T[\,, \\ -\kappa \frac{\partial y}{\partial \nu} = h\, y - g_\gamma & \text{on} & \Sigma_1 = \Gamma_1 \times]0,T[\,, \\ -\kappa \frac{\partial y}{\partial \nu} = h\, y & \text{on} & \Sigma_2 = \Gamma_2 \times]0,T[\,, \\ -\kappa \frac{\partial y}{\partial \nu} = h\,(y - y_B) & \text{on} & \Sigma_3 = \Gamma_3 \times]0,T[\,, \\ y(x,0) = y_0(x) & \text{for} & x \in \Omega. \end{array}\right. \tag{1.1}$$

Here $y(x,t)$ denotes the temperature at point $x \in \Omega$ and time $t \in]0,T[$, y_0 is the initial temperature, while $y_B \in L^2(\Gamma_3)$ corresponds to the temperature at the top of the previous layer. Here, $\Omega \subset \mathbb{R}^3$ is a bounded and simply connected domain with a connected Lipschitz boundary Γ, that is supposed to be split up

$$\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3},$$

where Γ_i, $i = 1,2,3$, are disjoint open disjoint subsets of Γ. In the SLM process, Γ_1 corresponds to the upper surface of the added layer and is supposed to be included in a plane, that without loss of generality we assume to be $\mathbb{R}^2$. By $\nu(x)$, we denote the outward normal direction at the point $x \in \Gamma$. The heat source g_γ is of the form

$$g_\gamma(x,t) = \alpha \frac{2P}{\pi R^2} \exp\left(-2\frac{|\, x - \gamma(t)\,|^2}{R^2}\right), \text{ for all } (x,t) \in \Sigma_1. \tag{1.2}$$

The parameters ρ, c, κ, h, α, P and R are positive constants. The control $\gamma : t \in [0,T] \to \Gamma_1$ represents the displacement of the laser beam center on Γ_1 with respect to time.

The cost functional is defined by

$$J(y,\gamma) := \frac{1}{2}\int_0^T \int_\Omega \mid \nabla y(x,t) \mid^2 dx\, dt + \frac{\lambda_Q}{2}\int_0^T \int_\Omega \mid y(x,t) - y_Q(x,t) \mid^2 dx\, dt + \frac{\lambda_\gamma}{2} \parallel \gamma \parallel^2_{H^1(0,T;\mathbb{R}^2)}, \tag{1.3}$$

where $\lambda_Q \geq 0$ and $\lambda_\gamma > 0$ are constants, while $y_Q \in L^2(Q)$ is a given function.

Our aim is to solve the following problem:

$$\min_{\gamma \in U_{ad}} J(y(\gamma),\gamma), \tag{OCP}$$

where $y(\gamma)$ denotes the weak solution of problem (1.1) associated with the control γ, and the set of admissible controls $U_{ad} \subset H^1(0,T;\mathbb{R}^2)$ is defined as follows. For $\epsilon \geq R$, and a fixed positive constant c, U_{ad} is defined by

$$U_{ad} \coloneqq \{\gamma \in H^1(0,T;\mathbb{R}^2);\ R(\gamma) \subset \Gamma_{1,-\epsilon},\ R_\epsilon(\gamma) = \Gamma_1 \text{ and } \mid \gamma'(t) \mid \leq c \text{ a.e. } t \in [0,T]\}, \tag{1.4}$$

where $R(\gamma) := \gamma([0,T])$,

$$\Gamma_{1,-\epsilon} = \{x \in \Gamma_1; dist\,(x, \partial\Gamma_1) \geq \epsilon\}, \tag{1.5}$$

and

$$R_\epsilon(\gamma) = \{x \in \Gamma_1;\ dist\,(x, R(\gamma)) \leq \epsilon\}. \tag{1.6}$$

The link between our non convex set of admissible controls and laser trajectory in SLM process is explained in section 2.2. In subsection 2.2.1 we prove that the optimal control problem has at least one optimal control. In subsection 2.2.2, we prove the differentiability of the control-to-state mapping, result from which we infer the differentiability of the reduced cost functional. In subsection 2.2.3, the adjoint state is introduced which allow us to compute the Fréchet derivative of this reduced cost functional.

The constraints on the displacement γ make the displacement space U_{ad} nonconvex, and that brings difficulties, that we overcome using Kuhn-Tucker conditions in subsection 2.2.3. Then we find a necessary condition that an optimal control satisfies. The proofs are classical and follow from usual stategies in optimal control theory [28, 53]. Section 2.2 corresponds to the paper [2] in collaboration with Serge Nicaise and Luc Paquet.

The two constraints $R(\gamma) \subset \Gamma_{1,-\epsilon}$, $R_\epsilon(\gamma) = \Gamma_1$ are strong, and do not seem appropriate for discretization. Hence, in section 2.3 we intend to replace the previous nonconvex constraints by other conditions on the trajectory γ, by adding a penalization term to the cost functional (1.3). Namely, given $\theta > 0$ a penalization parameter, we add to $J(y(\gamma),\gamma)$, the term

$$\frac{1}{\theta^2}\left(2R\int_0^T \sqrt{\mid \gamma'(t) \mid^2 + \theta^2}dt - \mid \Gamma_1 \mid\right)^2.$$

Formally as θ is close to zero, this will force the control to satisfy

$$2R\int_0^T \mid \gamma'(t) \mid\ dt \simeq \mid \Gamma_1 \mid, \tag{1.7}$$

which means that the area covered by the laser is close to the area of Γ_1. We consider the penalized optimal control problems: Given $\theta > 0$, find $\bar{\gamma}^\theta \in U_{ad}$ such that

$$(\mathrm{OCP}^\theta) \qquad J^\theta(y(\bar{\gamma}^\theta), \bar{\gamma}^\theta) = \min_{\gamma \in U_{ad}^p} J^\theta(y(\gamma), \gamma),$$

with,

$$J^\theta(y(\gamma), \gamma) := J(y(\gamma), \gamma) + \frac{1}{\theta^2}\left(2R\int_0^T \sqrt{|\ \gamma'(t)\ |^2 + \theta^2}dt - |\ \Gamma_1\ |\right)^2, \tag{1.8}$$

where

$$\begin{aligned} J(y(\gamma), \gamma) :=& \frac{1}{2}\int_0^T \int_\Omega |\ \nabla y(\gamma)(x,t)\ |^2\ dx\ dt + \frac{\lambda_Q}{2}\int_0^T \int_\Omega |\ y(\gamma)(x,t) - y_Q(x,t)\ |^2\ dx\ dt \\ &+ \frac{\lambda_\gamma}{2}\ \|\ \gamma\ \|^2_{H^2(0,T;\mathbb{R}^2)}\ . \end{aligned} \tag{1.9}$$

Here, we assume that Γ_1 is a convex subset of $\mathbb{R}^2$. If $\epsilon \geq R$ the set of admissible controls

$$\begin{aligned} U_{ad}^p = \{\gamma \in H^2(0,T;\Gamma_1);\ \exists\, c > 0 \text{ s.t } &|\ \gamma'(t)\ | \leq c \text{ a.e. } t \in [0,T] \\ &\text{and } 2R\int_0^T |\ \gamma'(t)\ |\ dt \leq |\ \Gamma_1\ | + 2\epsilon diam(\Gamma_1)\} \end{aligned} \tag{1.10}$$

will be convex. The choice of the control space $H^2(0,T;\mathbb{R}^2)$ is made for two main reasons: the first one is to guarantee the existence of an optimal control and the second one is to obtain smoother laser paths. From an industrial point of view, we believe that the use of C^1 curves will be more efficient than C^0 curves since smoother curves will diminish thermal gradients. Furthermore this kind of paths has been used in many additive manufacturing technologies [26, 63].

In subsection 2.3.1 we prove existence of at least one optimal control. In subsection 2.3.2 and 2.3.3 we derive the necessary optimality condition the optimal control satisfies, and we also determine explicitly $\nabla_{H^2}\hat{J}$ the gradient of the reduced cost functional in $H^2(0,T;\mathbb{R}^2)$.

Section 2.4 is devoted to the discrete penalized optimal control problem (OCP^θ). In subsection 2.4.1 we present the main optimization algorithms useful for our study. The two dimensional setup which we discretize is presented in subsection 2.4.2. In subsection 2.4.3 the parametrization of the laser path is introduced. We describe the path γ using cubic Hermite spline basis where the degree of freedom are both the value and the derivative at each value:

$$\begin{aligned} \gamma(t) =& \begin{pmatrix} \alpha_{k-1}^1 \\ \alpha_{k-1}^2 \end{pmatrix} H^0(t^{(k)}(t)) + \Delta_0 \begin{pmatrix} \beta_{k-1}^1 \\ \beta_{k-1}^2 \end{pmatrix} \hat{H}^0(t^{(k)}(t)) \\ &+ \begin{pmatrix} \alpha_k^1 \\ \alpha_k^2 \end{pmatrix} H^1(t^{(k)}(t)) + \Delta_0 \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} \hat{H}^1(t^{(k)}(t)),\ \text{for all } t \in [t_{k-1}, t_k],\ k = 1 \cdots, N. \end{aligned} \tag{1.11}$$

Here, $\Delta_0 = \frac{T}{N}$ is the step size of the uniform subdivision $t_0 = 0 < t_1 < \cdots < t_N = T$ of the time interval $[0,T]$. In subsection 2.4.4 we perform preliminary simulations to show how we coupled the Heat equation to the parametrized path. We discretize the two

dimensional heat equation by using the $\mathbb{P}^1$-finite element method in space and the implicit Euler method in time. In order to avoid the use of too many optimization parameters but to have a good approximation of the solution of the heat equation, we have decided to use a finer parametrization (t_{jk}), $1 \leq k \leq N$, $1 \leq j \leq n$, for the use of Euler's scheme. We consider the uniform subdivision $t_0 = 0 < t_1 < \cdots < t_{nN} = T$ of the time interval $[0, T]$ with step size $\Delta_1 = \frac{\Delta_0}{n}$ where n represents the number of points in which we want to simulate the temperature between the time interval $[t_{k-1}, t_k]$ for all $k = 1, \cdots, N$.

The discrete optimization problem is presented without constraint in subsection 2.4.5 to show how we handled the discretisation of the associated adjoint problem and the necessary optimality conditions in presence of the parametrized path. The adjoint system associated to the two dimensional heat equation is discretized using the $\mathbb{P}_1$-finite element method in space and the Euler implicit backward scheme in time. Similarly to the discrete state equation we have used a finer paramatrization (t_{jk}), $1 \leq k \leq N$, $1 \leq j \leq n$ for the Euler's scheme with step size Δ_1. $\nabla_{H^2}\hat{J}$ is the sum of a solution of an ODE and γ. In order to compute its discrete form we will solve the ODE using Hermite finite elements in one dimension. This finite element space is the same as the one used for the discrete curves which allow us to compute $\nabla_{H^2}\hat{J}$ by adding the solution of the ODE and γ in the same discrete functional space without passing by a transfer matrix.

In subsection 2.4.6 the fully constrained optimization problem is discretised: Firstly, by adding box constraints on the points $(\alpha_1, \cdots, \alpha_N)$ of γ to keep the path inside Ω. Secondly, by adding the geometric constraint (1.7) on the path γ.

Finally in section 2.5, we discuss another parametrization of the path γ using Bézier curves. In the end of this section we present ideas about how to control the power [14, 39] in the selective laser melting process.

For further information about laser path optimization in additive manufaturing with laser powder bed fusion and to consider a second numerical approach for path optimization, the reader is warmly invited to further look into the Allaire's and Boissier's work especially in [8] and [9].

1.2.2 Heat diffusion equations with volumic heat source induced by electromagnetic waves

In chapter 3 we are interested in heat diffusion equations with volumic heat source induced by electromagnetic waves. A typical heat-diffusion system involves a laser beam source irradiating locally a three dimensional medium Ω. This technique is used in laser melting process for additive metallic manufacturing [6, 62]. In [1], Maxwell's equations have also been considered to model accurately the interaction of the laser beam with biological tissues preferably to Beer's law or to the radiative transfer equation. To model accurately the laser interaction with the medium Ω, we consider in Ω the coupling between the heat diffusion equation and Maxwell's equations. In [59], time-harmonic electric and magnetic fields of some fixed frequency are considered, but the derived equations [59, Eq.(2.1)-(2.5)] lead to a contradiction as explained in [59, Remark 2.1].

Let us now describe our model. Let us fix some $T > 0$, we consider in the space-time cylinder $Q = \Omega \times]0, T[$, the 3-dimensional parabolic initial-boundary value problem:

$$\begin{cases} \partial_t y - \operatorname{div}(\alpha \nabla y) = S(y) & \text{in } Q, \\ -\alpha \frac{\partial y}{\partial n} = h(y - y_b) & \text{on } \Sigma, \\ y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases} \tag{1.12}$$

Here, y denotes the temperature, α the thermal diffusivity constant, n the outward unit normal vector field along the boundary Γ of Ω, $h > 0$ the heat transfer coefficient and y_b the temperature of the surrounding medium (air). By $\Sigma = \Gamma \times]0, T[$, we denote the lateral boundary of the space-time cylinder Q. The heat source function $S(y)$ in (1.12), represents the volumic power absorbed by the medium Ω from the electromagnetic field generated in Ω by an external source e.g. a laser beam. $S(y)(x,t)$ is defined as the multiplication of the absorption coefficient μ_a depending on $x \in \Omega$ and on the temperature $y(x,t)$, with the electric field intensity weighted around x:

$$S(y)(x,t) \coloneqq \mu_a(x, y(x,t)) \mid (\mathbf{E}(y) * \varphi_a)(x,t) \mid^2, \quad \text{for all } (x,t) \in Q. \tag{1.13}$$

$\varphi_a \in C_c^1(\mathbb{R}^3)$ represents a weight function which is supposed to be at least of class C^1 on $\mathbb{R}^3$ and with compact support. In formula (1.13), $\mathbf{E}$ represents the electric field in Ω solution of the following Maxwell's equations:

$$\begin{cases} \partial_t(\epsilon(\cdot, y)\mathbf{E}) - \operatorname{curl} \mathbf{H} + \sigma \mathbf{E} = 0 & \text{in } Q, \\ \partial_t(\mu \mathbf{H}) + \operatorname{curl} \mathbf{E} = 0 & \text{in } Q, \\ \operatorname{div}(\mu \mathbf{H}) = 0 & \text{in } Q, \\ \mathbf{E} \times n = \mathbf{E}_{ext} \times n & \text{on } \Sigma, \\ \mathbf{H} \cdot n = 0 & \text{on } \Sigma, \\ \mathbf{E}(\cdot, 0) = \mathbf{E}_0,\ \mathbf{H}(\cdot, 0) = \mathbf{H}_0 & \text{in } \Omega. \end{cases} \tag{1.14}$$

In these equations, $\mathbf{H}$ denotes the magnetic component of the electromagnetic field $(\mathbf{E}, \mathbf{H})$ in Ω, σ the electrical conductivity, μ the magnetic permeability, $\epsilon(x, y(x,t))$ the electric permittivity depending on the space and on the temperature y. $\mathbf{E}_{ext}$ denotes the electric field irradiating the boundary of Ω due to an external source. The first equation in the Maxwell system (1.14) is coupled to the heat diffusion initial boundary value problem (1.12) by the dependence of the permittivity ϵ with respect to the temperature y, and the heat diffusion initial boundary value problem (1.12) is coupled to the Maxwell system (1.14) by the right-hand side into the heat equation (1.12), the heat source term $S(y)$ (1.13) depending on the electric field $\mathbf{E}(y)$.

We begin by recalling the following standard functional spaces:

$$\mathbf{H}(\operatorname{curl}, \Omega) = \{\varphi \in \mathbf{L}^2(\Omega); \operatorname{curl} \varphi \in \mathbf{L}^2(\Omega)\},$$
$$\mathbf{H}(\operatorname{div}, \Omega) = \{\psi \in \mathbf{L}^2(\Omega); \operatorname{div} \psi \in \mathbf{L}^2(\Omega)\},$$

$$\mathbf{H}_0(\operatorname{curl}, \Omega) = \{\varphi \in \mathbf{L}^2(\Omega); \operatorname{curl} \varphi \in \mathbf{L}^2(\Omega), \varphi \times n = 0 \text{ on } \Gamma\},$$
$$\mathbf{H}_0(\operatorname{div}, \Omega) = \{\psi \in \mathbf{L}^2(\Omega); \operatorname{div} \psi \in L^2(\Omega), \psi \cdot n = 0 \text{ on } \Gamma\},$$
$$\mathbf{J}_n(\Omega, \mu) = \{\psi \in \mathbf{L}^2(\Omega); \operatorname{div}(\mu\psi) = 0, \psi \cdot n = 0 \text{ on } \Gamma\}.$$
$$\mathbf{J}_n^1(\Omega, \mu) = \mathbf{H}(\operatorname{curl}, \Omega) \cap \mathbf{J}_n(\Omega, \mu),$$
$$W_p^{2,1}(Q) = \{u \in L^p(Q) : \frac{\partial u}{\partial x_i} \in L^p(Q), \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(Q),\ i,j = 1,2,3$$
$$\text{and } \frac{\partial u}{\partial t} \in L^p(Q)\} \quad 1 < p < +\infty.$$

Our hypothesis on the coefficients appearing in (1.12), (1.13), and (1.14) are the following

- We assume that $y_0 \in C_b^4(\Omega)$. We assume $\alpha \in C_b^3(\Omega)$, $\alpha > 0$ on $\bar{\Omega}$, $h \in C^1(\partial\Omega)$, $h > 0$ on Γ and that $y_b \in C^2(\bar{\Sigma})$. We assume that the absorption coefficient $\mu_a(.,.) \in C_b^2(\Omega \times \mathbb{R})$ and $\varphi_a \in C_c^2(\mathbb{R}^3)$. Here, $C_b^k(\Omega)$ is the space of bounded continous functions with derivatives up to the order k bounded and continuous. $C_c^k(\Omega)$ is the space of compactly supported continuous functions with derivatives up to the order k compactly supported and continuous.

- The function $(x, y) \mapsto \epsilon(x, y)$ is real valued, positive, continuous on $\bar{\Omega} \times \mathbb{R}$ with first order partial derivatives with respect to the variables x_i $(i = 1, 2, 3)$ and y, also continuous on $\bar{\Omega} \times \mathbb{R}$. Also, there exist positive constants ϵ_1, ϵ_0 such that:

$$0 < \epsilon_0 \leq \epsilon(x, y) \leq \epsilon_1, \text{ for all } (x, y) \in \Omega \times \mathbb{R}.$$

- $\sigma \in L^\infty(\Omega)$ and the function $\mu \in W^{1,\infty}(\Omega)$. There are positive constants μ_0 and μ_1 such that:

$$0 < \mu_0 \leq \mu(x) \leq \mu_1, \text{ for all } x \in \Omega.$$

We suppose that Ω is a bounded domain of $\mathbb{R}^3$ with a boundary of class C^2. On Γ we have the boundary condition $\mathbf{E} \times n = \mathbf{E}_{ext} \times n$. Under the following hypothesis on $\mathbf{E}_{ext}$ and $(\mathbf{E}_0, \mathbf{H}_0)$

- $(\mathbf{E}_0, \mathbf{H}_0) \in \mathbf{H}(\mathrm{curl}\,, \Omega) \times \mathbf{J}_n^1(\Omega, \mu)$ and $\mathbf{E}_{ext} \in C^2([0, T]; \mathbf{H}^1(\mathbb{R}^3 \setminus \bar{\Omega}))$ such that $\mathrm{curl}\,\mathbf{E}_{ext} \cdot n_{|\Sigma} = 0$, $\mathbf{E}_{ext}(., 0) \times n_{|\Gamma} = \mathbf{E}_0 \times n_{|\Gamma}$,

we construct an extension $\mathbf{W}$ of $\mathbf{E}_{ext}$ to Ω. Introducing the new variable $\mathcal{E} := \mathbf{E} - \mathbf{W}$, problem (1.14) is reduced to the following Maxwell system with homogeneous boundary conditions of solutions $(\mathcal{E}, \mathbf{E})$. To alleviate the notations, $\mathcal{E}$ is denoted by $\mathbf{E}$:

$$\begin{cases} \partial_t \mathbf{E} - \hat{\epsilon}(\cdot, y)\,\mathrm{curl}\,\mathbf{H} + \hat{\epsilon}(\cdot, y)(\sigma + \partial_y \epsilon(\cdot, y)\partial_t y)\mathbf{E} = \mathbf{G}_1(t) & \text{in } Q, \\ \partial_t \mathbf{H} + \hat{\mu}\,\mathrm{curl}\,\mathbf{E} = \mathbf{G}_2(t) & \text{in } Q, \\ \mathbf{E} \times n = 0 & \text{on } \Sigma, \\ \mathbf{H} \cdot n = 0 & \text{on } \Sigma, \\ \mathrm{div}\,(\mu\mathbf{H}) = 0 & \text{in } Q, \\ \mathbf{E}(\cdot, 0) = \mathbf{E}_0,\ \mathbf{H}(\cdot, 0) = \mathbf{H}_0 & \text{in } \Omega, \end{cases} \tag{1.15}$$

with a right-hand side $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2)$ and $(\mathbf{E}_0, \mathbf{H}_0) \in \mathbf{H}_0(\mathrm{curl}\,, \Omega) \times \mathbf{J}_n^1(\Omega, \mu)$. Here

$$\hat{\epsilon}(x, z(x, t)) := \frac{1}{\epsilon(x, z(x, t))}, \quad \text{and} \quad \hat{\mu}(x) := \frac{1}{\mu(x)}, \text{ for all } x \in \Omega.$$

Moreover, supposing that the permittivity is independent of the temperature near the boundary of Ω, we may assume that the right-hand sides in the Maxwell system do not depend on the temperature.

This is for problem (1.15), coupled with the heat diffusion equation (1.12)-(1.13), that we want to prove the existence of a local solution. Firstly we consider in section 3.2 the Maxwell system (1.15), with a fixed distribution of temperature

$$z \in C^1([0, T]; C^1(\bar{\Omega})). \tag{1.16}$$

Consequently in (1.15), the permittivity depends not only on the space variable but also on the time variable, through its dependence with respect to the distribution of temperature z. In subsection 3.2.2, we apply the method of evolution systems in the hyperbolic case [44, Chapter 5, pp. 126-149] to establish the well posedness of our initial boundary value problem (1.15). Maxwell's system (1.15) (with z instead of y) can be written as an abstract Cauchy problem

$$\begin{cases} \frac{d}{dt}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}(t) = (\mathcal{A}_z(t)+\mathcal{M}_z(t))\begin{pmatrix}\mathbf{E}(t)\\ \mathbf{H}(t)\end{pmatrix} + \mathbf{G}(t) \\ \begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}(0) = \begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix} \end{cases} \tag{1.17}$$

where the operator $\mathcal{A}_z(t)$ and $\mathcal{M}_z(t)$ in the real Hilbert space

$$\mathcal{H} = \mathbf{L}^2(\Omega)\times \mathbf{J}_n(\Omega,\mu), \tag{1.18}$$

are defined as follow. For all $\phi=(\varphi,\psi)$ belonging to the domain independent of t

$$D(\mathcal{A}_z) \coloneqq \mathbf{H}_0(\operatorname{curl},\Omega)\times \mathbf{J}^1_n(\Omega,\mu), \tag{1.19}$$

we set

$$\mathcal{A}_z(t)\phi = \{\hat{\epsilon}(\cdot,z(\cdot,t))\operatorname{curl}\psi, -\hat{\mu}\operatorname{curl}\varphi\}\in\mathcal{H}, \tag{1.20}$$

$$\mathcal{M}_z(t)\phi = \{-\hat{\epsilon}(\cdot,z(\cdot,t))(\sigma+\partial_z\epsilon(\cdot,z(\cdot,t))\partial_t z)\varphi, 0\}, \tag{1.21}$$

with

$$\hat{\epsilon}(x,z(x,t)) := \frac{1}{\epsilon(x,z(x,t))}, \quad \text{and} \quad \hat{\mu}(x) := \frac{1}{\mu(x)}, \text{ for all } x\in\Omega. \tag{1.22}$$

In subsection 3.2.1, we prove that for each $t\in[0,T]$, $\mathcal{A}_z(t)$ generates a C_0-*group* of unitary operators in the Hilbert space $\mathcal{H}_t$: $\mathcal{H}$ endowed with the time dependent inner product with weights $\epsilon := \epsilon(\cdot,z(\cdot,t))$ and $\mu := \mu(\cdot)$ [40]

$$\left(\begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix},\begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix}\right)_{\mathcal{H}_t} = \int_\Omega \{\epsilon(x,z(x,t))\varphi_1(x)\cdot\varphi_2(x)+\mu(x)\psi_1(x)\cdot\psi_2(x)\}dx. \tag{1.23}$$

We then prove that the family of operators $(\mathcal{A}_z(t)+\mathcal{M}_z(t))_{t\in[0,T]}$ verifies the hypotheses from [44].

Consequently, by [44, Theorem 4.6, p.143] there exists a unique evolution system $U_z(t,s)$, $0\le s\le t\le T$ in $\mathcal{H}$. Also, for every initial condition $(\mathbf{E}_0;\mathbf{H}_0)\in Y := D(\mathcal{A}_z)$ and every right-hand side $\mathbf{G}\in C([0,T];Y)$, the initial value problem (1.17) possesses a unique Y -valued solution given by

$$\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}(t) = U_z(t,0)\begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix} + \int_0^t U_z(t,r)\mathbf{G}(r)dr. \tag{1.24}$$

By a Y-valued solution, we mean that

$$(\mathbf{E},\mathbf{H})\in C([0,T];Y)\cap C^1([0,T];\mathcal{H}),$$

verifies (1.17) such that $(\mathbf{E}_0, \mathbf{H}_0) \in Y$. Moreover in subsection 3.2.3, we show that for $\mathbf{G} \in C^1([0,T];\mathcal{H})$ this latest result remains true.

In section 3.3, we go back to the coupled problem (1.12)-(1.13)-(1.15). In subsection 3.3.1, we study the boundedness properties with respect to z of the evolution systems $(U_z(t,s))_{0\leq s\leq t\leq T}$ generated by the family of operators

$$(\mathcal{A}_z(t)+\mathcal{M}_z(t))_{0\leq t\leq T} \text{ for } z \in \bar{B}(0,R) \subset C^1([0,T];C^1(\bar{\Omega})),\ R>0.$$

We find consequently that

$$\| (\mathbf{E}_z, \mathbf{H}_z) \|_{C([0,T];Y)} \leq C(R) \tag{1.25}$$

and

$$\| \frac{d}{dt} (\mathbf{E}_z, \mathbf{H}_z) \|_{C([0,T];\mathcal{H})} \leq C(R). \tag{1.26}$$

From these bounds, we deduce that the family $\{(\mathbf{E}_z;\mathbf{H}_z)\}_{z\in B(0,R)}$ is bounded in $C([0,T];Y)$ and the family of their time derivatives in $C([0;T];\mathcal{H})$. Then, we study the continuity properties of the heat source term $S(z)$ in the heat equation (1.12), and of its time derivative $\frac{dS(z)}{dt}$ as a function of z from $C^1([0,T];C^1(\bar{\Omega}))$ into the space $L^p(0,T;C(\bar{\Omega}))$, for any $p \in]1,+\infty[$.

In subsection 3.3.2, our purpose is to prove that the initial nonlinear boundary value problem for the heat equation (1.12) admits at least one local solution. Firstly, we reduce our nonlinear initial non-homogeneous boundary value problem for the heat equation (1.12), to the homogeneous nonlinear initial boundary value problem with zero initial condition

$$\begin{cases} \partial_t \breve{y} - \operatorname{div}(\alpha \nabla \breve{y}) = S(\breve{y}+\omega) & \text{in } Q, \\ \alpha \frac{\partial \breve{y}}{\partial n} + h\breve{y} = 0 & \text{on } \Sigma, \\ \breve{y}(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{1.27}$$

with $\breve{y} := y - \omega$, where $\omega \in C^1([0,T];C^1(\bar{\Omega}))$ is solution of the auxilary linear initial boundary value problem for the heat equation (1.12)

$$\begin{cases} \partial_t \omega - \operatorname{div}(\alpha \nabla \omega) = 0 & \text{in } Q, \\ \alpha \frac{\partial \omega}{\partial n} + h\ \omega = h\ y_b & \text{on } \Sigma, \\ \omega(\cdot,0) = y_0 & \text{in } \Omega. \end{cases} \tag{1.28}$$

To prove the existence of a local solution to (1.27), we introduce the linear problem

$$\begin{cases} \partial_t \breve{y}_z - \operatorname{div}(\alpha \nabla \breve{y}_z) = S(z+\omega) & \text{in } Q, \\ \alpha \frac{\partial \breve{y}_z}{\partial n} + h\breve{y}_z = 0 & \text{on } \Sigma, \\ \breve{y}_z(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{1.29}$$

by fixing the distribution of temperature in the right-hand side to some arbitrary $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$ such that $z(0)=0$. (1.29) admits a solution in $W_p^{2,1}(Q)$ [32, Ch.IV-Sec.9, VII-Sec.10] ($p>5$) which is denoted by $\breve{y}_z$. For this last problem, we consider also its derivated problem with respect to time

$$\begin{cases} \partial_t v - \operatorname{div}(\alpha \nabla v) = \frac{d}{dt}(S(z+\omega)) & \text{in } Q, \\ \alpha \frac{\partial v}{\partial n} + hv = 0 & \text{on } \Sigma, \\ v(\cdot,0) = \mu_a(\cdot, y_0(\cdot)) \mid (\mathbf{E}_0 * \varphi_a)(\cdot) \mid^2 & \text{on } \Omega, \end{cases} \tag{1.30}$$

for which its solution is $\frac{d}{dt}\breve{y}_z$ [32]. From the continuous embedding of the Sobolev space [32, Lemma 3.3 p.80]

$$W_p^{2,1}(Q) \hookrightarrow C([0,T];C^1(\bar{\Omega})) \text{ for } p > 5,$$

it then follows that $\breve{y}_z \in C^1([0,T];C^1(\bar{\Omega}))$. But, the closed convex subset

$$K(0;R) := \bar{B}(0;R) \cap \{z \in \bar{B}(0;R); z(0) = 0\} \subset C^1([0,T];C^1(\bar{\Omega}))$$

is not stable by the mapping

$$z \mapsto \breve{y}_z,$$

so we can not apply to this mapping the Schauder fixed point theorem. In order to define an operator from $K(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$ into itself, we will need firstly to restrict $\breve{y}_z$ to some fixed subinterval $[0,t_f]$ $(t_f > 0)$ of $[0,T]$ of sufficiently small length and then to extend $\breve{y}_z|_{[0,t_f]}$ appropriately to the whole interval $[0,T]$ in order to obtain an element of $C^1([0,T];C^1(\bar{\Omega}))$ corresponding to z of norm less than or equal to R. This extension, denoted $\widetilde{\ }$, is built. For R sufficiently large we prove, that there exists $t_f \in]0,T/2]$ such that

$$\widetilde{(\breve{y}_z|_{[0,t_f]})} \in K(0;R) \text{ for all } z \in K(0;R) :$$

Using the boundedness and continuity properties established in subsection 3.3.1, we prove that the mapping which sends

$$z \in K(0;R) \mapsto \widetilde{(\breve{y}_z|_{[0,t_f]})} \in K(0;R)$$

is continuous and that its range is a relatively compact subset in $C^1([0,T];C^1(\bar{\Omega}))$, so it possesses a fixed point ζ by Schauder's point fix theorem. By the "causality principle", its restriction to the time interval $[0,t_f]$ is a solution to the homogeneous nonlinear initial boundary value problem (1.27) on the time interval $[0,t_f]$. Adding ω solution of (1.28) restricted to the interval $[0,t_f]$ to $\zeta|_{[0,t_f]}$, we obtain a solution to the nonlinear initial boundary value problem (1.12) on $[0,t_f]$.

Finally in subsection 3.3.3, we consider the case of a right-hand side in $C^1([0,T];\mathcal{H})$. We prove that the estimates (1.25) and (1.26) remain valid in this case. Also, all the reasonings of subsections 3.3.1 and 3.3.2 which follow from the estimates (1.25) and (1.26) remain valid. Consequently, the existence of a local weak solution to our coupled nonlinear initial boundary value problem (1.12) between the heat equation and the Maxwell system (1.17) is also valid when the right-hand side $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2)$ in the Maxwell system (1.17) belongs to $C^1([0,T];\mathcal{H})$.

In conclusion, the main result we have proved in chapter 3 is that for every

$$(y_0, \mathbf{E}_0, \mathbf{H}_0) \in C_b^4(\Omega) \times Y \text{ and } G \in C^1([0,T];\mathcal{H})$$

the initial boundary value problem (1.12)-(1.13)-(1.15) admits a local solution

$$(y, \mathbf{E}, \mathbf{H}) \in C^1([0,t_f];C^1(\bar{\Omega})) \times (C([0,t_f];Y) \cap C^1([0,t_f];\mathcal{H})).$$

Chapter 3 corresponds to the paper [3] in collaboration with Luc Paquet.

1.2.3 Controlling the electromagnetic external source

In chapter 4, we study a related optimal control problem to the Heat-Maxwell coupled system studied in chapter 3. To simplify matters, we suppose that the permittivity ϵ depends only on the space and not on the temperature. Consequently, the electric field $\mathbf{E}$ does not depend on the temperature y.

We suppose that the electric field $\mathbf{E}_{ext}$ of the exterior electromagnetic field $(\mathbf{E}_{ext}, \mathbf{H}_{ext})$ hitting Ω is in the complementary of Ω, $\mathbb{R}^3 \setminus \overline{\Omega}$, of the form

$$\mathbf{E}_{ext}(x,t) = \sum_{j=1}^{j=N} f_j(t)\mathbf{e}_{ext,j}(x,t),\ N \geq 1, \tag{1.31}$$

where

$$\mathbf{e}_{ext,j} \in \mathbf{C}^{1,1}([0,T]; \mathbf{H}(\mathrm{curl}\,, \mathbb{R}^3 \setminus \overline{\Omega})) \tag{1.32}$$

for $j = 1, \ldots, N$ are such that

$$(\mathrm{curl}\,\mathbf{e}_{ext,j}) \cdot \vec{n} = 0 \tag{1.33}$$

on the boundary Γ of Ω. Ω is supposed to be an open bounded subset of $\mathbb{R}^3$ with Lipschitz boundary. f_j for $j = 1, \ldots, N$ are given real-valued functions depending of the time variable t only. Our optimal control problem is the following:

$$\begin{aligned} \min J(y,v) := \frac{1}{2}\int_0^T \int_\Omega |\ \nabla y(x,t)\ |^2\ dx\ dt + \sum_{j=1}^{j=M} \frac{\lambda_{j,Q}}{2} \int_{T_{j,1}}^{T_{j,2}} \int_{\Omega_j} |\ y(x,t) \\ - y_{j,Q}(x,t)\ |^2\ dx\ dt + \frac{\lambda_\Omega}{2}\int_\Omega |\ y(x,T) - y_d(x)\ |^2\ dx + \frac{\lambda}{2}\sum_{k=1}^{k=N}\int_0^T |v_k(t)|^2 dt, \end{aligned} \tag{1.34}$$

with $\Omega_j \subset \Omega$, $T_{j,1} \leq T_{j,2}$, $[T_{j,1}, T_{j,2}] \subset [0,T]$, $\lambda_{j,Q} \geq 0$ $(j = 1, \ldots, M)$, $\lambda_\Omega \geq 0, \lambda > 0$, subject the following Heat-Maxwell system:

$$\begin{cases} \partial_t y - \mathrm{div}\,(\alpha\,\nabla y) = S(y) & \text{in } Q, \\ -\alpha\frac{\partial y}{\partial n} = h(y - y_b) & \text{on } \Sigma, \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases} \tag{1.35}$$

where the heat source $S(y)$ is defined by (1.13) and $\mathbf{E}$ the electric field in Ω is solution of the Maxwell system:

$$\begin{cases} \partial_t(\epsilon\mathbf{E}) - \mathrm{curl}\,\mathbf{H} + \sigma\mathbf{E} = 0 & \text{in } Q, \\ \partial_t\mathbf{H} + \hat{\mu}\ \mathrm{curl}\,\mathbf{E} = 0 & \text{in } Q, \\ \mathbf{E} \times n = \mathbf{E}_{ext} \times n & \text{on } \Sigma, \\ \mathbf{H} \cdot n = 0 & \text{on } \Sigma, \\ \mathbf{E}(\cdot,0) = \mathbf{E}_0,\ \mathbf{H}(\cdot,0) = \mathbf{H}_0 & \text{in } \Omega, \end{cases} \tag{1.36}$$

where $\alpha > 0$ belongs to $C(\bar{\Omega})$, $h > 0$ belongs to $C(\Gamma)$, $\epsilon \geq \epsilon_0 > 0$ belongs to $L^\infty(\Omega)$, $\hat{\epsilon} := \frac{1}{\epsilon} \in L^\infty(\Omega)$, $\sigma \geq 0$ belongs to $L^\infty(\Omega)$, $\mu \geq \mu_0 > 0$ belongs to $L^\infty(\Omega)$ and $\hat{\mu} := \frac{1}{\mu} \in L^\infty(\Omega)$ are only functions of x. We assume that the absorption coefficient $\mu_a \in C_b^1(\bar{\Omega} \times \mathbb{R})$ is monotone decreasing with respect to y. The initial condition $y_0 \in C(\bar{\Omega})$ and $y_b \in C(\overline{\Sigma})$. $\mathbf{E}_{ext}$ is given by (1.31), such that

$$\begin{cases} -f_j'' + f_j = v_j \quad in \quad]0,T[, \\ f_j(0) = f_{j,0}\,, \\ f_j'(0) = f_{j,1}\,, \end{cases} \tag{1.37}$$

where $v = (v_j)_{j=1}^N \subset \mathcal{U}$ are the controls belonging to the control space $\mathcal{U} := [L^2(0,T)]^N$. The set of admissible controls is the closed convex subset $\mathcal{U}_{ad} := \prod_{j=1}^N U_{ad,j} \subset \mathcal{U}$, where for $j = 1, \ldots, N$:

$$U_{ad,j} = \{v_j \in L^2(0,T); v_{j,a} \leq v_j(t) \leq v_{j,b} \text{ a.e. } t \in [0,T]\}. \tag{1.38}$$

The numbers $f_{j,0} \in \mathbb{R}$ and $f_{j,1} \in \mathbb{R}$ will remain fixed. $y_{j,Q}$ in the cost functional (1.34) is a given template temperature distribution in $L^2(Q)$.

Here, we have considered a more general cost functional than previously in chapter 2 by replacing the term

$$\frac{\lambda_Q}{2} \int_0^T \int_\Omega \mid y(x,t) - y_Q(x,t) \mid^2 dx\, dt$$

by the more flexible expression

$$\sum_{j=1}^M \frac{\lambda_{j,Q}}{2} \int_{T_{j,1}}^{T_{j,2}} \int_{\Omega_j} \mid y(x,t) - y_{j,Q}(x,t) \mid^2 dx\, dt,$$

the idea being that it is not clear how to choose adequaltly the function y_Q. For the functions $y_{j,Q}$ defined on $\Omega_j \times [T_{j,1}, T_{j,2}]$, we could choose for example $y_{j,Q}$ equals to a fixed temperature a little greater than the temperature of fusion of the powder e.g. 1600 ^0C [54] as we want the powder to have fused everywhere during a certain subinterval of the time of treatment. It seems also natural to require that $\bigcup_{j=1}^N \Omega_j = \Omega$ and the different pieces Ω_j to overlap near their boundaries to glue perfectly together.

Our aim in section 4.2 is to prove the well posedness of the heat-Maxwell system. To do this, the steps of the proof are shown in what follow. Under hypothesis (1.32) and (1.33) on $\mathbf{e}_{ext}$ and

$$\mathbf{E}_0 \times n = \sum_{j=1}^{j=N} f_{j,0}\ \mathbf{e}_{ext,j}(0) \times n \quad \text{ on } \Gamma, \tag{1.39}$$

we reduce the Maxwell problem (1.36) to an intial homogeneous boundary value problem by extending $\mathbf{e}_{ext,j}$ to Ω in a vector field

$$\mathbf{e}_j \in \mathbf{C}^{1,1}([0,T]; \mathbf{H}(\text{curl}\,, \Omega))$$

such that

$$\mathbf{e}_j \times \vec{n} = \mathbf{e}_{ext,j} \times \vec{n} \text{ and } (\text{curl}\, \mathbf{e}_j) \cdot \vec{n} = 0 \text{ on } \Gamma.$$

The new couple of vector fields $(\mathcal{E}, \mathbf{H})$ is solution of the following intial homogeneous boundary value problem

$$\begin{cases} \partial_t \mathcal{E} - \hat{\epsilon}\ \mathrm{curl}\, \mathbf{H} + \hat{\epsilon}\sigma\ \mathcal{E} = -\sum_{j=1}^{j=N} (f_j' \mathbf{e}_j + f_j \mathbf{e}_j' + \hat{\epsilon}\sigma f_j \mathbf{e}_j) & \text{in } Q, \\ \partial_t \mathbf{H} + \hat{\mu}\ \mathrm{curl}\, \mathcal{E} = -\hat{\mu} \sum_{j=1}^{j=N} f_j\ \mathrm{curl}\, \mathbf{e}_j & \text{in } Q, \\ \mathcal{E} \times n = 0 & \text{on } \Sigma := \Gamma \times]0,T[, \\ \mathbf{H} \cdot n = 0 & \text{on } \Sigma, \\ \mathcal{E}(0) = \mathbf{E}_0 - \sum_{j=1}^{j=N} f_{j,0} \mathbf{e}_j(0),\ \mathbf{H}(0) = \mathbf{H}_0 & \text{in } \Omega. \end{cases} \tag{1.40}$$

The new vector field $\mathcal{E}$ is given by

$$\mathcal{E}(x,t) = \mathbf{E}(x,t) - \sum_{j=1}^{j=N} f_j(t) \mathbf{e}_j(x,t), \text{ for all } (x,t) \in \Omega \times [0,T].$$

We also suppose that $\mathbf{H}_0 \in \mathbf{J}_n^1(\Omega, \mu)$.

Consequently, the initial condition $(\mathcal{E}_0, \mathbf{H}_0)$ to our initial boundary value problem (1.40) belongs to the domain $D(\mathcal{A})$ (given by (1.19)) of the infinitesimal generator $A = \mathcal{A} + \mathcal{M}$ in the Hilbert space $\mathcal{H}$ (given by (1.18)). The operators $\mathcal{A}$, $\mathcal{M}$ are given respectively by (1.20) and (1.21), with one difference that $\hat{\epsilon}$ depends only on the space. We prove that the right hand side of (1.40)

$$\begin{aligned} \mathbf{G} : t \in [0,T] \mapsto \mathbf{G}(t) := (-\sum_{j=1}^{j=N} (f_j'(t) \mathbf{e}_j(t) + f_j(t) \mathbf{e}_j'(t) + \hat{\epsilon}\sigma f_j(t) \mathbf{e}_j(t)), \\ - \hat{\mu} \sum_{j=1}^{j=N} f_j(t)\ \mathrm{curl}\, \mathbf{e}_j(t)) \in \mathcal{H} \end{aligned} \tag{1.41}$$

is a Lipschitz continuous function.

Using the theory of semi-groups [44, Corollary 2.11 p.109], we prove that the initial boundary value problem for the Maxwell's equations (1.36) possesses one and only one strong solution $(\mathbf{E}, \mathbf{H})$ for any given control $v = (v_j)_{j=1}^N \in \mathcal{U}_{ad}$, and any given initial condition $(\mathbf{E}_0, \mathbf{H}_0) \in \mathbf{H}(\mathrm{curl}\,, \Omega) \times \mathbf{J}_n^1(\Omega, \mu)$ verifying (1.39), $\mathbf{E}_{ext}$ being given by (1.31) and (1.37) i.e. possesses one and only one function

$$(\mathbf{E}, \mathbf{H}) \in W^{1,1}(0,T; \mathcal{H}) := \{\mathbf{U} \in L^1(0,T; \mathcal{H}) \text{ such that } \frac{d\mathbf{U}}{dt} \in L^1(0,T; \mathcal{H})\}$$

verifying $(\mathbf{E}, \mathbf{H})(0) = (\mathbf{E}_0, \mathbf{H}_0)$ and equations (1.36) for a.e. $t \in [0,T]$. This solution is given by

$$(\mathbf{E}, \mathbf{H})(t) = (\mathcal{E}(t) + \sum_{j=1}^{j=N} f_j(t) \mathbf{e}_j(t), \mathbf{H}(t)) \text{ for all } t \in [0,T],$$

where $(\mathcal{E}(t), \mathbf{H}(t))$ is given by the equation:

$$(\mathcal{E}(t), \mathbf{H}(t)) = T_t\, (\mathcal{E}_0, \mathbf{H}_0) + \int_0^t T_{t-s} \mathbf{G}(s) ds, \tag{1.42}$$

with $(T_t)_{t\geq 0}$ the semigroup generated by the operator $A = \mathcal{A} + \mathcal{M}$ in $\mathcal{H}$, $\mathbf{G}$ given by (1.41), a Lipschitz continuous function, and

$$\mathcal{E}_0 := \mathbf{E}_0 - \sum_{j=1}^{j=N} f_{j,0}\mathbf{e}_j(0) \in \mathbf{H}_0(\mathrm{curl}\,,\Omega). \tag{1.43}$$

We prove that the semilinear parabolic initial boundary value problem (1.35) with electromagnetic heat source (1.13), possesses a unique weak solution $y \in W(0,T)\cap C(\bar{Q})$ using [53, Theorem 5.5 p.268] about existence and uniqueness of the weak solution of general semilinear parabolic initial-boundary value problems of the form [53, (5.1) p.265]:

$$\begin{cases} \partial_t y - \mathrm{div}\,(\alpha\,\nabla y) + d(x,t,y) = 0 & \text{in } Q := \Omega\times]0,T[, \\ \alpha\frac{\partial y}{\partial \nu} + b(x,t,y) = g & \text{on } \Sigma := \Gamma\times]0,T[, \\ y(\cdot,0) = y_0 & \text{in } \Omega. \end{cases} \tag{1.44}$$

Thus, we first check that the nonlinear term

$$d(x,t,y) := -S(x,t,y) = -\mu_a(x,y)\ |(\mathbf{E}*\varphi_a)(x,t)|^2,$$

is monotone increasing with respect to y and locally Lipschitz continuous with respect to y for almost every $(x,t)\in Q$. Furthermore, d satisfies for a positive constant K the boundedness condition

$$|\ d(x,t,0)\ | \leq K \quad (x,t)\in\Sigma. \tag{1.45}$$

Then we prove that the boundary term

$$b(x,t,y) := h(x)(y - y_b(x,t)),$$

satisfies the same properties as d. Consequently, the semi-linear parabolic initial boundary value problem (1.35), (1.13) has a unique weak solution $y \in W(0,T)\cap C(\bar{Q})$ for any $\mathbf{E}\in C([0,T];L^2(\Omega)^3)$, any $y_b \in C(\bar{\Sigma})$ and any initial condition $y_0 \in C(\bar{\Omega})$. Moreover for any $r > 2.5$ and any $s > 4$, there exists a constant $C(r,s)$ such that

$$\begin{aligned} \|y\|_{W(0,T)} + \|y\|_{C(\bar{Q})} \leq & C(r,s)(\big\|\mu_a(\cdot,0)|\mathbf{E}*\varphi_a|^2(\cdot,\cdot)\big\|_{L^r(Q)} \\ & + \|h(\cdot)y_b(\cdot,\cdot)\|_{L^s(\Sigma)} + \|y_0\|_{C(\bar{\Omega})}). \end{aligned} \tag{1.46}$$

Let us point out that the temperature y depends on the electric field $\mathbf{E}$, which depends on $\mathbf{E}_{ext}$, thus on $f = (f_j)_{j=1}^N \in L^2(0,T)^N$ which itself depends on the control $v = (v_j)_{j=1}^N \in \mathcal{U}_{ad}$.

In section 4.3 we prove existence of an optimal control $v = (v_j)_{j=1}^N \in \mathcal{U}_{ad}$. We indicate this dependence of the temperature y with respect to the control v, by writing y_v. Let us introduce the reduced cost functional:

$$\hat{J} : \mathcal{U}_{ad} \to \mathbb{R} : v \mapsto J(y_v, v).$$

The idea of the proof is to show that $\hat{J}$ is weakly lower semi-continuous. More precisely, let us consider a minimizing sequence $(v^{(k)})_{k\geq 0}$ in $\mathcal{U}_{ad}$, such that

$$\hat{J}(v^{(k)}) \to \inf_{v\in U_{ad}} \hat{J}(v),$$

we can prove that the sequence $(\mathbf{E}_{v^{(k)}})_{k\geq 0}$ is bounded in $C([0,T];\mathbf{L}^2(\Omega))$, so that $(\mathbf{E}_{v^{(k)}} * \varphi_a)_{k\geq 0}$ is bounded in $C([0,T];\mathbf{C}(\bar{\Omega})) = \mathbf{C}(\bar{Q})$. Therefore this latest result with estimate (1.46) show us, that $(y_{v^{(k)}})_{k\geq 0}$ is bounded in $W(0,T)$. Modulo extracting a new subsequence, we prove that $(y_{v^{(k)}})_{k\geq 0}$ converges weakly in $W(0,T)$. By the compact embedding from $H^1(\Omega)$ into $L^p(\Omega)$ for $1 < p < 6$ and the Lions-Aubin compactness lemma [47, p.106], $(y_{v^{(k)}})_{k\geq 0}$ is strongly convergent in $L^2(Q)$. In conclusion we have existence of an optimal control, $\bar{v}$ such that

$$\hat{J}(\bar{v}) = \inf_{w\in\mathcal{U}_{ad}} \hat{J}(w).$$

In order to determine a first order necessary optimality condition for our problem, we prove in section 4.4 that the reduced cost functional $\hat{J}$ is Fréchet differentiable. The proof reduces essentially to show that the control-to-state mapping is Fréchet differentiable. We find that the control-to-state mapping

$$\mathcal{S} : L^2(0,T)^N \to W(0,T)\cap C(\bar{Q}) \tag{1.47}$$

which sends $v = (v_j)_{j=1}^N \in L^2(0,T)^N$ onto y weak solution of the semilinear initial parabolic boundary value problem (1.35)-(1.13), is Fréchet differentiable at any point $v = (v_j)_{j=1}^N \in L^2(0,T)^N$. Its Fréchet derivative at the point $v = (v_j)_{j=1}^N \in L^2(0,T)^N$ is the linear continuous mapping

$$D\mathcal{S}(v) : \delta v = (\delta v_j)_{j=1}^N \in L^2(0,T)^N \mapsto \delta y \in W(0,T)\cap C(\bar{Q}) \tag{1.48}$$

where δy is a solution of the linear initial boundary value problem

$$\begin{cases} \frac{\partial \delta y}{\partial t} - \operatorname{div}(\alpha\nabla\delta y) - \frac{\partial \mu_a}{\partial y}(\cdot,y)|\mathbf{E}*\varphi_a|^2\delta y & \\ = 2\mu_a(\cdot,y)\,(\mathbf{E}*\varphi_a)\cdot(\delta\mathbf{E}*\varphi_a) & \text{in} \quad Q, \\ \alpha\frac{\partial\delta y}{\partial n} + h\delta y = 0 & \text{on} \quad \Sigma, \\ \delta y(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.49}$$

with $(\delta\mathbf{E},\delta\mathbf{H}) \in W^{1,1}(0,T;\mathcal{H})$ strong solution of (1.36)-(1.37). The Fréchet derivative of $\hat{J}$ is

$$\begin{aligned} D\hat{J}(v)\delta v = &\int_0^T\int_\Omega \nabla y_v(x,t)\cdot\nabla(D\mathcal{S}(v)\delta v)(x,t)\;dxdt \\ +&\sum_{j=1}^M \lambda_{j,Q}\int_{T_{j,1}}^{T_{j,2}}\int_{\Omega_j}(y_v(x,t)-y_{j,Q}(x,t))(D\mathcal{S}(v)\delta v)(x,t)\;dxdt \\ +\lambda_\Omega&\int_\Omega (y_v(x,T)-y_d(x))(D\mathcal{S}(v)\delta v)(x,T)\;dx + \lambda\sum_{k=1}^N\int_0^T v_k(t)\delta v_k(t)dt. \end{aligned} \tag{1.50}$$

The adjoint system for our problem presented in section 4.5 is the following

$$\begin{cases} \frac{\partial p}{\partial t} + \operatorname{div}(\alpha\nabla p) + \frac{\partial\mu_a}{\partial y_v}(x,y)|E*\varphi_a|^2 p = & \\ \Delta y_v - \sum_{j=1}^M \lambda_{j,Q}(y_v - y_{j,Q})1_{\Omega_j\times]T_{j,1},T_{j,2}[} & \text{in} \quad Q, \\ \alpha\frac{\partial p}{\partial n} + hp = \frac{\partial y_v}{\partial n} & \text{on} \quad \Sigma, \\ p(\cdot,T) = \lambda_\Omega(y_v(.,T) - y_d) & \text{on} \quad \Omega, \end{cases} \tag{1.51}$$

By a weak solution $p \in W(0,T)$ of the backward parabolic boundary value problem (1.51), we mean that for every $\varphi \in L^2(0,T;H^1(\Omega))$:

$$\begin{aligned}
&\int_0^T \int_\Omega \tfrac{\partial p}{\partial t}(x,t)\varphi(x,t)dxdt - \int_0^T \int_\Omega \alpha(x)\nabla p(x,t)\nabla\varphi(x,t)dxdt \\
&\quad + \int_0^T \int_\Omega \tfrac{\partial \mu_a}{\partial y_v}(x,y(x,t))|\mathbf{E}*\varphi_a|^2(x,t)p(x,t)\varphi(x,t)dxdt \\
&= -\int_0^T \int_\Omega \nabla y(x,t)\nabla\varphi(x,t)dxdt - \sum_{j=1}^M \lambda_{j,Q}\int_{T_{j,1}}^{T_{j,2}}\int_{\Omega_j}(y_v - y_{j,Q})(x,t)\varphi(x,t)dxdt \\
&\quad + \int_0^T \int_\Gamma h(x)p(x,t)\varphi(x,t)dS(x)dt,
\end{aligned} \tag{1.52}$$

and

$$p(x,T) = \lambda_\Omega(y_v(x,T) - y_d(x)), \text{ for a.e. } x \in \Omega. \tag{1.53}$$

By formulas (1.48), (1.50) and (1.52), we can determine the first order necessary condition. If $\bar{v} \in \mathcal{U}_{ad}$ is an optimal control, then

$$\begin{aligned}
&2\int_0^T \int_\Omega \mu_a(x,\bar{y}_v(x,t))\bar{p}(x,t)(\bar{E}*\varphi_a)(x,t)\cdot(\delta E*\varphi_a)(x,t)\ dxdt \\
&\quad + \lambda\sum_{k=1}^N \int_0^T \bar{v}_k(t)\delta v_k(t)dt \geq 0,
\end{aligned} \tag{1.54}$$

for all $\delta v = v - \bar{v}$, $v \in \mathcal{U}_{ad}$, δE being deduced from δv by solving (1.36)-(1.37).

Chapter 2

Laser path optimization in the selective laser melting process (SLM) using optimal control theory

2.1 Introduction

Selective laser melting (SLM) is an Additive Layer Manufacturing process used to produce three-dimensional objects from metal powders by melting the material in a layer-by-layer manner. First, a thin layer of powder is spread onto a build platform and simultaneously levelled or compacted to the required thickness. The laser beam scans the powder surface at an appropriate speed, heating the surface according to the desired scanning pattern and part profile. The mechanisms of SLM have been discussed in [29, 52, 48].

Thermal distortion of the fabricated part is one serious problem in SLM process [16], because of its fast laser scan rates and material transformations (solidification and liquifation) in a very short time frame. The temperature field was found to be inhomogeneous by many previous researchers [57, 61]. Meanwhile, the temperature evolution history in SLM process has significant effects on the quality of the final parts, such as density, dimensions, mechanical properties, microstructure, etc. For metals, rapid repeated heating and cooling cycles of the powder during SLM build process is responsible for large temperature gradients resulting in hight residual stresses and deformations, and may even lead to crack formation in the fabricated part.

Our aim in this chapter is to propose a laser path optimization model minimizing thermal gradients in SLM. First we will perform a litterature survey on the effect of scanning strategy (laser trajectory) on temperature gradients and residual stresses (since large temperature gradient generates residual stresses and cracks) in SLM. In subsection 2.1.1, we will discuss temperature distribution in SLM and we will present the thermal model coupled with the laser trajectory which we adopted in our study. In subsection 2.1.2, we will mention some works whose authors have noticed and tested the effect of laser scanning strategy in SLM, even with random trajectories. In section 2.4.4, we will present preliminary results on the effect of parametric curves on the distribution of temperature and thermal gradient in SLM. In section 2.2 we will study mathematically the laser path optimization model using optimal control theory. In section 2.3, we will present a penalized optimization model related to the problem presented in section 2.2 but with different interpretation of the laser path constraints. Section 2.4 will be devoted

to present the discretization of our optimization model.

2.1.1 Temperature distribution

Since temperature distribution in laser sintering is important, many researchers have put their efforts toward understanding the SLM process and formulating models to describe SLM thermal evolution. The laser scans on the top of the powder bed following a prescribed scan pattern. The heat transfer process consists of powder bed radiation, convection between the powder bed and environment, and heat conduction inside the powder bed and between the powder bed and substrate. The latent heat of fusion is large in SLM. The complexity brought about by the powder phase change and the corresponding variation of the thermal properties during SLM also complicates the heat transfer problem. In order to better reflect the SLM process, a lot of research on key process variables such as laser beam characteristics and powder thermal properties has been conducted. The simplest laser beam has been assumed to be a point source which is not in conjunction with reality. It has been found that the laser beam can be characterized using three parameters namely diameter, power, and intensity distribution. The most widely adopted model in literature is the Gaussian laser beam model [17]:

$$q(r) := \frac{2P}{\pi R^2} e^{\frac{-2r^2}{R^2}} \tag{2.1}$$

where P is the laser power, R the spot radius and r the radial distance.

The thermal model that describes the propagation of heat through a single layer of powder irradiated by the laser is the following one [31, 18, 54, 50]. Let $\Omega \subset \mathbb{R}^3$ be a layer of thickness δ and boundary $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3}$. The thermal balance in Ω is described by the following heat equation (see figure 2.1):

$$\begin{cases} \rho(y)\, c(y)\, \partial_t y - div(\kappa(y)\, \nabla y) = 0 & \text{in} \quad Q = \Omega \times]0, T[\,, \\ -\kappa(y) \frac{\partial y}{\partial \nu} = h(y)\,(y - y_{ex}) + \varepsilon(y)\, \sigma_{SB}\,(y^4 - y_{ex}^4) - g_\gamma(y) & \text{on} \quad \Sigma_1 = \Gamma_1 \times]0, T[\,, \\ -\kappa(y) \frac{\partial y}{\partial \nu} = h(y)\,(y - y_{ex}) & \text{on} \quad \Sigma_2 = \Gamma_2 \times]0, T[\,, \\ y = y_{ex} & \text{on} \quad \Sigma_3 = \Gamma_3 \times]0, T[\,, \\ y(x, 0) = y_0(x) & \text{for} \quad x \in \Omega. \end{cases} \tag{2.2}$$

More precisely, Ω is a Lipschitz bounded open set of $\mathbb{R}^3$, with boundary Γ devided in three parts $\overline{\Gamma_1}$, $\overline{\Gamma_2}$ and $\overline{\Gamma_3}$.

- $y := y(x, t)$ the temperature,
- y_0 the initial temperature,
- y_{ex} the ambient temperature,
- Γ_1 the part of the boundary irradiated by the laser,
- Γ_2 the part of the boundary on which the convection is only applied,
- Γ_3 the part of the boundary in contact with the built platform or with the previous layer, on which a constant temperature y_{ex} is applied
- ρ the density,

- c the calorific capacity,
- κ the thermal conductivity,
- $g_\gamma(y) := \alpha(y)\, q_\gamma$ the Gaussian laser flux model where α is the absorbance of the material and

$$q_\gamma(x,t) = \frac{2P}{\pi R^2} \exp\left(-2\frac{|\, x-\gamma(t)\,|^2}{R^2}\right), \text{ for all } (x,t) \in \Sigma_1$$

  P is the laser power, R is the radius of the laser spot, $x = (x_1, x_2) \in \Gamma_1$ the surface irradiated by the laser and $\gamma : t \in [0,T] \to \Gamma_1$ the laser path which represents the displacement of the laser beam center with respect to time,
- h the heat exchange coefficient,
- σ_{SB} the Stefan Bolzman constant ($5.67036 \times 10^{-8}\, W \cdot m^{-2} \cdot K^{-4}$),
- ε the material emissivity,
- ν the outgoing normal of Γ,
- $h(y)\,(y - y_{ex})$ the surface convection heat loss described using Newton's law of cooling,
- $\varepsilon(y)\,\sigma_{SB}\,(y^4 - y_{ex}^4)$ the power radiation described using Stefan-Boltzmann law.

Remark 1. From the heating of metal powders above their melting temperature, and subsequent cooling, result big variations of material thermal properties with respect to temperature, so that we consider that ρ, c, κ, h and ε depend on temperature.

Remark 2. Note that in (2.2) the boundary condition on Σ_3 can be replaced by a Robin type boundary condition

$$-\kappa(y)\,\frac{\partial y}{\partial \nu} = h(y)\,(y - y_{ex}) \quad \text{on} \quad \Sigma_3.$$

Remark 3. A two-dimensional version of (2.2) with volumic Gaussian laser source also exists, see [9, 38, 43, 51, 55], and can be treated with similar arguments used here.

In appendix A we have studied the existence of solutions to the non linear thermal model (2.2).

2.1.2 Effect of scanning strategies in SLM

Our main interest is to prove that laser scanning strategies influence temperature distribution. Indeed highly localized heat inputs may result in large temperature gradient and thus lead to deformations during SLM process. We will briefly discuss some results in literature describing the relation between scan strategies and thermal distribution in SLM.

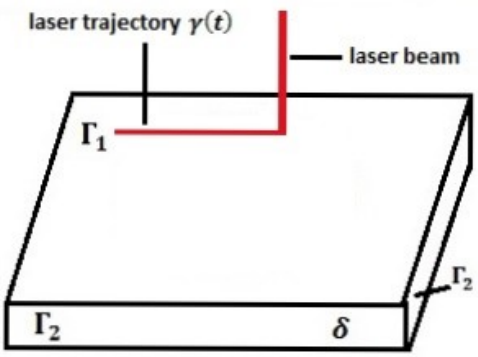


Figure 2.1: One layer Ω of thickness δ with boundary $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3}$.

In [54], six laser paths were compared: a continuous round-trip trajectory of the laser (denoted by AR), a spaced round-trip trajectory (referred to as ARE), a spiral trajectory that runs from the corner of the piece towards the inside of this one (noted SPI) and another going from the center of the piece towards the outside (noted SPE), a trajectory in zigzag (noted ZZ) and a last point-by-point (noted PP) where one merges the material by spaced points on the surface of the powder (figure 2.2 shows the different laser paths studied in [54]). The simulation results show that thermal gradients are very disparate from one type of trajectory to another, as well as the maximum temperatures reached. The point-by-point manufacturing strategy has lower temperature levels than in the other cases. These temperatures depend strongly on the position of the laser, they will be all more important as the laser heats areas closer and closer. For example, the temperature reached at the point of impact of the laser for the zigzag trajectory is around 1815°C. For the point-by-point trajectory, it reaches 1500°C. For the point by point path, temperature gradients and the maximum amplitude of the temperatures reached are the lowest compared to the other cases. The simulation results in [54] of the cumulative plastic deformation fields in the parts at the end of the process for the 6 laser trajectories shows that the point-by-point scanning of the laser generates the least amount of plastic deformation. On the contrary, the spiral-shaped trajectories, inwards (SPI), is the one that leads to the highest plastic deformation.

Also Nickel [41] and Klingbeil [30], studied the metal deposition process for three laser scanning strategies including a round trip trajectory (AR), a spiral inward (SPI) and another outward (SPE). Their results show that spiral trajectories are the ones that induce the most deformation. From their results we can deduce that the paths which contributes to large temperature gradient also lead to hight residual stresses and deformations.

Cheng et al. [12] also investigate the influence of scanning strategies on temperature gradient. They studied six scanning strategies with distinct difference between each other. They have been summarized in figure 2.3; Case (a) is a random island scanning, the whole scanning domain has been divided into 9 islands and with a scanning direction rotation process on the subsequent layer, the completion sequence of the single island is randomly selected. A prime degree rotation is used to avoid repeated scan vector in just a few layers, e.g., the scanning vector will be exactly the same after 4 layers for 90° rotation angle.

The temperature distribution contours for laser scanning on the third layer have been

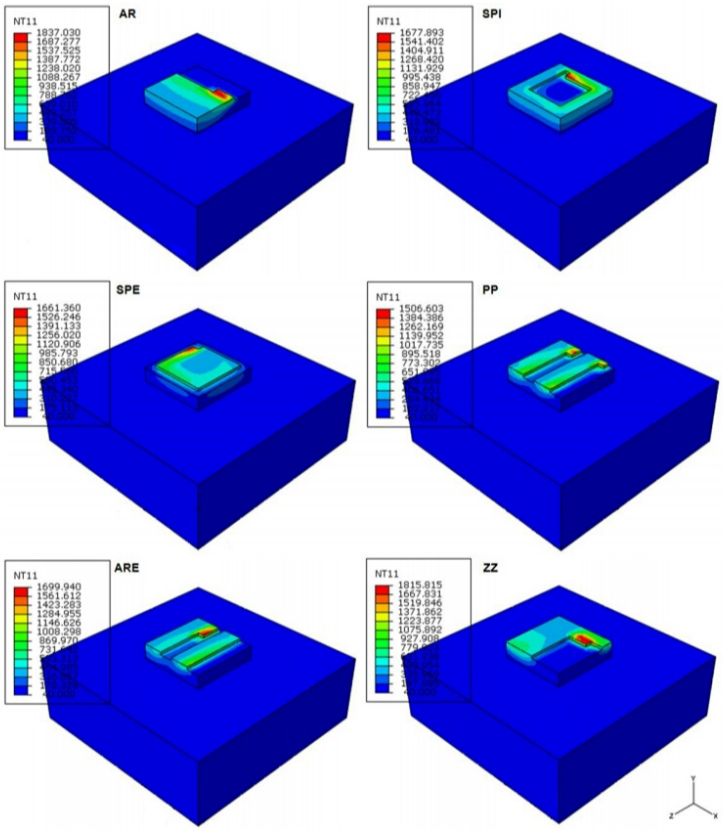


Figure 2.2: Cartographies of temperature fields for different laser paths at time t studied in [54].

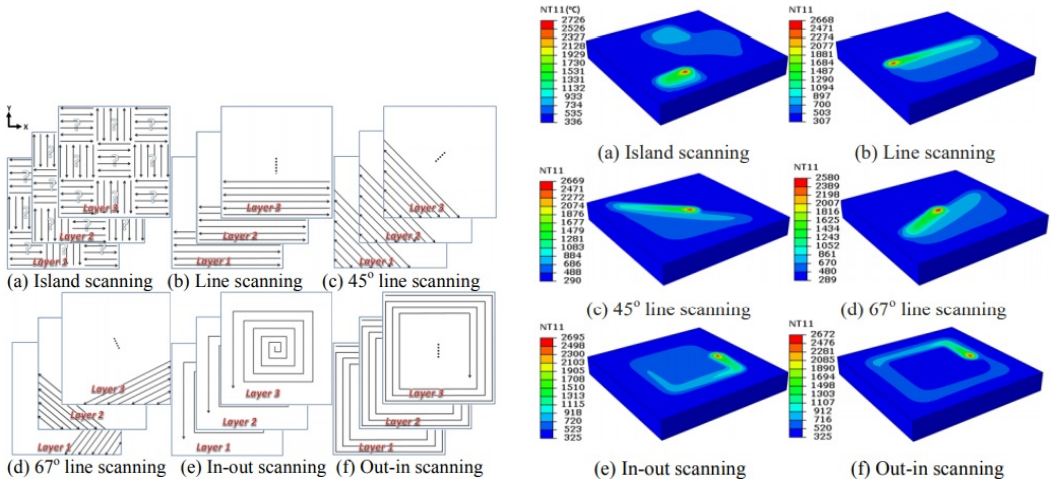


Figure 2.3: The different laser paths studied in [12].

collected in [12] at the time point which is half of the total time needed for one layer scanning, and are shown in figure 2.3. The minimum temperature of part substrate did not maintain at room temperature of 20°C. It reached to around 300°C when laser beam traveled on the third layer for all cases because of small part size and high beam power.

It is noticed that the three line scanning cases have comparatively lower minimum substrate temperature while the island scanning case has the highest minimum substrate temperature. Since the island scanning case needs comparatively more time to complete a layer, more heat is introduced to the substrate. The maximum temperatures of different scanning strategies are also different from each other at this time point. Generally speaking, the island scanning case has the highest maximum temperature which is caused by residual heat effect from short scanning path in a single island. In addition, steady state maximum temperature is very hard to reach in these cases due to frequently change of scanning direction, path angle and length. Concerning residual stresses and deformations Cheng et al. [12] have found that the out-in scanning case has the maximum stresses while the 45° inclined scanning case can reduce residual stress in both directions among all tested cases. The 45° inclined scanning case has a smaller deformation while the in-out scanning case has a larger deformation in build direction than other cases.

In mathematical analysis, a space-filling curve is a curve whose range contains the entire 2-dimensional unit square (or more generally an n-dimensional unit hypercube). Giuseppe Peano (1858-1932) was the first to discover one, space-filling curves in the 2-dimensional plane are sometimes called Peano curves. It was Hilbert (1891) who first popularized their existence and gave an insight into their generation. Space filling curves belong to the class of FASS curves; an acronym for space-filling, self-avoiding, simple, and self-similar. In [37] and [11], fractal scanning strategies based upon mathematical fill curves, namely the Hilbert and Peano-Gosper curve were explored.

In [37] a comparision between S scanning and Fractal scanning (figure 2.4) to investigate the effect of laser scanning patterns on temperature, residual thermal stresses and distortion. It was shown that the stresses of a layer processed by a moving laser beam is decreased with fractal scanning pattern. Compared to the S scanning pattern, much more symmetrical temperature field and smaller distortion can be gained with the fractal scanning pattern.

In [11] an island scan sample was used for comparison with fractal scan strategies based upon Hilbert and Peano-Gosper curves (figure 2.5). The result shows that Fractal scanning reduce cracking when compared to the island scanning but fractal scan strategies have limited scope for optimisation, relying on laser power and scan speed.

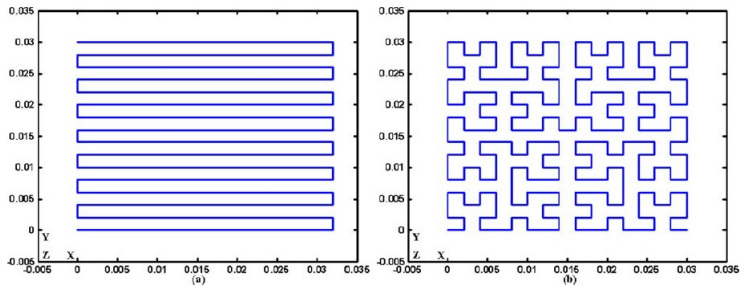


Figure 2.4: S scanning pattern and fractal scanning pattern based upon Hilbert curve studied in [37].

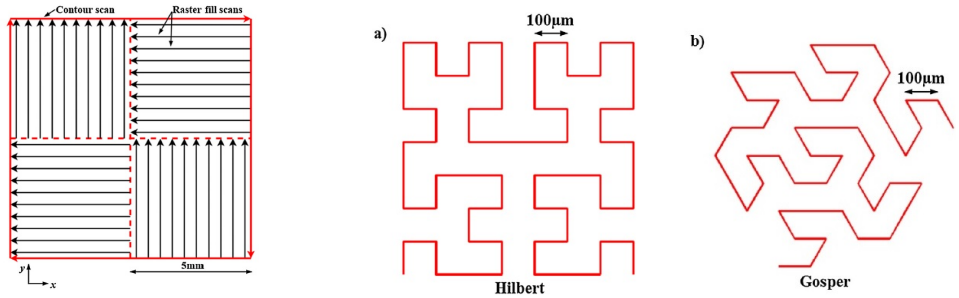


Figure 2.5: Illustration of island scan control sample and scan strategies generated using (a) Hilbert and (b) Peano-Gosper mathematical space filling curves as studied in [11].

2.2 The optimal control problem

In this section we present a model incorporating trajectory optimization to minimize thermal gradients in SLM. Our aim is to propose a mathematical model to find an optimal trajectory minimizing thermal gradients within the produced part using optimal control theory of PDE's. Thus, we introduce the appropriate cost functional and the set of admissible controls taking into account the constraints on laser trajectory. To the best of our knowledge, we know only one recent paper and thesis dealing with this topic using shape optimization tools [9, 8]. The main difference between our model and [9] is that our approach is based on geometry: the geometrical constraint is imposed on the trajectories to cover the built structure, the optimal arbitrary and parametrizable trajectories being chosen to minimize the gradient temperature, while their approach is based on physics: the minimization functional is chosen in order that the temperature must attain a melting value and the paths are broken lines. As mentioned in [9] such an optimization could seem too costly to be used straightly in the industry but it may give some intuitions about an optimal path satisfying the industrial constraints, validating some patterns or proposing new ones. Furthermore, using a parametrizable control trajectories allow us to use different types of initial curves (for instance the ones employed by industry), hence the chosen algorithm will furnish different "optimal" curves that can be compared in order to chose the best one.

We consider the optimal control of a linear heat equation that models the distribution of temperature within one layer Ω heated on its upper surface by a Gaussian laser beam [31, 18, 54] with a linear heat transfer with the bottom layer:

$$\begin{cases} \rho\, c\, \partial_t y - \kappa\, \Delta y = 0 & \text{in} \quad Q = \Omega \times]0,T[\,, \\ -\kappa \frac{\partial y}{\partial \nu} = h\, y - g_\gamma & \text{on} \quad \Sigma_1 = \Gamma_1 \times]0,T[\,, \\ -\kappa \frac{\partial y}{\partial \nu} = h\, y & \text{on} \quad \Sigma_2 = \Gamma_2 \times]0,T[\,, \\ -\kappa \frac{\partial y}{\partial \nu} = h\,(y - y_B) & \text{on} \quad \Sigma_3 = \Gamma_3 \times]0,T[\,, \\ y(x,0) = y_0(x) & \text{for} \quad x \in \Omega. \end{cases} \tag{2.3}$$

These equations are the linear version of (2.2). Here $y(x,t)$ denotes the temperature at point $x \in \Omega$ and time $t \in]0,T[$, y_0 is the initial temperature, while $y_B \in L^2(\Gamma_3)$ corresponds to the temperature at the top of the previous layer. We may always suppose that the ambient temperature y_{ex} is zero by taking as new dependent variable $y - y_{ex}$ so that we are led to the system (2.3). Here, $\Omega \subset \mathbb{R}^3$ is a bounded and simply connected domain with a connected Lipschitz boundary Γ, that is supposed to be split up

$$\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3},$$

where Γ_i, $i = 1, 2, 3$, are disjoint open disjoint subsets of Γ. In the SLM process, Γ_1 corresponds to the upper surface of the added layer and is supposed to be included in a plane, that without loss of generality we assume to be $\mathbb{R}^2$. By $\nu(x)$, we denote the outward normal direction at the point $x \in \Gamma$. The heat source g_γ is of the form

$$g_\gamma(x,t) = \alpha \frac{2P}{\pi R^2} \exp\left(-2 \frac{|\, x - \gamma(t)\,|^2}{R^2}\right), \text{ for all } (x,t) \in \Sigma_1. \tag{2.4}$$

The parameters ρ, c, κ, h, α, P and R are positive constants. The control $\gamma : t \in [0,T] \to \Gamma_1$ represents the displacement of the laser beam center on Γ_1 with respect to time. Note that g_γ depends nonlinearly on γ.

We have chosen the Robin type boundary condition on Σ_3 rather than a non-homogeneous Dirichlet boundary condition (see problem (2.2)) because it is more realistic in the practical point of view.

As suggested before, our main goal is to find a trajectory γ (the control) in such a way as to minimize temperature gradients inside the layer Ω with the constraints that the laser beam runs over the whole surface Γ_1 and does not leave it. From a mathematical point of view, these constraints lead to a non convex admissible set of controls, which is the main difficulty to overcome.

The outline of this section is as follows. In subsection 2.2.1, we introduce the optimal control problem. We explain the link between our non convex set of admissible controls and laser trajectory in SLM process. Then we prove existence of a solution to the optimal control problem. In subsection 2.2.2, we prove the differentiability of the control-to-state mapping, result from which we infer the differentiability of the reduced cost functional. In subsection 2.2.3, the adjoint state is introduced which allow us to compute the Fréchet derivative of this reduced cost functional. Therefore, we derive a first order necessary condition for a control to be optimal in the form of a variational inequality. The main difficulty is in the non convex constraints required on the control γ.

2.2.1 Existence of an optimal control

For further purposes, we introduce the following (Hilbert) space:

$$W(0,T) := \{u \in L^2(0,T; H^1(\Omega)) \text{ such that } \frac{du}{dt} \in L^2(0,T; (H^1(\Omega))^*)\}.$$

Recall that by Theorem 3.12 in [53], if y_0 belongs to $L^2(\Omega)$, $y_B \in L^2(\Sigma_3)$ and $\gamma \in H^1(0,T;\mathbb{R}^2)$ then the initial-boundary value problem (2.3) has a unique solution y in $W(0,T) \cap L^\infty(0,T;L^2(\Omega))$ in the sense that

$$\begin{aligned}
&\rho\, c \int_0^T \langle \frac{dy}{dt}(\cdot,t)\,,\, \varphi(\cdot,t)\rangle_{(H^1(\Omega))^*,H^1(\Omega)}\, dt + \kappa \int_0^T \int_\Omega \nabla y(x,t)\cdot\nabla\varphi(x,t)\, dx\, dt \\
&+ h\int_0^T \int_\Gamma y(x,t)\,\varphi(x,t)\, dS(x)\, dt - \int_0^T \int_{\Gamma_3} h y_B(x,t)\,\varphi(x,t)\, dS(x)\, dt \\
&- \int_0^T \int_{\Gamma_1} g_\gamma(x,t)\varphi(x,t)\, dS(x)\, dt = 0,
\end{aligned} \tag{2.5}$$

for all $\varphi \in L^2(0,T;H^1(\Omega))$.

Our goal is to find a trajectory γ (a control) in such a way as to minimize the temperature gradient inside the layer Ω. We also want to allow the control to be chosen in such a way that the corresponding temperature distribution y in Q (the state) is the best possible approximation to a given temperature distribution $y_Q \in L^2(Q)$. To meet all requirements, we define the following cost functional

$$\begin{aligned}
J(y,\gamma) :=& \frac{1}{2}\int_0^T \int_\Omega |\, \nabla y(x,t)\,|^2\, dx\, dt + \frac{\lambda_Q}{2}\int_0^T \int_\Omega |\, y(x,t) - y_Q(x,t)\,|^2\, dx\, dt \\
&+ \frac{\lambda_\gamma}{2}\,\|\,\gamma\,\|^2_{H^1(0,T;\mathbb{R}^2)},
\end{aligned} \tag{2.6}$$

where $\lambda_Q \geq 0$ and $\lambda_\gamma > 0$ are constants, while $y_Q \in L^2(Q)$ is a given function. Note that λ_γ is a regularization parameter and that if $\lambda_Q = 0$, the only goal is to minimize the temperature gradient. The optimal control problem is

$$\text{(OCP)} \qquad \min_{\gamma\in U_{ad}} J(y(\gamma),\gamma),$$

where $y(\gamma)$ denotes the weak solution of problem (2.3) associated with the control γ, and the set of admissible controls $U_{ad} \subset H^1(0,T;\mathbb{R}^2)$ is defined as follows. For $\epsilon \geq R$, and a fixed positive constant c, U_{ad} is defined by

$$\begin{aligned}
U_{ad} \coloneqq \{\gamma \in H^1(0,T;\mathbb{R}^2);\ & R(\gamma) \subset \Gamma_{1,-\epsilon},\ R_\epsilon(\gamma) = \Gamma_1 \\
& \text{and } |\,\gamma'(t)\,| \leq c \text{ a.e. } t \in [0,T]\},
\end{aligned} \tag{2.7}$$

where $R(\gamma) := \gamma([0,T])$,

$$\Gamma_{1,-\epsilon} = \{x \in \Gamma_1; dist\,(x,\partial\Gamma_1) \geq \epsilon\}\,, \tag{2.8}$$

and

$$R_\epsilon(\gamma) = \{x \in \Gamma_1;\ dist\,(x,R(\gamma)) \leq \epsilon\}. \tag{2.9}$$

Note that the condition $dist(x,\partial\Gamma_1) \geq \epsilon$ has a physical meaning because the control $t \mapsto \gamma(t)$ is nothing else than the path traced by the laser beam center which has R for

radius. Practically, ϵ should be chosen in function of R. The constraint $R(\gamma) \subset \Gamma_{1,-\epsilon}$ is chosen to describe that γ must stay from an ϵ distance from the boundary of Γ_1. Moreover, $R_\epsilon(\gamma) = \Gamma_1$ is to constrain γ to cover Γ_1. Note that U_{ad} is not convex due to the two constraints $R(\gamma) \subset \Gamma_{1,-\epsilon}$ and $R_\epsilon(\gamma) = \Gamma_1$ (see figure 2.6). By the theory of "tubes" [7], if $\partial\Gamma_1 \in C^2(\mathbb{R}^2)$, U_{ad} will be non void if $\epsilon > 0$ is chosen sufficently small and the constant c in definition (2.7) is chosen large enough.

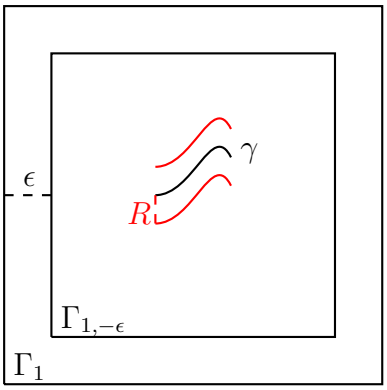


Figure 2.6: The surface Γ_1 scanned by the laser

Let us prove some preliminary results that will allow us to show that (OCP) has at least one optimal control.

Proposition 1. *U_{ad} is a weakly closed subset of $H^1(0,T;\mathbb{R}^2)$.*

Proof. Let $(\gamma_n)_{n\in\mathbb{N}} \subset U_{ad}$ be a weakly convergent sequence in $H^1(0,T;\mathbb{R}^2)$ and let us call γ its weak limit. As the embedding from $H^1(0,T;\mathbb{R}^2)$ into $C([0,T];\mathbb{R}^2)$ is compact, the weak convergence in $H^1(0,T;\mathbb{R}^2)$ of $(\gamma_n)_{n\in\mathbb{N}}$ to γ implies the strong convergence of $(\gamma_n)_{n\in\mathbb{N}}$ in $C([0,T];\mathbb{R}^2)$.

Given $x \in \Gamma_1$, there exists $(t_n)_{n\in\mathbb{N}} \subset [0,T]$ such that

$$\gamma_n(t_n) \in \Gamma_{1,-\epsilon} \quad \text{for every } n \in \mathbb{N},$$

$$|x - \gamma_n(t_n)| \leq \epsilon \quad \text{for every } n \in \mathbb{N}.$$

As $[0,T]$ is compact there exists a subsequence $(t_{n_j})_{j\in\mathbb{N}} \subset [0,T]$ convergent to some $t \in [0,T]$.
Thus we have

$$\begin{aligned} |\, \gamma_{n_j}(t_{n_j}) - \gamma(t) \,| &\leqslant |\, \gamma_{n_j}(t_{n_j}) - \gamma(t_{n_j}) \,| + |\, \gamma(t_{n_j}) - \gamma(t) \,| \\ &\leqslant \| \gamma_{n_j} - \gamma \|_\infty + |\, \gamma(t_{n_j}) - \gamma(t) \,| \longrightarrow 0 \text{ as } j \longrightarrow \infty. \end{aligned} \tag{2.10}$$

This implies that $|x - \gamma(t)| \leq \epsilon$. Thus, $R_\epsilon(\gamma) = \Gamma_1$ and $R(\gamma) \subset \Gamma_{1,-\epsilon}$ (recalling that $\Gamma_{1,-\epsilon}$ is closed) .

Using Mazur's theorem [60], for all $j \in \mathbb{N}^*$ there exists a convex combination

$$u_{n_j} = \sum_{k=1}^{n_j} \alpha_k \gamma_k, \left(\alpha_k \geq 0, \ \sum_{k=1}^{n_j} \alpha_k = 1 \right), \text{ such that } \| \gamma - u_{n_j} \|_{H^1(0,T;\mathbb{R}^2)} \leq \frac{1}{j}.$$

This implies that

$$\| \gamma' - u'_{n_j} \|_{L^2(0,T;\mathbb{R}^2)} \to 0 \text{ as } j \to \infty.$$

In particular, there exists a subsequence $(u'_{n_{j_k}})_{k\in\mathbb{N}}$ such that

$$u'_{n_{j_k}}(t) \to \gamma'(t) \text{ as } k \to \infty \text{ for a.e. } t \in [0,T].$$

As $| u'_{n_{j_k}}(t) | \leq c$ for a.e. $t \in [0,T]$, then also

$$| \gamma'(t) | \leq c \text{ for a.e. } t \in [0,T].$$

Thus $\gamma \in U_{ad}$. □

Proposition 2. *The control-to-state mapping $G : \gamma \in U_{ad} \longmapsto y(\gamma) \in W(0,T)$ is weakly sequentially continuous.*

Proof. Let $(\gamma_n)_{n\in\mathbb{N}} \subset U_{ad}$ be a weakly convergent sequence in $H^1(0,T;\mathbb{R}^2)$ and let $\gamma \in U_{ad}$ its weak limit. By the compact embedding from $H^1(0,T;\mathbb{R}^2)$ into $C([0,T];\mathbb{R}^2)$, $(\gamma_n)_{n\in\mathbb{N}}$ strongly converges to γ in $C([0,T];\mathbb{R}^2)$. From Theorem 3.13 in [53] it follows that the sequence $(y(\gamma_n))_{n\in\mathbb{N}}$ is a bounded sequence in the space $W(0,T)$. Consequently, it possesses a weakly convergent subsequence $(y(\gamma_{n_j}))_{j\in\mathbb{N}}$ in the space $W(0,T)$. Let y be the weak limit of $(y(\gamma_{n_j}))_{j\in\mathbb{N}}$.

For $\frac{1}{2} < \varepsilon < 1$, the embedding from $W(0,T)$ into $L^2(0,T;H^\varepsilon(\Omega))$ is a linear continuous compact mapping by the Lions-Aubin compactness Lemma [34, p.57]. The trace mapping

$$\begin{array}{ccc} L^2(0,T;H^\varepsilon(\Omega)) & \longrightarrow & L^2(0,T;H^{\varepsilon-1/2}(\Gamma)) \\ y & \longmapsto & y_{|\Sigma}. \end{array}$$

is linear and continuous [35], and thus the trace mapping from $L^2(0,T;H^\varepsilon(\Omega))$ into $L^2(\Sigma) = L^2(0,T;L^2(\Gamma))$ is also linear and continuous. This implies that the sequence of traces on Σ of $(y(\gamma_{n_j}))_{j\in\mathbb{N}}$ strongly converges to $y_{|\Sigma}$ in $L^2(\Sigma)$.

Now we recall that each $y(\gamma_{n_j})$ satisfies the equivalent weak formulation of problem (2.3), namely

$$\begin{aligned} &\rho\, c \int_0^T \langle \frac{dy}{dt}(\gamma_{n_j})(.,t)\,,\, v\rangle_{(H^1(\Omega))^*,H^1(\Omega)}\, \varphi(t)dt + \kappa \int_0^T \int_\Omega \nabla y(\gamma_{n_j})(x,t)\cdot\nabla v(x)\varphi(t)\,dx\,dt \\ &+h\int_0^T\int_\Gamma y(\gamma_{n_j})(x,t)\,v(x)\,\varphi(t)\,dS(x)\,dt - h\int_0^T\int_{\Gamma_3} y_B(x,t)\,v(x)\,\varphi(t)\,dS(x)dt \\ &-\alpha\frac{2P}{\pi R^2}\int_0^T\int_{\Gamma_1}\exp\left(-2\frac{|\, x-\gamma_{n_j}(t)\,|^2}{R^2}\right) v(x)\,\varphi(t)\,dS(x)\,dt = 0, \end{aligned}$$

for all $v \in H^1(\Omega)$, and all $\varphi \in L^2(0,T)$.

By the Lebesgue convergence theorem we have,

$$\begin{aligned} &\int_0^T\int_{\Gamma_1} exp\left(-2\frac{|\, x-\gamma_{n_j}(t)\,|^2}{R^2}\right) v(x)\varphi(t)dS(x)dt \longrightarrow \\ &\qquad\qquad \int_0^T\int_{\Gamma_1} exp\left(-2\frac{|\, x-\gamma(t)\,|^2}{R^2}\right) v(x)\,\varphi(t)dS(x)dt \end{aligned}$$

as $j \to \infty$, for all $v \in H^1(\Omega)$ and all $\varphi \in L^2(0,T)$.

Using all the previous convergence properties to pass to the limit in the above equation as $j \to \infty$, we obtain $y = y(\gamma)$. Thus $(y(\gamma_{n_j}))_{j\in\mathbb{N}}$ weakly converges to $y(\gamma)$ in $W(0,T)$. Therefore any subsequence of $(y(\gamma_n))_{n\in\mathbb{N}}$ contains a further subsequence which converges weakly to $y(\gamma)$ in $W(0,T)$. This implies that the whole sequence itself $(y(\gamma_n))_{n\in\mathbb{N}}$ converges weakly to $y(\gamma)$ in $W(0,T)$. This proves the proposition. □

Definition 3. The reduced cost functional is defined by

$$\begin{array}{rccc} \hat{J}: & U_{ad} & \longrightarrow & \mathbb{R} \\ & \gamma & \longmapsto & J(G(\gamma),\gamma). \end{array}$$

We are ready to prove our main result, namely the existence of at least one optimal control.

Theorem 4 (Existence of an optimal control)**.** *Supposing $U_{ad} \neq \emptyset$, then the optimal control problem* (OCP) *admits at least one optimal control $\bar{\gamma} \in U_{ad}$.*

Proof. Since $\hat{J}(\gamma) \geq 0$, the infimum

$$L := \inf_{\gamma\in U_{ad}} \hat{J}(\gamma),$$

exists and there is a sequence $(\gamma_n)_{n\in N} \subset U_{ad}$ such that $\hat{J}(\gamma_n) \to L$ as $n \to +\infty$.

The sequence $(\gamma_n)_{n\in\mathbb{N}} \subset U_{ad}$ is bounded in $H^1(0,T;\mathbb{R}^2)$, because $\|\gamma_n\|^2_{H^1(0,T;\mathbb{R}^2)} \leq \frac{2}{\lambda_\gamma}\hat{J}(\gamma_n)$ for all $n \in \mathbb{N}$. Hence, it possesses a subsequence $(\gamma_{n_j})_{j\in\mathbb{N}}$ weakly convergent to some element $\bar{\gamma} \in U_{ad}$. This implies that

$$\| \bar{\gamma} \|_{H^1(0,T;\mathbb{R}^2)} \leqslant \liminf_{j\to\infty} \| \gamma_{n_j} \|_{H^1(0,T;\mathbb{R}^2)} \leq \sqrt{\frac{2L}{\lambda_\gamma}}. \tag{2.11}$$

By proposition 2 $G(\gamma_{n_j}) \rightharpoonup G(\bar{\gamma})$ in $W(0,T)$ which implies that $G(\gamma_{n_j}) \rightharpoonup G(\bar{\gamma})$ in $L^2(0,T;H^1(\Omega))$, and thus

$$\| G(\bar{\gamma}) \|_{L^2(0,T;H^1(\Omega))} \leqslant \liminf_{j\to\infty} \| G(\gamma_{n_j}) \|_{L^2(0,T;H^1(\Omega))} . \tag{2.12}$$

The embedding from $W(0,T)$ into $L^2(0,T;L^2(\Omega))$ being compact [34, p.57], the sequence $G(\gamma_{n_j})$ also strongly converges to $G(\bar{\gamma})$ in $L^2(0,T;L^2(\Omega))$.

Using all the previous convergence properties and formula (2.6) we have

$$\begin{aligned} L \geqslant \frac{1}{2}\liminf_{j\to\infty} \int_0^T \int_\Omega | \nabla G(\gamma_{n_j})(x,t) |^2 \, dx \, dt + \frac{\lambda_Q}{2}\liminf_{j\to\infty} \| G(\gamma_{n_j}) - y_Q \|^2_{L^2(Q)} \\ + \frac{\lambda_\gamma}{2}\liminf_{j\to\infty} \| \gamma_{n_j} \|^2_{H^1(0,T;\mathbb{R}^2)} \geqslant \hat{J}(\bar{\gamma}). \end{aligned}$$

By the definition of L we have also that $L \leqslant \hat{J}(\bar{\gamma})$. Thus $L = \hat{J}(\bar{\gamma})$.

□

2.2.2 Differentiability of the control-to-state mapping.

Our aim is to derive necessary optimality conditions for an admissible control to be optimal. We first have to discuss the differentiability of the control-to-state mapping.

Lemma 5. *The mapping*

$$\begin{array}{rccc} G: & U_{ad} & \longrightarrow & L^2(0,T;H^1(\Omega)) \\ & \gamma & \longmapsto & y(\gamma) \end{array}$$

is Fréchet differentiable.

Proof. We can write G as the restriction to U_{ad} of the composition of the Fréchet differentiable mappings w, g and q, where w, g and q are defined as follows:

$$\begin{array}{rccc} w: & H^1(0,T;\mathbb{R}^2) & \longrightarrow & C(\bar{\Gamma}_1 \times [0,T]) \\ & \gamma & \longmapsto & -c_R \mid \tilde{\gamma}(\gamma) \mid^2 \end{array} \tag{2.13}$$

where $c_R = \frac{2}{R^2}$ and $\tilde{\gamma}(\gamma)(x,t) := x - \gamma(t),\ \forall (x,t) \in \bar{\Gamma}_1 \times [0,T]$,

$$\begin{array}{rccc} g: & C(\bar{\Gamma}_1 \times [0,T]) & \longrightarrow & L^2(\Sigma_1) \\ & u & \longmapsto & a \exp(u) \end{array} \tag{2.14}$$

where $a = \alpha \frac{2P}{\pi R^2}$, and

$$\begin{array}{rccc} q: & L^2(\Sigma_1) & \longrightarrow & L^2(0,T;H^1(\Omega)) \\ & g & \longmapsto & y \end{array} \tag{2.15}$$

where y denotes the weak solution of the initial boundary value problem:

$$\left\{ \begin{array}{lll} \rho\, c\, \partial_t y - \kappa\, \Delta y = 0 & \text{in} & Q, \\ -\kappa \frac{\partial y}{\partial \nu} = h\, y - g & \text{on} & \Sigma_1, \\ -\kappa \frac{\partial y}{\partial \nu} = h\, y & \text{on} & \Sigma_2, \\ -\kappa \frac{\partial y}{\partial \nu} = h\, (y - y_B) & \text{on} & \Sigma_3, \\ y(x,0) = y_0(x) & \text{for} & x \in \Omega, \end{array} \right. \tag{2.16}$$

$y_0 \in L^2(\Omega)$ denoting a fixed initial condition.

• We start by proving that w is Fréchet differentiable, when $C(\bar{\Gamma}_1 \times [0,T])$ is endowed with its natural norm $\| u \|_\infty := \sup_{(x,t) \in \bar{\Gamma}_1 \times [0,T]} |u(x,t)|$. For all $\delta\gamma \in H^1(0,T;\mathbb{R}^2)$ we have:

$$\begin{aligned} [w(\gamma + \delta\gamma) - w(\gamma)]\,(x,t) &= -c_R\, [\, \mid x - (\gamma + \delta\gamma)(t) \mid^2 - \mid x - \gamma(t) \mid^2] \\ &= c_R\, \delta\gamma(t) \cdot (2x - 2\gamma(t) - \delta\gamma(t)) \\ &= 2\, c_R\, (x - \gamma(t)) \cdot \delta\gamma(t) - c_R \mid \delta\gamma(t) \mid^2,\ \forall (x,t) \in \bar{\Gamma}_1 \times [0,T], \end{aligned}$$

where $\cdot$ denotes here the inner product in $\mathbb{R}^2$. Using the fact that $H^1(0,T;\mathbb{R}^2) \hookrightarrow C([0,T];\mathbb{R}^2)$, there exists a constant $\eta \geq 0$ such that

$$\frac{\| |\delta\gamma|^2 \|_\infty}{\| \delta\gamma \|_{H^1(0,T;\mathbb{R}^2)}} \leq \eta \frac{\| \delta\gamma \|^2_{H^1(0,T;\mathbb{R}^2)}}{\| \delta\gamma \|_{H^1(0,T;\mathbb{R}^2)}} = \eta \| \delta\gamma \|_{H^1(0,T;\mathbb{R}^2)} \to 0 \text{ as } \| \delta\gamma \|_{H^1(0,T;\mathbb{R}^2)} \to 0.$$

Hence, w is Fréchet differentiable with Fréchet derivative

$$Dw(\gamma) \cdot \delta\gamma = 2\, c_R\, \tilde{\gamma}(\gamma) \cdot \delta\gamma, \text{ for all } \delta\gamma \in H^1(0,T;\mathbb{R}^2).$$

• The mapping q is Fréchet differentiable being an affine mapping and continuous by [53, (3.26) p.140]. The mapping g being a superposition operator (also called Nemytskii operator) is known to be Fréchet differentiable by [53, p.202] or [4]. We therefore skip the details and only give their Fréchet derivatives, which are respectively given by

$$Dq(\tilde{g}) = \tau, \text{ for all } \tilde{g} \in L^2(\Sigma_1),$$

with

$$\begin{array}{rccc} \tau\colon & L^2(\Sigma_1) & \longrightarrow & L^2(0,T;H^1(\Omega)) \\ & g & \longmapsto & \tau(g) := y_2, \end{array}$$

where y_2 is the unique solution of

$$\begin{cases} \rho\, c\, \partial_t y_2 - \kappa\, \Delta y_2 = 0 & \text{in} \quad Q, \\ -\kappa \frac{\partial y_2}{\partial \nu} = h y_2 - g & \text{on} \quad \Sigma_1, \\ -\kappa \frac{\partial y_2}{\partial \nu} = h y_2 & \text{on} \quad \Sigma_2 \cup \Sigma_3, \\ y_2(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.17}$$

and

$$Dg(u) \cdot \delta u = a \exp(u) \delta u, \text{ for all } \delta u \in C(\bar{\Gamma}_1 \times [0,T]).$$

In conclusion, G is Fréchet differentiable with Fréchet derivative

$$\begin{aligned} DG(\gamma) \cdot \delta\gamma &= D(q \circ g \circ w)(\gamma) \cdot \delta\gamma \\ &= Dq(g(w(\gamma))) \cdot (D(g \circ w)(\gamma) \cdot \delta\gamma) \\ &= (Dq(g(w(\gamma))) \circ Dg(w(\gamma))) \cdot (Dw(\gamma) \cdot \delta\gamma) \\ &= \tau((Dg(w(\gamma)) \circ Dw(\gamma)) \cdot \delta\gamma) \\ &= 2ac_R\, \tau(\exp(w(\gamma))\, \tilde{\gamma}(\gamma) \cdot \delta\gamma), \text{ for all } \delta\gamma \in H^1(0,T;\mathbb{R}^2). \end{aligned}$$

□

From the previous lemma and by composition we deduce that the reduced cost functional $\gamma \in H^1(0,T;\mathbb{R}^2) \mapsto J(G(\gamma),\gamma) \in \mathbb{R}$ is Fréchet differentiable. Let us denote by

$$v(\gamma, \delta\gamma) := 2ac_R\, \tau(\exp(w(\gamma))\, \tilde{\gamma}(\gamma) \cdot \delta\gamma) \tag{2.18}$$

the Fréchet derivative of the control-to-state mapping $\gamma \in U_{ad} \mapsto G(\gamma) \in L^2(0,T;H^1(\Omega))$. Using the previous result, we obtain

$$D\hat{J}(\gamma)\cdot\delta\gamma = \int_0^T\int_\Omega \nabla G(\gamma)(x,t)\cdot\nabla v(\gamma,\delta\gamma)(x,t)\,dxdt + \lambda_Q\,(G(\gamma)-y_Q,\,v(\gamma,\delta\gamma))_{L^2(Q)}$$
$$+\lambda_\gamma\,(\gamma,\delta\gamma)_{H^1(0,T;\mathbb{R}^2)} = \int_0^T\int_\Omega \nabla G(\gamma)(x,t)\cdot\nabla v(\gamma,\delta\gamma)(x,t)\,dxdt$$
$$+\lambda_Q\int_0^T\int_\Omega G(\gamma)(x,t)\,v(\gamma,\delta\gamma)(x,t)\,dxdt - \lambda_Q\int_0^T\int_\Omega y_Q(x,t)\,v(\gamma,\delta\gamma)(x,t)\,dxdt$$
$$+\lambda_\gamma\int_0^T\gamma(t)\cdot\delta\gamma(t)\,dt + \lambda_\gamma\int_0^T\gamma'(t)\cdot\delta\gamma'(t)\,dt,\ \text{ for all } \delta\gamma\in H^1(0,T;\mathbb{R}^2). \tag{2.19}$$

2.2.3 Adjoint equation and necessary optimality conditions

It is well known that an optimal control $\bar\gamma$ minimizing $\hat J$ in U_{ad} has to obey the variational inequality

$$D\hat{J}(\bar\gamma)(\gamma-\bar\gamma)\geq 0 \text{ for all } \gamma\in U_{ad}, \tag{2.20}$$

provided that $\hat J$ is Gâteaux differentiable at $\bar\gamma$ and U_{ad} convex. In our case $\hat J$ is Fréchet differentiable but U_{ad} is not convex, thus (2.20) is no more true. Therefore, we introduce at any point $\gamma\in U_{ad}$ the cone of admissible directions and we use the Kuhn-Tucker conditions. More precisely, we recall from [15, p.211] the following definition and result.

Definition 6. Let V a normed vector space and U_{ad} a non-empty subset of V. For every $\gamma\in U_{ad}$, the cone of admissible directions at γ is

$$C(\gamma) := \{0\}\ \cup\ \{w\in V\setminus\{0\};\ \exists(\gamma_k)_{k\geqslant 0}\subset U_{ad},\gamma_k\neq\gamma\ \forall k\geq 0 \text{ s.t. } \lim_{k\to\infty}\gamma_k=\gamma \text{ and } \lim_{k\to\infty}\frac{\gamma_k-\gamma}{\|\,\gamma_k-\gamma\,\|}=\frac{w}{\|\,w\,\|}\}. \tag{2.21}$$

Theorem 7 (Kuhn-Tucker)**.** *Let V be a normed vector space and U_{ad} a non-empty subset of V. Let $J: O\subset V\to\mathbb{R}$, a function defined on an open set O of V such that $U_{ad}\subset O$. If J has at $\bar\gamma\in U_{ad}$ a relative minimum compared to the subset U_{ad}, and if J is Fréchet differentiable at $\bar\gamma$ then*

$$DJ(\bar\gamma)\cdot(\delta\bar\gamma)\geq 0 \text{ for every } \delta\bar\gamma\in C(\bar\gamma), \tag{2.22}$$

i.e. $DJ(\bar\gamma)$ belongs to the dual cone of the cone of admissible directions $C(\bar\gamma)$ at $\bar\gamma$.

Since $\hat J$ is Fréchet differentiable, Theorem 7 and (2.19) allow us to derive a necessary condition for an admissible control to be optimal. However, this necessary condition would not be practical due to the appearance of $v(\bar\gamma,\delta\bar\gamma)$ in (2.19) which should be computed by solving the initial boundary value problem (2.17) for $g=\exp(w(\bar\gamma))\,\tilde\gamma(\bar\gamma)\cdot\delta\bar\gamma$ each time we consider another $\delta\bar\gamma\in C(\bar\gamma)$ to check if the necessary condition (2.22) is true at $\bar\gamma$. To resolve this difficulty, it is classical to introduce the "adjoint system" whose solution is called the adjoint state [53]. We claim that the adjoint system for our problem is the following linear backward boundary value problem

$$\begin{cases} \rho\,c\,\partial_t p+\kappa\,\Delta p=\Delta G(\bar\gamma)-\lambda_Q(G(\bar\gamma)-y_Q) & \text{in} \quad Q,\\ \kappa\frac{\partial p}{\partial\nu}+h\,p=\frac{\partial G(\bar\gamma)}{\partial\nu} & \text{on} \quad \Gamma\times]0,T[,\\ p(.,T)=0 & \text{in} \quad \Omega. \end{cases} \tag{2.23}$$

Definition 8. Let $\bar{\gamma}$ be an optimal control of (OCP) with associated state $G(\bar{\gamma})$. A function $p \in W(0,T)$ is said to be a weak solution to (2.23) if $p(\cdot,T)=0$ in Ω and

$$\begin{aligned} &-\rho c\int_0^T <\partial_t p(\cdot,t),\varphi(\cdot,t)>_{H^1(\Omega)^*,H^1(\Omega)} dt+\kappa\int_0^T\int_\Omega \nabla p(x,t)\cdot\nabla\varphi(x,t)dxdt \\ &+h\int_0^T\int_\Gamma p(x,t)\varphi(x,t)dS(x)dt=\int_0^T\int_\Omega \nabla G(\bar{\gamma})(x,t)\nabla\varphi(x,t)dxdt \\ &+\lambda_Q\int_0^T\int_\Omega (G(\bar{\gamma})-y_Q)(x,t)\varphi(x,t)dxdt \end{aligned} \tag{2.24}$$

for every $\varphi \in L^2(0,T;H^1(\Omega))$.

Let us notice that (2.23) admits a unique weak solution in $W(0,T)$, see [19, pp. 512-513] for instance.

Theorem 9. *If $\bar{\gamma} \in U_{ad}$ is an optimal control of (OCP) with associated state $G(\bar{\gamma})$, and $p \in W(0,T)$ the corresponding adjoint state that solves (2.23), then the variational inequality*

$$\begin{aligned} &\lambda_\gamma\int_0^T \bar{\gamma}(t)\cdot(\delta\bar{\gamma})(t)dt+\lambda_\gamma\int_0^T \bar{\gamma}'(t)\cdot(\delta\bar{\gamma})'(t)dt \\ &+2ac_R\int\int_{\Sigma_1}\exp(w(\bar{\gamma})(x,t))\tilde{\gamma}(\bar{\gamma})(x,t)\cdot(\delta\bar{\gamma})(t)p(x,t)dS(x)dt\geq 0 \end{aligned} \tag{2.25}$$

holds for all $\delta\bar{\gamma} \in C(\bar{\gamma})$.

Proof. If $\bar{\gamma}$ is an optimal control for the problem (OCP) then by Theorem 7

$$D\hat{J}(\bar{\gamma})\cdot(\delta\bar{\gamma})\geq 0 \text{ for all } \delta\bar{\gamma}\in C(\bar{\gamma}).$$

By (2.19) and (2.24) we have

$$\begin{aligned} D\hat{J}(\bar{\gamma})\cdot(\delta\bar{\gamma}) &= -\rho c\int_0^T <\partial_t p(\cdot,t),v(\bar{\gamma},\delta\bar{\gamma})(\cdot,t)>_{H^1(\Omega)^*,H^1(\Omega)} dt \\ &+\kappa\int_0^T\int_\Omega \nabla p(x,t)\cdot\nabla v(\bar{\gamma},\delta\bar{\gamma})(x,t)dxdt+h\int_0^T\int_\Gamma p(x,t)v(\bar{\gamma},\delta\bar{\gamma})(x,t)dS(x)dt \\ &+\lambda_{\bar{\gamma}}\int_0^T \bar{\gamma}(t)\cdot(\delta\bar{\gamma})(t)\,dt+\lambda_{\bar{\gamma}}\int_0^T \bar{\gamma}'(t)\cdot(\delta\bar{\gamma})'(t)\,dt \end{aligned} \tag{2.26}$$

where we recall that $v(\bar{\gamma},\delta\bar{\gamma})$ given by (2.18) is the weak solution of

$$\begin{cases} \rho\, c\,\partial_t v(\bar{\gamma},\delta\bar{\gamma})-\kappa\,\Delta v(\bar{\gamma},\delta\bar{\gamma})=0 & \text{in} \quad Q, \\ \kappa\frac{\partial v(\bar{\gamma},\delta\bar{\gamma})}{\partial\nu}+h\,v(\bar{\gamma},\delta\bar{\gamma})=2ac_R\exp(w(\bar{\gamma}))\tilde{\gamma}(\bar{\gamma})\cdot\delta\bar{\gamma} & \text{on} \quad \Sigma_1, \\ \kappa\,\frac{\partial v(\bar{\gamma},\delta\bar{\gamma})}{\partial\nu}+h\,v(\bar{\gamma},\delta\bar{\gamma})=0 & \text{on} \quad \Sigma_2\cup\Sigma_3, \\ v(\bar{\gamma},\delta\bar{\gamma})(x,0)=0 & \text{for} \quad x\in\Omega, \end{cases} \tag{2.27}$$

and p is the weak solution of (2.23). Using the fact that $v(\bar{\gamma},\delta\bar{\gamma})$ is the weak solution of (2.27), taking $p(\cdot,\cdot)$ as test function in the weak formulation of (2.27) and using the integration by parts formula in $W(0,T)$ [53, p.148] taking into account that $v(\bar{\gamma},\delta\bar{\gamma})(\cdot,0)=0$ and $p(\cdot,T)=0$, we obtain:

$$
\begin{aligned}
&- \rho c \int_0^T < \partial_t p(\cdot,t), v(\bar\gamma,\delta\bar\gamma)(\cdot,t) >_{H^1(\Omega)^*,H^1(\Omega)} dt + \kappa \int_0^T \int_\Omega \nabla p(x,t)\cdot\nabla v(\bar\gamma,\delta\bar\gamma)(x,t)dxdt \\
&+ h\int_0^T \int_\Gamma p(x,t)v(\bar\gamma,\delta\bar\gamma)(x,t)dS(x)dt \\
&= 2ac_R \int_0^T \int_{\Gamma_1} \exp(w(\bar\gamma)(x,t))\tilde\gamma\,(\bar\gamma)\,(x,t)\cdot(\delta\bar\gamma)(t)p(x,t)dS(x)dt.
\end{aligned}
$$

By replacing this last identity in (2.26) we get

$$
\begin{aligned}
&D\hat J(\bar\gamma)\cdot(\delta\bar\gamma) = 2ac_R \int\int_{\Sigma_1} \exp(w(\bar\gamma)(x,t))\tilde\gamma\,(\bar\gamma)\,(x,t)\cdot(\delta\bar\gamma)(t)p(x,t)dS(x)dt \\
&+\lambda_\gamma \int_0^T \bar\gamma(t)\cdot(\delta\bar\gamma)(t)dt + \lambda_\gamma \int_0^T \bar\gamma'(t)\cdot(\delta\bar\gamma)'(t)dt \geq 0.
\end{aligned}
$$

This concludes the proof of the Theorem.

□

2.3 The penalized control problem

The previous approach has two drawbacks: First the set of admissible controls is not convex. Secondly the constraints on the control $R(\gamma) \subset \Gamma_{1,-\epsilon}$, $R_\epsilon(\gamma) = \Gamma_1$ do not seem appropriate for discretization. Therefore, in this section we intend to replace the previous nonconvex constraints by other conditions on the trajectory γ, by adding a penalization term to the cost functional (2.6). Namely, given $\theta > 0$ a penalization parameter, we add to $J(y(\gamma),\gamma)$, the term

$$
\frac{1}{\theta^2}\left(2R\int_0^T \sqrt{|\ \gamma'(t)\ |^2 + \theta^2}dt - |\ \Gamma_1\ |\right)^2.
$$

Formally as θ is close to zero, this will force the control to satisfy

$$
2R\int_0^T |\ \gamma'(t)\ |\ dt \simeq |\ \Gamma_1\ |, \tag{2.28}
$$

which means that the area covered by the laser is close to the area of Γ_1.

2.3.1 Existence of an optimal control

We consider the penalized optimal control problem: Given $\theta > 0$, find $\bar\gamma^\theta \in U_{ad}^p$ such that

$$
(\mathrm{OCP}^\theta) \qquad J^\theta(y(\bar\gamma^\theta),\bar\gamma^\theta) = \min_{\gamma\in U_{ad}^p} J^\theta(y(\gamma),\gamma),
$$

with,

$$
J^\theta(y(\gamma),\gamma) \coloneqq J(y(\gamma),\gamma) + \frac{1}{\theta^2}\left(\int_0^T \sqrt{|\ \gamma'(t)\ |^2 + \theta^2}dt - |\ \Gamma_1\ |\right)^2, \tag{2.29}
$$

where we take here

$$J(y(\gamma),\gamma) := \frac{1}{2}\int_0^T \int_\Omega \mid \nabla y(\gamma)(x,t) \mid^2 dx\, dt + \frac{\lambda_Q}{2}\int_0^T \int_\Omega \mid y(\gamma)(x,t) - y_Q(x,t) \mid^2 dx\, dt + \frac{\lambda_\gamma}{2} \parallel \gamma \parallel^2_{H^2(0,T;\mathbb{R}^2)} . \tag{2.30}$$

In this section, we assume that Γ_1 to be a convex subset of $\mathbb{R}^2$. If $\epsilon \geq R$ the set of admissible controls

$$U^p_{ad} = \{\gamma \in H^2(0,T;\Gamma_1);\ \exists\, c > 0 \text{ s.t } \mid \gamma'(t) \mid \leq c \text{ a.e. } t \in [0,T] \text{ and } 2R\int_0^T \mid \gamma'(t) \mid dt \leq \mid \Gamma_1 \mid + 2\epsilon diam(\Gamma_1)\} \tag{2.31}$$

will be convex.

Proposition 10. *U^p_{ad} is closed.*

Proof. Let $(\gamma_n)_{n\in\mathbb{N}} \subset H^2(0,T;\mathbb{R}^2)$ be a convergent sequence in U^p_{ad} and let us call γ its limit. This implies that γ'_n converge to γ' in $L^2(0,T;\mathbb{R}^2)$, so that there exists a subsequence

$$\gamma'_{n_k} \longrightarrow \gamma' \text{ as } k \to \infty \text{ a.e.} \tag{2.32}$$

As $\mid \gamma'_{n_k}(t) \mid \leq c$ for almost every $t \in [0,T]$, by passing to the limit we obtain

$$\mid \gamma'(t) \mid \leq c \text{ for almost every } t \in [0,T]. \tag{2.33}$$

Also, we have that $\gamma'_n \to \gamma'$ in $L^1(0,T;\mathbb{R}^2)$ so we can pass to the limit in the second constraint defining U^p_{ad} obtaining that

$$2R\int_0^T \mid \gamma'(t) \mid dt \leq \mid \Gamma_1 \mid + 2\epsilon\, diam(\Gamma_1). \tag{2.34}$$

□

It is clear that since U^p_{ad} is convex and closed that it is weakly closed.

Proposition 11. *The control-to-state mapping $\gamma \in U_{ad} \longmapsto y(\gamma) \in W(0,T)$ is weakly sequentially continuous.*

Proof. Similar to the proof of proposition 2. □

The reduced cost functional $\hat{J}^\theta(\cdot)$, is now defined by

$$\hat{J}^\theta(\gamma) := \hat{J}(\gamma) + \frac{1}{\theta^2}\left(2R\int_0^T \sqrt{\mid \gamma'(t) \mid^2 + \theta^2} dt - \mid \Gamma_1 \mid\right)^2. \tag{2.35}$$

Theorem 12 (Existence of an optimal control)**.** *The optimal control problem* (OCP$^\theta$) *admits at least one optimal control $\bar{\gamma}^\theta \in U^p_{ad}$.*

Proof. Since $\hat{J}^\theta(\gamma) \geq 0$, the infimum

$$L^\theta := \inf_{\gamma \in U_{ad}^p} \hat{J}^\theta(\gamma),$$

exists and there is a minimizing sequence $(\gamma_n)_{n\in N} \subset U_{ad}^p$ such that $\hat{J}(\gamma_n) \to L^\theta$ as $n \to \infty$. The sequence $(\gamma_n)_{n\in\mathbb{N}} \subset U_{ad}^p$ being bounded in $H^2(0,T;\mathbb{R}^2)$ it posseses a subsequence $(\gamma_{n_j})_{j\in\mathbb{N}}$ weakly convergent to some element $\bar{\gamma}^\theta \in U_{ad}^p$. This implies

$$\| \bar{\gamma}^\theta \|_{H^2(0,T;\mathbb{R}^2)} \leqslant \liminf_{j\to\infty} \| \gamma_{n_j} \|_{H^2(0,T;\mathbb{R}^2)} \leq \sqrt{\frac{2L^\theta}{\lambda_\gamma}}. \tag{2.36}$$

Since $H^2(0,T;\mathbb{R}^2)$ is compactly embedded in $C^1([0,T];\mathbb{R}^2)$ then $(\gamma_{n_j})_{j\in\mathbb{N}}$ strongly converges to $\bar{\gamma}^\theta \in C^1([0,T];\mathbb{R}^2)$. This implies $(\gamma'_{n_j})_{n\in\mathbb{N}}$ converges uniformly to $\bar{\gamma}^{\theta\prime}$ on $[0,T]$, and thus that

$$\int_0^T \sqrt{| \gamma'_{n_j}(t) |^2 + \theta^2} dt \to \int_0^T \sqrt{| \gamma'(t) |^2 + \theta^2} dt \text{ as } j \to +\infty.$$

By the previous proposition $y(\gamma_{n_j}) \rightharpoonup y(\bar{\gamma}^\theta)$ in $W(0,T)$ this implies that $y(\gamma_{n_j}) \rightharpoonup y(\bar{\gamma}^\theta)$ in $L^2(0,T;H^1_{\Gamma_3}(\Omega))$ thus

$$\| y(\bar{\gamma}^\theta) \|_{L^2(0,T;H^1_{\Gamma_3}(\Omega))} \leqslant \liminf_{j\to\infty} \| y(\gamma_{n_j}) \|_{L^2(0,T;H^1_{\Gamma_3}(\Omega))}. \tag{2.37}$$

The embedding from $W(0,T)$ into $L^2(0,T;L^2(\Omega))$ being compact [34], the sequence $y(\gamma_{n_j})$ also converge strongly to $y(\bar{\gamma}^\theta)$ in $L^2(0,T;L^2(\Omega))$.

Using all the previous convergence properties we have

$$\begin{aligned} L^\theta \geqslant & \liminf_{j\to\infty} \| y(\gamma_{n_j}) \|^2_{L^2(0,T;H^1_{\Gamma_3}(\Omega))} + \frac{\lambda_d}{2} \lim_{j\to\infty} \| y(\gamma_{n_j}) - y_d \|^2_{L^2(Q)} \\ & + \liminf_{j\to\infty} \| \gamma_{n_j} \|_{H^2(0,T;\mathbb{R}^2)} \\ & + \frac{1}{\theta^2} \left(2R \int_0^T \lim_{j\to\infty} \sqrt{| \gamma'_{n_j}(t) |^2 + \theta^2} dt - | \Gamma_1 | \right)^2 \geqslant \hat{J}(\bar{\gamma}^\theta) \end{aligned}$$

and by the definition of L^θ we have $L^\theta \leqslant \hat{J}(\bar{\gamma}^\theta)$. Thus $L^\theta = \hat{J}(\bar{\gamma}^\theta)$.

□

Proposition 13. *Let $(\theta_j)_{j\in\mathbb{N}}$ a sequence of positive numbers tending to 0. Let $\bar{\gamma}^{\theta_j}$ be an optimal control of the penalized control problem $(\mathrm{OCP}^{\theta_j})$. If there exists $\gamma_1 \in U_{ad}$ and a constant c independant of θ_j such that $\hat{J}^{\theta_j}(\gamma_1) \leq c$, then there is a subsequence $(\bar{\gamma}^{\theta_{j_k}})_{k\in\mathbb{N}}$ such that $\bar{\gamma}^{\theta_{j_k}}$ converges strongly to some $\gamma \in H^1(0,T;\mathbb{R}^2)$ as $k \to +\infty$ and*

$$2R \int_0^T | \gamma'(t) | \, dt = | \Gamma_1 | .$$

Proof. Since $\bar{\gamma}^{\theta_j}$ is a minimum then

$$\hat{J}^{\theta_j}(\bar{\gamma}^{\theta_j}) \leq \hat{J}^{\theta_j}(\gamma_1) \leq c, \tag{2.38}$$

hence

$$\| \bar{\gamma}^{\theta_j} \|_{H^2(0,T;\mathbb{R}^2)} \leq \hat{J}^{\theta_j}(\bar{\gamma}^{\theta_j}) \leq c. \tag{2.39}$$

$\bar{\gamma}^{\theta_j}$ being bounded in $H^2(0,T;\mathbb{R}^2)$ it possesses a subsequence $(\bar{\gamma}^{\theta_{j_k}})_{k\in\mathbb{N}}$ weakly convergent to some element γ in $H^2(0,T;\mathbb{R}^2)$. Since $H^2(0,T;\mathbb{R}^2) \overset{c}{\hookrightarrow} C^1([0,T];\mathbb{R}^2)$ then $(\bar{\gamma}^{\theta_{j_k}})_{k\in\mathbb{N}}$ converges strongly to γ in $C^1([0,T];\mathbb{R}^2)$ and therefore in $H^1(0,T;\mathbb{R}^2)$. Now multiplying $\hat{J}^{\theta_j}(\bar{\gamma}^{\theta_{j_k}})$ by $\theta_{j_k}^2$ we get

$$\left(2R\int_0^T \sqrt{\mid \bar{\gamma}^{\theta_{j_k}\prime(t)} \mid^2 +\theta_{j_k}^2}dt - \mid \Gamma_1 \mid\right)^2 \leq \theta_{j_k}^2 c, \tag{2.40}$$

by tending k to $+\infty$ we get

$$2R\int_0^T \mid \gamma'(t) \mid dt = \mid \Gamma_1 \mid . \tag{2.41}$$

□

2.3.2 Differentiability of the control-to-state mapping.

Lemma 14. *The mapping*

$$\begin{array}{rccc} G: & H^2(0,T;\mathbb{R}^2) & \longrightarrow & L^2(0,T;H^1(\Omega)) \\ & \gamma & \longmapsto & y(\gamma) \end{array}$$

is Fréchet differentiable.

Proof. It is the same proof of Lemma 5. It is sufficient to notice that G can be written as a composition of the Fréchet differentiable mappings w, g and q (see [33, p.262]), where w, g and q are defined as follows:

$$\begin{array}{rccc} w: & H^2(0,T;\mathbb{R}^2) & \longrightarrow & C(\bar{\Gamma}_1\times[0,T]) \\ & \gamma & \longmapsto & -c_R \mid \tilde{\gamma}(\gamma) \mid^2 \end{array} \tag{2.42}$$

where $c_R = \frac{2}{R^2}$ and $\tilde{\gamma}(\gamma)(x,t) := x - \gamma(t),\ \forall (x,t) \in \bar{\Gamma}_1 \times [0,T]$,

$$\begin{array}{rccc} g: & C(\bar{\Gamma}_1\times[0,T]) & \longrightarrow & L^2(\Sigma_1) \\ & u & \longmapsto & a\exp(u) \end{array} \tag{2.43}$$

where $a = \alpha\frac{2P}{\pi R^2}$, and

$$\begin{array}{rccc} q: & L^2(\Sigma_1) & \longrightarrow & L^2(0,T;H^1(\Omega)) \\ & g & \longmapsto & y \end{array} \tag{2.44}$$

where y denotes the weak solution of the initial boundary value problem:

$$\begin{cases} \rho\, c\, \partial_t y - \kappa\, \theta y = 0 & \text{in} \quad Q, \\ -\kappa\frac{\partial y}{\partial \nu} = h\, y - g & \text{on} \quad \Sigma_1, \\ -\kappa\frac{\partial y}{\partial \nu} = h\, y & \text{on} \quad \Sigma_2, \\ y = 0 & \text{on} \quad \Sigma_3, \\ y(x,0) = y_0(x) & \text{for} \quad x \in \Omega, \end{cases} \tag{2.45}$$

$y_0 \in L^2(\Omega)$ denoting a fixed initial condition. Therefore we can conclude that

$$\begin{aligned} DG(\gamma)\cdot\delta\gamma &= D(q\circ g\circ w)(\gamma)\cdot\delta\gamma \\ &= v(\gamma,\delta\gamma),\ \text{for all } \delta\gamma\in H^2(0,T;\mathbb{R}^2), \end{aligned}$$

where $v(\gamma,\delta\gamma)$ is a solution of (2.27). □

Lemma 15. *The mapping*

$$l:\gamma\in H^1(0,T;\mathbb{R}^2)\mapsto\int_0^T\sqrt{\mid\gamma'(t)\mid^2+\theta^2}dt \tag{2.46}$$

is Fréchet differentiable.

Proof. We firstly prove that the mapping l is Gâteaux differentiable at every point $\gamma\in H^1(0,T;\mathbb{R}^2)$. Let $\delta\gamma\in H^1(0,T;\mathbb{R}^2)$ and $h>0$

$$\begin{aligned} \frac{l(\gamma+h\delta\gamma)-l(\gamma)}{h} &= \int_0^T\frac{\sqrt{\mid\gamma'(t)+h\delta\gamma'(t)\mid^2+\theta^2}-\sqrt{\mid\gamma'(t)\mid^2+\theta^2}}{h}dt \\ &= \int_0^T\frac{\mid\gamma'(t)+h\delta\gamma'(t)\mid^2-\mid\gamma'(t)\mid^2}{h\left(\sqrt{\mid\gamma'(t)+h\delta\gamma'(t)\mid^2+\theta^2}+\sqrt{\mid\gamma'(t)\mid^2+\theta^2}\right)}dt \\ &= \int_0^T\frac{2h\gamma'(t)\cdot\delta\gamma'(t)+h^2\mid\delta\gamma'(t)\mid^2}{h\left(\sqrt{\mid\gamma'(t)+h\delta\gamma'(t)\mid^2+\theta^2}+\sqrt{\mid\gamma'(t)\mid^2+\theta^2}\right)}dt \\ &= \int_0^T\frac{2\gamma'(t)\cdot\delta\gamma'(t)+h\mid\delta\gamma'(t)\mid^2}{\sqrt{\mid\gamma'(t)+h\delta\gamma'(t)\mid^2+\theta^2}+\sqrt{\mid\gamma'(t)\mid^2+\delta^2}}dt. \end{aligned} \tag{2.47}$$

We have that

$$\begin{aligned} h\int_0^T\frac{\mid\delta\gamma'(t)\mid^2}{\sqrt{\mid\gamma'(t)+h\delta\gamma'(t)\mid^2+\theta^2}+\sqrt{\mid\gamma'(t)\mid^2+\theta^2}} &\le \frac{h}{\theta}\int_0^T\mid\delta\gamma'(t)\mid^2dt \\ &\le \frac{h}{\theta}\parallel\delta\gamma\parallel_{H^1(0,T;\mathbb{R}^2)}\to 0 \text{ as } h\to 0^+. \end{aligned} \tag{2.48}$$

Let $(h_n)_{n\in\mathbb{N}}$ such that $h_n\to 0^+$, since γ' and $\delta\gamma'\in L^2(0,T;\mathbb{R}^2)$ then by Cauchy-Schwartz we have

$$\frac{\gamma'\cdot\delta\gamma'}{\sqrt{\mid\gamma'+h_n\delta\gamma'\mid^2+\theta^2}+\sqrt{\mid\gamma'\mid^2+\theta^2}}\le\frac{1}{\theta}\mid\gamma'\cdot\delta\gamma'\mid\in L^1(0,T),$$

thus by the dominated Lebesgue's convergence theorem we have

$$2\int_0^T\frac{\gamma'(t)\cdot\delta\gamma'(t)dt}{\sqrt{\mid\gamma'(t)+h_n\delta\gamma'(t)\mid^2+\theta^2}+\sqrt{\mid\gamma'(t)\mid^2+\theta^2}}\to\int_0^T\frac{\gamma'(t)\cdot\delta\gamma'(t)}{\sqrt{\mid\gamma'(t)\mid^2+\theta^2}}\text{ as } n\to+\infty.$$

Therefore the Gâteau derivative of l is

$$l'(\gamma;\delta\gamma)=\int_0^T\frac{\gamma'(t)\cdot\delta\gamma'(t)}{\sqrt{\mid\gamma'(t)\mid^2+\theta^2}}dt, \tag{2.49}$$

as it is a continuous linear form in $\delta\gamma$ on $H^1(0,T;\mathbb{R}^2)$.

Let us prove that $\gamma \in H^1(0,T;\mathbb{R}^2) \mapsto l'(\gamma;\cdot) \in (H^1(0,T;\mathbb{R}^2))^*$ is continuous. This will implies that l is Fréchet differentiable [27]. Let $(\gamma_n)_{n\in\mathbb{N}}$ a sequence of $H^1(0,T;\mathbb{R}^2)$ convergent to γ. In fact

$$\begin{aligned} &\sup_{\|\delta\gamma\|_{H^1(0,T;\mathbb{R}^2)}\leq 1} \left| \int_0^T \frac{\gamma_n'(t)\cdot\delta\gamma'(t)}{\sqrt{|\ \gamma_n'(t)\ |^2+\theta^2}}dt - \int_0^T \frac{\gamma'(t)\cdot\delta\gamma'(t)}{\sqrt{|\ \gamma'(t)\ |^2+\theta^2}}dt \right| \\ &\leq \sup_{\|\delta\gamma\|_{H^1(0,T;\mathbb{R}^2)}\leq 1} \left\| \frac{\gamma_n'}{\sqrt{|\ \gamma_n'\ |^2+\theta^2}} - \frac{\gamma'}{\sqrt{|\ \gamma'\ |^2+\theta^2}} \right\|_{L^2(0,T;\mathbb{R}^2)}. \end{aligned} \tag{2.50}$$

By the hypothesis on $(\gamma_n)_{n\in\mathbb{N}}$ and the dominated convergence theorem we have

$$\begin{aligned} &\left\| \frac{\gamma_n'}{\sqrt{|\ \gamma_n'\ |^2+\theta^2}} - \frac{\gamma'}{\sqrt{|\ \gamma'\ |^2+\theta^2}} \right\|^2_{L^2(0,T;\mathbb{R}^2)} \leq \left\| \frac{\gamma_n'}{\sqrt{|\ \gamma_n'\ |^2+\theta^2}} - \frac{\gamma'}{\sqrt{|\ \gamma_n'\ |^2+\theta^2}} \right\|^2_{L^2(0,T;\mathbb{R}^2)} \\ &+ \left\| \frac{\gamma'}{\sqrt{|\ \gamma_n'\ |^2+\theta^2}} - \frac{\gamma'}{\sqrt{|\ \gamma'\ |^2+\theta^2}} \right\|^2_{L^2(0,T;\mathbb{R}^2)} \\ &\leq \frac{1}{\theta^2}\ \|\ \gamma_n-\gamma\ \|_{H^1(0,T;\mathbb{R}^2)} + \int_0^T \left| \frac{1}{\sqrt{|\ \gamma_n'\ |^2+\theta^2}} - \frac{1}{\sqrt{|\ \gamma'\ |^2+\theta^2}} \right|^2 |\ \gamma'(t)\ |^2\ dt \\ &\to 0 \text{ as } n\to+\infty. \end{aligned} \tag{2.51}$$

Therefore by (2.50) and (2.51) $l'(\gamma_n,\cdot) \to l'(\gamma,\cdot)$ in $H^1(0,T;\mathbb{R}^2)^*$. Thus l is Fréchet differentiable.

□

Lemma 16. *The mapping*

$$q : \gamma \in H^2(0,T;\mathbb{R}^2) \mapsto \frac{1}{\theta^2}(2R\int_0^T \sqrt{|\ \gamma'(t)\ |^2+\theta^2}dt - |\ \Gamma_1\ |)^2 \tag{2.52}$$

is Fréchet differentiable.

Proof. We can write q as the composition of the Fréchet differentiable mappings w and q defined as follow:

$$\begin{array}{rccc} w: & H^2(0,T;\mathbb{R}^2) & \longrightarrow & \mathbb{R} \\ & \gamma & \longmapsto & 2R\int_0^T \sqrt{|\ \gamma'(t)\ |^2+\theta^2}dt - |\ \Gamma_1\ |, \end{array} \tag{2.53}$$

and

$$\begin{array}{rccc} g: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & u & \longmapsto & \frac{1}{\theta^2}u^2. \end{array} \tag{2.54}$$

w is Fréchet differentiable by the previous lemma and its Fréchet derivative is given by (2.49).

For all $\delta u \in \mathbb{R}$

$$(u+\delta u)^2 = u^2 + \delta u^2 + 2\,u\,\delta u, \tag{2.55}$$

so it is clear that

$$Dg(u)\cdot\delta u = \frac{2}{\theta^2} u\,\delta u. \tag{2.56}$$

Thus

$$\begin{aligned} Dq(\gamma)\cdot\delta\gamma &= D(g\circ w)(\gamma)\cdot\delta\gamma \\ &= Dg(w(\gamma))\cdot Dw(\gamma)\cdot\delta\gamma \\ &= \frac{2}{\theta^2}\left(2R\int_0^T \sqrt{\mid\gamma'(t)\mid^2+\theta^2}dt - \mid\Gamma_1\mid\right)\left(2R\int_0^T \frac{\gamma'(t)\cdot\delta\gamma'(t)}{\sqrt{\mid\gamma'(t)\mid^2+\theta^2}}dt\right) \end{aligned} \tag{2.57}$$

□

Using the previous lemmas, we obtain:

$$\begin{aligned} D\hat{J}^\theta(\gamma)\cdot\delta\gamma = &\int_0^T\int_\Omega \nabla G(\gamma)(x,t)\cdot\nabla v(\gamma,\delta\gamma)(x,t)\,dxdt \\ &+\lambda_Q\int_0^T\int_\Omega G(\gamma)(x,t)\,v(\gamma,\delta\gamma)(x,t)\,dxdt - \lambda_Q\int_0^T\int_\Omega y_Q(x,t)\,v(\gamma,\delta\gamma)(x,t)\,dxdt \\ &+\lambda_\gamma\int_0^T\gamma(t)\cdot\delta\gamma(t)\,dt + \lambda_\gamma\int_0^T\gamma'(t)\cdot\delta\gamma'(t)\,dt + \lambda_\gamma\int_0^T\gamma''(t)\cdot\delta\gamma''(t)dt, \\ &+\frac{2}{\theta^2}\left(2R\int_0^T \sqrt{\mid\gamma'(t)\mid^2+\theta^2}dt - \mid\Gamma_1\mid\right)\left(2R\int_0^T \frac{\gamma'(t)\cdot\delta\gamma'(t)}{\sqrt{\mid\gamma'(t)\mid^2+\theta^2}}dt\right). \end{aligned} \tag{2.58}$$

where $v(\gamma,\delta\gamma)$ is a solution of (2.27).

2.3.3 Adjoint equation and necessary optimality conditions

The adjoint system of our problem is the following linear backward boundary value problems

$$\begin{cases} \rho\,c\,\partial_t p + \kappa\,\Delta p = \Delta G(\bar\gamma^\theta) - \lambda_d(G(\bar\gamma^\theta) - y_Q) & \text{in} \quad Q, \\ \kappa\frac{\partial p}{\partial\nu} + h\,p = \frac{\partial G(\bar\gamma^\theta)}{\partial\nu} & \text{on} \quad \Gamma\times]0,T[, \\ p(.,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.59}$$

Theorem 17. *If $\bar\gamma^\theta \in U_{ad}$ is an optimal control of (OCP$^\theta$) with associated state $G(\bar\gamma^\theta)$, and $p \in W(0,T)$ the corresponding adjoint state that solves (2.59), then the variational inequality*

$$\begin{aligned} &D\hat{J}^\theta(\gamma)\cdot\delta\gamma = \lambda_\gamma\int_0^T\bar\gamma^\theta(t)\cdot(\delta\gamma)(t)dt + \lambda_\gamma\int_0^T(\bar\gamma^\theta)'(t)\cdot(\delta\gamma)'(t)dt + \lambda_\gamma\int_0^T(\bar\gamma^\theta)''(t)\cdot(\delta\gamma)''(t)dt, \\ &+\frac{2}{\theta^2}\left(2R\int_0^T \sqrt{\mid(\bar\gamma^\theta)'(t)\mid^2+\theta^2}dt - \mid\Gamma_1\mid\right)\left(2R\int_0^T \frac{(\bar\gamma^\theta)'(t)\cdot(\delta\gamma)'(t)}{\sqrt{\mid(\bar\gamma^\theta)'(t)\mid^2+\theta^2}}dt\right) \\ &+2ac_R\int\int_{\Sigma_1}\exp(w(\bar\gamma^\theta)(x,t))\tilde\gamma(\bar\gamma^\theta)(x,t)\cdot(\delta\gamma)(t)p(x,t)dS(x)dt \geq 0 \end{aligned} \tag{2.60}$$

holds for all $\delta\gamma \in U_{ad}$.

Proof. For the proof see theorem 9. □

$D\hat{J}^\theta(\gamma)$ is a continuous linear mapping on $H^2(0,T;\mathbb{R}^2)$. Thus by Riesz representation theorem it is the scalar product by a unique element of $H^2(0,T;\mathbb{R}^2)$ that we denote by $\nabla_{H^2}\hat{J}(\gamma)$:

$$D\hat{J}^\theta(\gamma)\cdot\delta\gamma = (\nabla_{H^2}\hat{J}^\theta(\gamma),\delta\gamma)_{H^2(0,T;\mathbb{R}^2)} \text{ for all } \delta\gamma \in H^2(0,T;\mathbb{R}^2).$$

Proposition 18. $\nabla_{H^2}\hat{J}^\theta(\gamma)$ *can be obtained by solving the following boundary value problem on the time interval* $(0,T)$

$$\begin{cases} v^{(4)}(t) - v''(t) + v(t) = f(t) \\ v'(0) = v^{(3)}(0), \\ v'(T) = v^{(3)}(T), \\ v''(0) = 0, \\ v''(T) = 0. \end{cases} \tag{2.61}$$

and

$$\begin{cases} u^{(4)}(t) - u''(t) + u(t) = -q''(t) \\ -u'(0) + u^{(3)}(0) = -q'(0), \\ -u'(T) + u^{(3)}(T) = -q'(T), \\ u''(0) = 0, \\ u''(T) = 0. \end{cases} \tag{2.62}$$

with

$$f(t) = 2ac_R \int_{\Gamma_1} exp(w(\gamma)(x,t))\tilde{\gamma}(\gamma)(x,t)p(x,t)dS(x) \tag{2.63}$$

and

$$q'(t) = \frac{2}{\theta^2}\left(2R\int_0^T \sqrt{\mid(\bar{\gamma}^\theta)'(t)\mid^2 + \theta^2}dt - \mid\Gamma_1\mid\right)\left(2R\frac{(\bar{\gamma}^\theta)'(t)}{(\sqrt{\mid(\bar{\gamma}^\theta)'(t)\mid^2+\theta^2}}\right). \tag{2.64}$$

Then $\nabla_{H^2}\hat{J}^\theta(\gamma) = v + u + \lambda_\gamma\gamma$.

Proof. Let us set

$$\tilde{J} : U_{ad} \to \mathbb{R} : \gamma \mapsto \frac{1}{2}\int\int_Q \mid\nabla y(\gamma)\mid^2(x,t)dxdt + \frac{\lambda_Q}{2}\int\int_Q \mid y(\gamma) - y_Q\mid^2(x,t)dxdt \tag{2.65}$$

By formula (2.60)

$$D\tilde{J}(\gamma)\cdot\delta\gamma = 2ac_R\int_0^T\int_{\Gamma_1} exp(w(\gamma)(x,t))\tilde{\gamma}(\gamma)(x,t)\cdot(\delta\gamma)(t)p(x,t)dS(x)dt, \tag{2.66}$$

for all $\delta\gamma \in H^2(0,T;\mathbb{R}^2)$. This latest expression shows immediately that $\tilde{J}'(\gamma)\cdot\delta\gamma$ is equal to the L^2-scalar product of $\delta\gamma \in H^2(0,T;\mathbb{R}^2) \subset L^2(0,T;\mathbb{R}^2)$ with the square integrable function on the time interval $(0,T)$

$$f : t \mapsto 2ac_R\int_{\Gamma_1} exp(w(\gamma)(x,t))\tilde{\gamma}(\gamma)(x,t)p(x,t)dS(x). \tag{2.67}$$

Now, we want to write $\delta\gamma \mapsto \tilde{J}'(\gamma)\cdot\delta\gamma$ as the $H^2(0,T;\mathbb{R}^2)$-scalar product of $\delta\gamma$ with some $H^2(0,T;\mathbb{R}^2)$-function v. Thus we must have

$$(v,\delta\gamma)_{H^2(0,T;\mathbb{R}^2)} = (f,\delta\gamma)_{L^2(0,T;\mathbb{R}^2)}, \quad \forall\delta\gamma \in H^2(0,T;\mathbb{R}^2) \tag{2.68}$$

i.e.

$$\int_0^T v(t)\cdot\delta\gamma(t)dt + \int_0^T v'(t)\cdot\delta\gamma'(t)dt + \int_0^T v''(t)\cdot\delta\gamma''(t) = \int_0^T f(t)\delta\gamma(t)dt \tag{2.69}$$

for all $\delta\gamma \in H^2(0,T;\mathbb{R}^2)$. That v exists and is unique folllows from the Riesz representation theorem. Taking $\delta\gamma \in C_c^\infty(0,T;\mathbb{R}^2)$ and integrating by parts, we obtain

$$v^{(4)} - v'' + v = f. \tag{2.70}$$

f being a square integrable function on $(0,T)$, it follows from (2.70) that $v \in H^4(0,T;\mathbb{R}^2)$. Making an integration by part in (2.69), we obtain that

$$\begin{aligned} v'(0) &= v^{(3)}(0),\\ v'(T) &= v^{(3)}(T),\\ v''(0) &= 0,\\ v''(T) &= 0. \end{aligned} \tag{2.71}$$

Thus $\tilde{J}'(\gamma)\cdot\delta\gamma = (v,\delta\gamma)_{H^2(0,T;\mathbb{R}^2)}$.

Let us set

$$q'(t) = \frac{2}{\theta^2}\left(2R\int_0^T \sqrt{|\ (\bar{\gamma}^\theta)'(t)\ |^2 + \theta^2}dt - |\ \Gamma_1\ |\right)\left(2R\frac{(\bar{\gamma}^\theta)'(t)}{(\sqrt{|\ (\bar{\gamma}^\theta)'(t)\ |^2 + \theta^2}}\right). \tag{2.72}$$

We want to find $u \in H^2(0,T;\mathbb{R}^2)$ such that

$$(u,\delta\gamma)_{H^2(0,T;\mathbb{R}^2)} = (q'(t),\delta\gamma')_{L^2(0,T;\mathbb{R}^2)}. \tag{2.73}$$

Taking $\delta\gamma \in C_c^\infty(0,T;\mathbb{R}^2)$, we obtain

$$u^{(4)}(t) - u''(t) + u(t) = -q''(t)$$

in the weak sense q'' being square integrable on $(0,T)$. This implies that $u \in H^4(0,T;\mathbb{R}^2)$. Performing an integration by parts in (2.73) we obtain the boundary conditions

$$\begin{cases} -u'(0) + u^{(3)}(0) = -q'(0),\\ -u'(T) + u^{(3)}(T) = -q'(T),\\ u''(0) = 0,\\ u''(T) = 0. \end{cases} \tag{2.74}$$

Knowing that

$$\hat{J}^\theta(\gamma) \coloneqq \tilde{J}(\gamma) + \frac{\lambda_\gamma}{2}\parallel\gamma\parallel^2_{H^2(0,T;\mathbb{R}^2)} + \frac{1}{\theta^2}\left(\int_0^T \sqrt{|\ \gamma'(t)\ |^2 + \theta^2}dt - |\ \Gamma_1\ |\right)^2,$$

we have

$$\begin{aligned} D\hat{J}^\theta(\gamma)\cdot\delta\gamma &= D\tilde{J}(\gamma)\cdot\delta\gamma + \lambda_\gamma(\gamma,\delta\gamma)_{H^2(0,T;\mathbb{R}^2)} + (u,\delta\gamma)_{H^2(0,T;\mathbb{R}^2)}\\ &= (v+u+\lambda_\gamma\gamma,\delta\gamma)_{H^2(0,T;\mathbb{R}^2)}. \end{aligned}$$

Thus

$$\nabla_{H^2}\hat{J}^\theta = v + u + \lambda_\gamma\gamma.$$

□

2.4 Numerical approximations

This section is devoted to the discretization of the penalized optimal control problem (OCP^{θ}). In subsection 2.4.1 we present the main optimization algorithms useful for our study. Our numerical tests are in a two dimensional setup which is presented in subsection 2.4.2. In subsection 2.4.2 the parametrization of the laser path is introduced. In subsection 2.4.4 , we perform preliminary simulations to show how we coupled the Heat equation to the parametrized path. The discretization optimization problem is presented without constraint in subsection 2.4.5 to show how we handled the discretisation of the associated adjoint problem and the necessary optimality condition in presence of the parametrized path. Finally, in subsection 2.4.6 the fully constrained optimization problem is discretized.

2.4.1 Optimization algorithms

In this section we recall the main optimization algorithms we have used. We will not discuss convergence analysis for these algorithms. Details about these methods can be found in the following references: [28, Chapter 2], [42, Chapter 17].

The gradient descent algorithm

We present a descent method for simply constrained problem of the form

$$\min_{x \in U} f(x)$$

with $U = \mathbb{R}^n$, $f : U \to \mathbb{R}$ continuously Fréchet differentiable. We know in order to minimize a convex function, we need to find a stationary point. One possible approach is to start at an arbitrary point, and move in the opposite side of the gradient at that point towards the next point, and repeat until converging to a stationary point. In general, one can consider a search for a stationary point as having two components: the direction and the step size. The direction decides which direction we search next, and the step size determines how far we go in that particular direction. Such methods can be generally described as starting at some arbitrary point x_0 and then at every step $k \geq 0$ iteratively moving in the direction d_k by step size σ_k to the next point, $x_{k+1} = x_k + \sigma_k d_k$. In gradient descent, the direction is the negative gradient at the point, i.e. $d = -\nabla f(x)$. Thus, the iterative search of gradient descent can be described through the following recursive rule:

$$x_{k+1} = x_k - \sigma_k \nabla f(x_k).$$

Since our objective is to minimize the function, one reasonable approach is to choose the step size in manner that will deminish the value at the new point, i.e. find the step size that minimizes $f(x_{k+1})$. Namely, choose a descent direction σ_k such that

$$f(x_{k+1} + \sigma_k d_k) < f(x_k).$$

Formally, given a desired precision $\epsilon > 0$, we define the gradient descent algorithm as described below.

Algorithm 1: Gradient descent

0. Choose an initial point $x_0 \in U$;
while $\| \nabla f(x_k) \| \geq \epsilon$ **do**
  1. Set $d_k = -\nabla f(x_k)$;
  2. Choose $\sigma_k > 0$ such that $f(x_k + \sigma_k d_k) < f(x_k)$;
  3. Set $x_{k+1} = x_k + \sigma_k d_k$;
  4. $k \leftarrow k + 1$;
end
5. return x_k;

The projected gradient algorithm

We now present a descent method for simply constrained problem of the form

$$\begin{aligned} & \min_{x \in U} f(x), \\ \text{subject to} \quad & a \leq x \leq b, \end{aligned}$$

with $U = \mathbb{R}^n$, $f : U \to \mathbb{R}$ is a Fréchet differentiable function and a, $b \in \mathbb{R}^n$, $a \leq b$ are positive constants. The presence of the constraint set $[a, b]$ requires to take care that we stay feasible with respect to $[a, b]$. The projected gradient descent algorithm uses an initial point x_0, and then updates it for $k = 1, 2, 3, \cdots$ by first performing gradient descent on the current solution and then projecting it back onto the constraint set. This can be expressed as

$$x_{k+1} = \mathbb{P}_{[a,b]}(x_k + \sigma_k d_k)$$

where $\mathbb{P}_{[a,b]}$ is the corresponding orthogonal projection into $[a, b]$:

$$\mathbb{P}_{[a,b]}(y) = \max(a, \min(y, b)). \tag{2.75}$$

Formally, given a desired precision $\epsilon > 0$, we define the gradient descent algorithm as described below.

Algorithm 2: Projected Gradient descent

0. Choose an initial point $x_0 \in U$;
while $\| x_k - \mathbb{P}_{[a,b]}(x_k + \sigma_k d_k) \| \geq \epsilon$ **do**
  1. Set $d_k = -\nabla f(x_k)$;
  2. Choose $\sigma_k > 0$ such that $f(x_k + \sigma_k d_k) < f(x_k)$;
  3. Set $x_{k+1} = \mathbb{P}_{[a,b]}(x_k + \sigma_k d_k)$;
  4. $k \leftarrow k + 1$;
end
5. return x_k;

Remark 4. The motivation of the stopping criterium in algorithm 2, is that if x_k would be equal to $\mathbb{P}_{[a,b]}(x_k + \sigma_k d_k)$, this would mean that

$$(\nabla f(x_k), y - x_k)_{\mathbb{R}} \geq 0, \text{ for all } y \in U \text{ such that } y \in [a, b].$$

This is an approximation of the necessary condition $f'(\bar{x})(y - x) \geq 0$ for all $y \in [a, b]$, that every local minimum $\bar{x}$ must satisfy.

The augmented-Lagrangian algorithm

Augmented Lagrangian methods are a certain class of algorithms for solving constrained optimization problems. They have similarities to quadratic penalty methods in that they replace a constrained optimization problem by a series of unconstrained problems and add a penalty term to the objective. Namely, the unconstrained objective is the Lagrangian of the constrained problem, with an additional penalty term (the augmentation). The augmented-Lagrangian method reduces the possibility of ill conditioning of the subproblems that are generated in the quadratic penalty approach (see section 17.1 and 17.4 of [42] for more explanation about the ill conditioning). We are interesred in the following optimization problem with inequalities constraint.

$$\begin{aligned} & \min_{x\in U} f(x), \\ \text{subject to} \quad & c(x) \geq 0, \end{aligned}$$

with $U = \mathbb{R}^n$ and the functions $f : U \to \mathbb{R}$, $c : U \to \mathbb{R}$ are Fréchet differentiable. We consider the Lagrangian function

$$\begin{array}{rccl} \mathcal{L}: & (U, \mathbb{R}, \mathbb{R}^+) & \longrightarrow & \mathbb{R} \\ & (x, \lambda; \mu) & \longmapsto & \mathcal{L}(x, \lambda; \mu) := f(x) + \psi(c(x), \lambda; \mu) \end{array}$$

with

$$\psi(c(x), \lambda; \mu) = \begin{cases} \lambda c(x) + \frac{1}{2\mu}(c(x))^2, & if \quad -c(x) - \mu\lambda \leq 0 \\ \frac{\mu}{2}(\lambda)^2, & otherwise. \end{cases} \tag{2.76}$$

We want to solve the following unconstrained optimization problem:

$$\min_x \mathcal{L}(x, \lambda; \mu). \tag{2.77}$$

Once the approximate solution x_k is obtained, we use the following formulas to update the Lagrange multipliers

$$\lambda_{k+1} = \max(\lambda_k - c(x_k)/\mu_k, 0), \tag{2.78}$$

Algorithm 4 gives the augmented Lagrangian method used in this section.

Algorithm 3: Augmented Lagrangian

0. Choose an initial point $x_0^s \in U$, $\mu_0 > 0$, tolerance $\tau > 0$ and $\lambda_k > 0$;
for $k = 0, 1, 2, \cdots$ **do**
  1. Starting from x_k^s, use the gradient descent minimization algorithm 1 to find an approximate minimizer x_k of $\mathcal{L}(\cdot, \lambda_k; \mu_k)$;
  if $\| \nabla_x \mathcal{L}(x_k, \lambda_k; \mu_k) \| \leq \tau$ **then**
    iteration accepted **STOP** with approximate solution x_k;
  else
    2. Update Lagrange multiplier by formula (2.78);
    3. Choose new penalty parameter $\mu_{k+1} \in (0, \mu_k)$;
    4. Set starting point for the next iteration with $x_{k+1}^s = x_k$;
  end
end

The Lancelot algorithm

We consider an optimization problem of the form

$$\begin{aligned} & \min_{x\in U} f(x), \\ \text{subject to} \quad & c(x) \geq 0 \text{ and } a \leq x \leq b, \end{aligned}$$

with $U = \mathbb{R}^n$, the function $f : U \to \mathbb{R}$, $c : U \to \mathbb{R}$ are Fréchet differentiable functions and a, b are positive constants. The algorithm designed to solve this optimization problem is a combination between the projected gradient descent algorithm 2 and the augmented Lagrangien algorithm 4.

Algorithm 4: Augmented Lagrangian

0. Choose an initial point $x_0^s \in U$, $\mu_0 > 0$, tolerance $\tau > 0$ and $\lambda_k > 0$;
for $k = 0, 1, 2, \cdots$ **do**
  1. Starting from x_k^s, use the projected gradient descent minimization algorithm 2 to find an approximate minimizer x_k of $\mathcal{L}(\cdot, \lambda_k; \mu_k)$;
  if $\| \nabla_x \mathcal{L}(x_k, \lambda_k; \mu_k) \| \leq \tau$ **then**
    iteration accepted STOP with approximate solution x_k;
  else
    2. Update Lagrange multiplier by formula (2.78);
    3. Choose new penalty parameter $\mu_{k+1} \in (0, \mu_k)$;
    4. Set starting point for the next iteration to $x_{k+1}^s = x_k$;
  end
end

2.4.2 Two dimensional laser path optimization model

In order to simplify the numerical tests we restrict ourselves to the two dimensional model proposed in [9, 38] with a slight difference that we have considered non-homogenous Robin boundary condition on Γ to take into account heat exchange with the exterior environment. The model describing temperature distribution within a single layer of thickness δ is given by:

$$\begin{cases} \rho\, c\, \partial_t y - \kappa\, \Delta y = -\frac{h}{\delta}\,(y - y_0) + \frac{g_\gamma}{\delta} & \text{in} \quad Q = \Omega \times]0, T[\,, \\ -\kappa \frac{\partial y}{\partial \nu} = \frac{h}{\delta}\,(y - y_0) & \text{on} \quad \Sigma = \Gamma \times]0, T[\,, \\ y(x, 0) = y_0(x) & \text{for} \quad x \in \Omega, \end{cases} \tag{2.79}$$

Here Ω is supposed to be equal to $[0, 1]^2$ and g_γ represents the Gaussian laser beam given by:

$$g_\gamma(x, t) = \alpha \frac{2P}{\pi R^2} \exp\left(-2\frac{|\, x - \gamma(t)\,|^2}{R^2}\right), \text{ for all } (x, t) \in \Omega \times [0, T], \tag{2.80}$$

where $\gamma : t \in [0, T] \to \Omega$ represents the displacement of the laser beam center on Ω with respect to time. We want to compute the discrete solution of the following optimization problem.

Given $\theta > 0$, we want to find a discrete solution of the following minimization problem

$$\min_{\gamma \in U_{ad}^p} \hat{J}^\theta(\gamma). \tag{2.81}$$

where,

$$\hat{J}^{\theta}(\gamma) := \frac{1}{2}\int_0^T \int_\Omega \mid \nabla y(\gamma)(x,t) \mid^2 dx\, dt + \frac{1}{\theta^2}\left(2R\int_0^T \sqrt{\mid \gamma'(t) \mid^2 + \theta^2} dt - \mid \Omega \mid\right)^2 + \frac{\lambda_\gamma}{2} \parallel \gamma \parallel^2_{H^2(0,T;\mathbb{R}^2)}, \tag{2.82}$$

and U^p_{ad} is given (2.31). The existence of an optimal control and the determination of the first order optimality condition can be done using the same arguments as those mentionned in section 2.3. We give the adjoint equation and the first order necessary optimality condition without proofs.

The adjoint system of (2.79) is the following linear backward boundary value problem

$$\begin{cases} \rho\, c\, \partial_t p + \kappa\, \Delta p = -\frac{h}{\delta}(p - y_0) + \Delta y(\bar{\gamma}^\theta) & \text{in} \quad Q, \\ \frac{\partial y(\bar{\gamma}^\theta)}{\partial \nu} - \kappa \frac{\partial p}{\partial \nu} = \frac{h}{\delta}(p - y_0) & \text{on} \quad \Sigma, \\ p(.,T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.83}$$

where $G(\bar{\gamma}^\theta)$ is the associated state to an optimal control $\bar{\gamma}^\theta \in U_{ad}$.

If $\bar{\gamma}^\theta \in U^p_{ad}$ is an optimal control with associated state $y(\bar{\gamma}^\theta)$, and $p \in W(0,T)$ the corresponding adjoint state that solves (2.83), then the variational inequality

$$\begin{aligned} D\hat{J}^{\theta}(\gamma)\cdot\delta\gamma &= \lambda_\gamma \int_0^T \bar{\gamma}^\theta(t)\cdot(\delta\gamma)(t)dt + \lambda_\gamma \int_0^T (\bar{\gamma}^\theta)'(t)\cdot(\delta\gamma)'(t)dt + \lambda_\gamma \int_0^T (\bar{\gamma}^\theta)''(t)\cdot(\delta\gamma)''(t)dt, \\ &+ \frac{2}{\theta^2}\left(2R\int_0^T \sqrt{\mid (\bar{\gamma}^\theta)'(t) \mid^2 + \theta^2} dt - \mid \Gamma_1 \mid\right)\left(2R\int_0^T \frac{(\bar{\gamma}^\theta)'(t)\cdot(\delta\gamma)'(t)}{\sqrt{\mid (\bar{\gamma}^\theta)'(t) \mid^2 + \theta^2}} dt\right) \\ &+ 2ac_R \int\int_\Omega \exp(w(\bar{\gamma}^\theta)(x,t))\tilde{\gamma}(\bar{\gamma}^\theta)(x,t)\cdot(\delta\gamma)(t)p(x,t)dS(x)dt \geq 0 \end{aligned} \tag{2.84}$$

holds for all $\delta\gamma \in U^p_{ad}$.

Therefore, we can now compute $\nabla_{H^2}\hat{J}^{\theta}(\gamma)$ which is given by

$$\nabla_{H^2}\hat{J}^{\theta}(\gamma) = v + u + \lambda_\gamma \gamma, \tag{2.85}$$

where v is a solution of (2.61) ans u is a solution of (2.62).

2.4.3 Path and Heat equation discretization

To parametrize the laser path we use cubic Hermite spline basis. This choice will allow us to obtain smoother laser paths and therefore to have control on the path derivative. From an industrial point of view, we believe that the use of C^1 curves will be more efficient than C^0 curves since smoother curves will diminish thermal gradients. Furthermore this kind of paths has been used in many additive manufacturing technologies [26, 63].

We start by recalling the cubic Hermite spline basis on the unit interval $[0,1]$. Given a starting point p_0 at $t=0$ and an ending point p_1 at $t=1$ with starting tangent m_0 at $t=0$ and ending tangent m_1 at $t=1$, the polynomial of interpolation is given by

$$P(t) = H^0(t)p_0 + \hat{H}^0(t)m_0 + H^1(t)p_k + \hat{H}^1(t)m_1, \tag{2.86}$$

where H refers to the basis functions (see figure 2.7):

- $H^0(t) = 2t^3 - 3t^2 + 1$,
- $\hat{H}^0(t) = t^3 - 2t^2 + t$,
- $H^1(t) = -2t^3 + 3t^2$,
- $\hat{H}^1(t) = t^3 - t^2$.

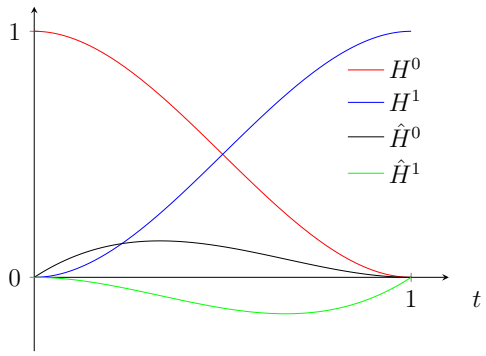


Figure 2.7: Hermite basis functions.

Interpolation in an arbitrary interval $[t_{k-1}, t_k]$ is done using the following formula [22]

$$\begin{aligned} P(t) =& H^0(t^{(k)}(t))p_{k-1} + \hat{H}^0(t^{(k)}(t))(t_k - t_{k-1})m_{k-1} + H^1(t^{(k)}(t))p_k \\ &+ \hat{H}^1(t^{(k)}(t))(t_k - t_{k-1})m_k, \end{aligned} \tag{2.87}$$

with

$$t^{(k)}(t) = \frac{(t - t_{k-1})}{(t_k - t_{k-1})} \text{ for all } t \in [t_{k-1}, t_k], \tag{2.88}$$

$$p_{k-1} = P(t_{k-1}),\ p_k = P(t_k),\ m_{k-1} = P'(t_{k-1}),\ m_k = P'(t_k). \tag{2.89}$$

Let us consider the uniform subdivision $t_0 = 0 < t_1 < \cdots < t_N = T$ of the time interval $[0, T]$ with uniform step size $\Delta_0 = \frac{T}{N}$. We describe the path γ using cubic Hermite spline basis where the degree of freedom are both the value and the derivative at each value:

$$\begin{aligned} \gamma(t) =& \begin{pmatrix} \alpha^1_{k-1} \\ \alpha^2_{k-1} \end{pmatrix} H^0(t^{(k)}(t)) + \Delta_0 \begin{pmatrix} \beta^1_{k-1} \\ \beta^2_{k-1} \end{pmatrix} \hat{H}^0(t^{(k)}(t)) \\ &+ \begin{pmatrix} \alpha^1_k \\ \alpha^2_k \end{pmatrix} H^1(t^{(k)}(t)) + \Delta_0 \begin{pmatrix} \beta^1_k \\ \beta^2_k \end{pmatrix} \hat{H}^1(t^{(k)}(t)), \text{ for all } t \in [t_{k-1}, t_k],\ k = 1 \cdots, N. \end{aligned} \tag{2.90}$$

Then we discretize (2.79) by using, the $\mathbb{P}^1$-finite element method in space and the implicit Euler method in time. Let us consider the uniform subdivision $t_0 = 0 < t_1 < \cdots < t_{nN} = T$ of the time interval $[0, T]$ with step size $\Delta_1 = \frac{\Delta_0}{n}$ where n represents the number of points in which we want to simulate the temperature between the time interval $[t_{k-1}, t_k]$ for all $k = 1, \cdots, N$. In order to avoid to use too many optimization parameters but to have a good approximation of the solution of the heat equation, we have decided to use

a rough paramatrization t_k, $1 \leq k \leq N$, for the path and a finer parametrization (t_{jk}), $1 \leq k \leq N$, $1 \leq j \leq n$, for the use of Euler's scheme. We then discretize (2.79) using the implicit Euler method where we denote by:

$$Y^j \simeq y(x, j\,\Delta_1) \quad , \tag{2.91}$$

where $\frac{\partial y}{\partial t}(x, j\Delta_1)$ is replaced by

$$\frac{Y^j - Y^{j-1}}{\Delta_1}$$

we are reduced to find Y^j For given $((\alpha_k^1, \alpha_k^2, \beta_k^1, \beta_k^2))_{k\in\{1,\cdots,N\}}$ using the path γ given by formula (2.90) then we find Y^j solution of

$$\begin{cases} \rho\, c\, \frac{Y^j - Y^{j-1}}{\Delta_1} - \kappa\, \Delta Y^j = -\frac{h}{\delta}\,(Y^j - Y_0) + \frac{g_\gamma}{\delta}(x, j\Delta_1) & \text{for all} \quad j = 1, \cdots, nN, \\ -\kappa \frac{\partial Y^j}{\partial \nu} = \frac{h}{\delta}\,(Y^j - Y_0) & \text{for all} \quad j = 1, \cdots, nN, \\ Y^0 = y_0. \end{cases} \tag{2.92}$$

Considering a regular family of triangulation $(\mathcal{T}_h)_{h>0}$ on $\bar{\Omega}$, using the $\mathbb{P}_1$- finite element method we arrive at the fully discrete approximation of (2.79). Find $Y_h^j \in V_h$ solution of

$$\begin{aligned} &\rho\, c \int_\Omega Y_h^j\, v_h dx + \Delta_1 \kappa \int_\Omega \nabla Y_h^j \cdot \nabla v_h dx + \Delta_1 \frac{h}{\delta} \int_\Omega (Y_h^j - Y_{0,h})\, v_h dx \\ &+ \Delta_1 \frac{h}{\delta} \int_\Gamma (Y_h^j - Y_{0,h})\, v_h dS - \Delta_1 \int_\Omega \frac{g_\gamma}{\delta}(x, j\Delta_1)\, v_h dS \\ &- \rho\, c \int_\Omega Y_h^{j-1}\, v_h dx = 0 \quad \text{for all } v_h \in V_h, \end{aligned} \tag{2.93}$$

where the subset V_h is a conforming approximation of $H^1(\Omega)$.

Remark 5. The choice of the implicit Euler method is to avoid a CFL condition, thus the error will only depend on the time and space step size.

2.4.4 Preliminary numerical results

Our aim is to study the influence of some parametric curves on the distribution of temperature and thermal gradient in the selective laser melting process. Namely, given a parametric curve in the right hand side of the heat equation we compute the maximum temperature reached during the process and the L^2 norm of the temperature gradient.

For the thermal properties we have used the data mentionned in [12], where they used Titanium Alloy Ti6Al4V powder (see table 2.1).

We have tested four types of paths given by (2.90), we mention in table 2.2 all the data used for the FEM and the geometry. The numerical tests presented in this chapter are performed with the help of Python [56] and the open-source software package Netgen/NGSolve [46], a FEM library with Python interface.

When we compare the result presented in table 2.3, we can conclude that the spiral paths has the largest maximum temperature while the interior spiral path induce the most important thermal gradients. On the other hand we notice that the basic path has the lowest maximum temperature and the zigzag path induce the lowest thermal gradients. Therefore these results are coherent with those presented by Vanbelle et al. [55, 54] and Cheng et al. [12].

Titanium Alloy Ti6Al4V	
Laser power	$P = 200\,W$
Laser velocity	$1\,m.s^{-1}$
Laser radius	$R = 50 \times 10^{-6}\,m$
Layer thickness	$\delta = 40 \times 10^{-6}\,m$
Initial temperature	$y_0 = 273^\circ\,C$
Heat exchange coefficient	$h = 10\,W.m^{-2}.^\circ C$
Powder density	$\rho = 4000\,kg.m^{-3}$
Thermal conductivity	$\kappa = 0.25\,W.m^{-1}.^\circ C^{-1}$
Calorific capacity	$c = 450 J.kg^{-1}.^\circ C^{-1}$

Table 2.1: Thermal parameters.

FEM & Geometry	
Fem step size	$h = \frac{2}{100}$
Number of points	$N = 50$
Number of intermediate points	$n = 10$
Final time	$t_f = 1$
Dimension of Ω	$[0,1] \times [0,1]$

Table 2.2: Data used for the FEM and the geometry.

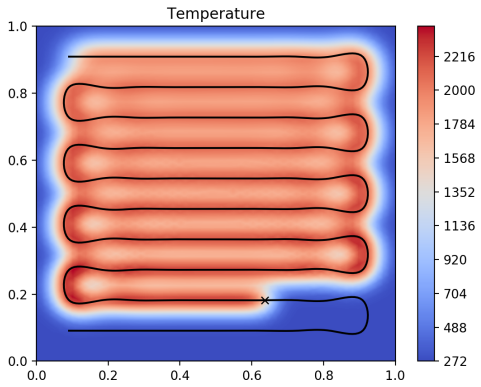


Figure 2.8: Basic path at time $t = 0.87$.

2.4.5 Discretization of the unconstraint optimization problem

In order to explain how we discretize the adjoint equation and $\nabla_{H^2}\hat{J}$ we will consider the optimization problem presented in subsection 2.4.2 without the constraints on γ and the geometrical constraint

$$\frac{1}{\theta^2}\left(2R\int_0^T \sqrt{\mid \gamma'(t) \mid^2 + \theta^2}dt - \mid \Omega \mid\right)^2, \ \theta > 0.$$

The control space $U := H^2(0,T;\mathbb{R}^2)$ is discretized using the cubic Hermite spline basis . Let us consider the uniform subdivision $t_0 = 0 < t_1 < \cdots < t_N = T$ of the time interval

Path	max. temperature	$\| \nabla y \|_{L^2(\Omega)}$
Basic (figure 2.8)	2446	258485
Interior spiral (figure 2.9)	2763	326636
Exterior spiral (figure 2.10)	2745	276869
Zigzag (figure 2.11)	2714	234869

Table 2.3: Different paths tested.

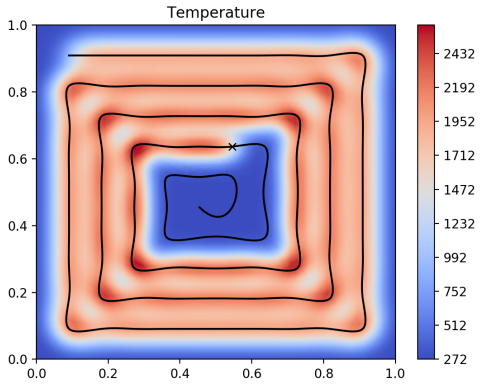


Figure 2.9: Interior spiral path at time t=0.87.

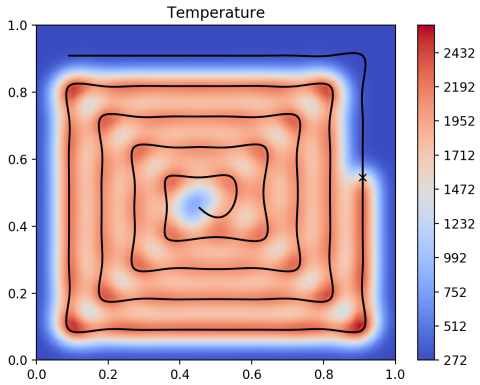


Figure 2.10: Exterior spiral path at time t=0.87.

$[0, T]$.

We replace U by

$$U^N := \{\gamma \in C^1([0,T];\mathbb{R}^2);\ \gamma(t) \text{ is given by (2.90)} ;((\alpha_k^1, \alpha_k^2, \beta_k^1, \beta_k^2))_{k\in\{1,\cdots,N\}} \subset \mathbb{R}^{4N}\}. \tag{2.94}$$

The adjoint system associated to (2.79) is discretized using the $\mathbb{P}_1$-finite element method in space and the Euler implicit backward scheme in time. Similarly to the discrete state equation (2.92) we have used a finer paramatrization (t_{jk}), $1 \leq k \leq N$, $1 \leq j \leq n$ for the

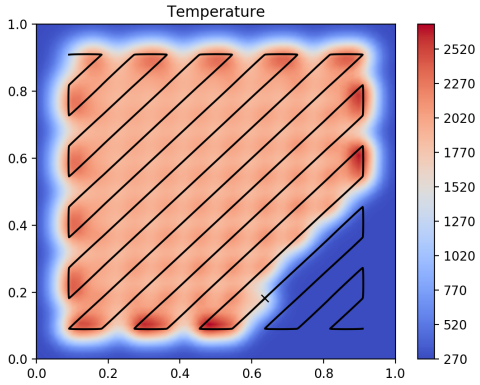


Figure 2.11: Zigzag path at time $t = 0.87$.

Euler's scheme. We denote by

$$P^j \simeq p(x, j\Delta_1).$$

$\frac{\partial p}{\partial t}(x, j\Delta_1)$ is replaced by

$$\frac{P^{j+1} - P^j}{\Delta_1}.$$

For given Y^j solution of (2.92), we want to find P^j solution to

$$\begin{cases} \rho\, c\, \frac{P^{j+1}-P^j}{\Delta_1} + \kappa\, \Delta P^j = -\frac{h}{\delta}\,(P^j - Y_0) + \Delta Y^j & \text{for all} \quad j = 1, \cdots, nN, \\ \frac{\partial Y^j}{\partial \nu} - \kappa \frac{\partial P^j}{\partial \nu} = \frac{h}{\delta}\,(P^j - Y_0) & \text{for all} \quad j = 1, \cdots, nN, \\ P^T = 0. \end{cases} \tag{2.95}$$

Considering a regular family of triangulation $(\mathcal{T}_h)_{h>0}$ on $\bar{\Omega}$, using the $\mathbb{P}_1$- finite element method we arrive at the fully discrete approximation of (2.95). Find $P_h^j \in V_h$ solution of

$$\begin{aligned} &\rho\, c \int_\Omega P_h^{j+1}\, v_h dx - \Delta_1 \kappa \int_\Omega \nabla P_h^j \cdot \nabla v_h dx + \Delta_1 \frac{h}{\delta} \int_\Omega (P_h^j - Y_{0,h})\, v_h dx \\ &- \rho\, c \int_\Omega P_h^{j-1}\, v_h dx + \int_\Omega \nabla Y_h^j \nabla v_h dx \\ &- \Delta_1 \frac{h}{\delta} \int_\Gamma (P_h^j - Y_{0,h})\, v_h dS(x) = 0 \quad \text{for all } v_h \in V_h, \end{aligned} \tag{2.96}$$

where the subset V_h is a conforming approximation of $H^1(\Omega)$. In order to compute the discrete form of

$$\nabla_{H^2} \hat{J}(\gamma) = v + \lambda_\gamma \gamma$$

with v solution of (2.61) we will solve (2.61) using Hermite finite elements in one dimension. We therefore build a mesh of $[0, T]$ corresponding to a division into N elements. An Hermite finite element is a triplet (K, Σ, P) where the reference finite element triad $(\hat{K}, \hat{\Sigma}, \hat{P})$ is defined by

$$\begin{aligned} &\hat{K} = [0, 1], \\ &\hat{\Sigma} = \{P(0), P(1), P'(0), P'(1)\}, \\ &\hat{P} = \mathbb{P}_3. \end{aligned}$$

Therefore, this approximation uses two degrees of freedom per node: for each node k of coordinate t_k these are the nodal value of the function $v_k = v_h(t_k)$ and the nodal value of the derivative $dv_k = \frac{dv_h}{dt}(t_k)$. This makes it possible to uniquely define the approximation v_h, which therefore has $2N$ degrees of freedom (the values at node $k = 0$ and $k = T$ are imposed by the boundary condition (2.71)). This finite element space is a conforming approximation of $H^2(0, T)$. This approximation is written as a linear combination of the nodal values of the function and of its derivative. By noting $\Phi_k(t)$ the base functions associated with the nodal values of the function v_k and $\Psi_k(t)$ the base functions associated with the nodal values of the derivative dv_k defined as follow (see figure 2.12)

$$\begin{aligned}\Phi_k(t) &= \mathbb{1}_{[t_{k-1},t_k[}(t)(H^1(t^{(k)}(t)) + \mathbb{1}_{[t_k,t_{k+1}[}(t)(H^0(t^{(k+1)}(t)),\\ \Psi_k(t) &= \mathbb{1}_{[t_{k-1},t_k[}(t)((t_k - t_{k-1})\hat{H}^1(t^{(k)}(t)) + \mathbb{1}_{[t_k,t_{k+1}[}(t)((t_{k+1} - t_k)\hat{H}^0(t^{(k+1)}(t)),\end{aligned} \tag{2.97}$$

where

$$t^{(k)}(t) = \frac{(t - t_{k-1})}{(t_k - t_{k-1})} \text{ for all } t \in [t_{k-1}, t_k]. \tag{2.98}$$

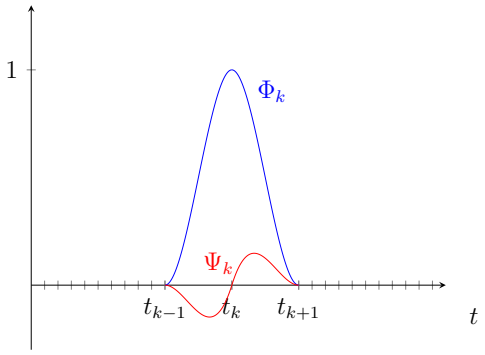


Figure 2.12: Hermite element finite basis functions.

Thus, a function v_h in this finite element space is written by:

$$v_h(t) = \sum_{k=1}^{N} v_k \Phi_k(t) + \sum_{k=1}^{N} dv_k \Psi_k(t).$$

On an element $[t_{k-1}, t_k]$, this approximation is written:

$$v_h(t) = v_{k-1}\Phi_{k-1}(t) + dv_{k-1}\Psi_{k-1}(t) + v_k\Phi_k(t) + dv_k\Psi_k(t)$$

v_h is a Hermite polynomial of degree 3, the four associated shape functions, which are Hermite polynomials of degree 3 are given by $H^0(t^{(k)}(t))$, $\hat{H}^0(t^{(k)}(t))$, $H^1(t^{(k)}(t))$, $\hat{H}^1(t^{(k)}(t))$ already defined in subsection 2.4.2.

Remark 6. This finite element space is the same as the one used for the discrete curves which allow us to compute $\nabla_{H^2}\hat{J}$ by adding v and γ in the same discrete functional space without passing by a transfer matrix.

Algorithm 5: Unsconstrained optimization

0. Initial path γ^0, $\epsilon = 0.1$;
1. Computation of an approximated solution of the heat equation and the adjoint equations along the path γ_0;
2. Computation of the objective function $\hat{J}$ and $\nabla_{H_2}\hat{J}$ along the path γ_0;

while $\| \nabla_{H_2}\hat{J}(\gamma_k) \| \geq \epsilon$ **do**

3. Set $d_k = -\nabla_{H_2}\hat{J}(\gamma_k)$;
4. Choose $\sigma_k > 0$ such that

$$\hat{J}(\gamma_k + \sigma_k d_k) < \hat{J}(\gamma_k) \tag{2.99}$$

;

5. Set $\gamma_{k+1} = \gamma_k + \sigma_k d_k$;
6. Computation of an approximated solution of the heat and the adjoint equations along the new path γ_{k+1};
7. Update $\hat{J}$ and $\nabla_{H_2}\hat{J}$;
8. $k \leftarrow k + 1$;

end

9. return γ_k;

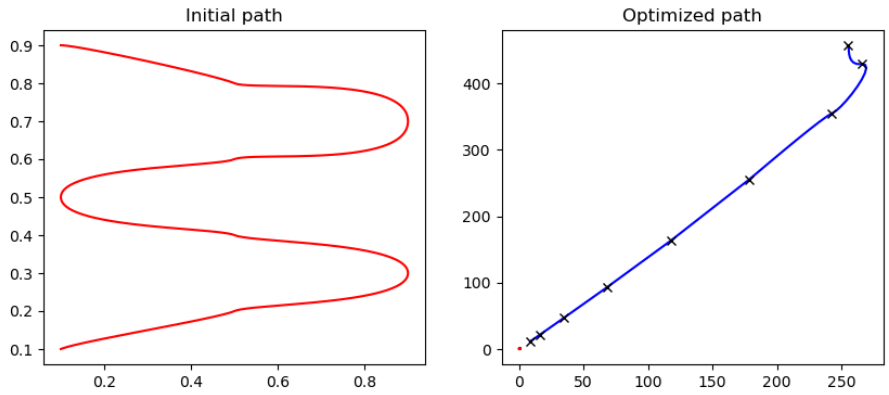


Figure 2.13: Initial and optimized path

A preliminary test has been carried out. The initial curve is given by figure 2.13 and it is composed of nine optimization points. The thermal properties we used are given by table 2.1. The finite elements and geometry data we used are given by table 2.2. We have chosen $\lambda_\gamma = 10^{-5}$ and σ_k in algorithm 5 was computed by solving the minimisation problem (2.99). After 200 iterations we obtain the optimized path which go out of the domain (the red point show the location of the initial path) which is quite consistent since we have no constraints on the path and we demand to minimize $\| \nabla y(\gamma) \|_{L^2}$. Further, the temperature in the domain for the optimized path is $265°C$. After 100 iterations, we may notice that $\| \nabla y(\gamma) \|_{L^2}$ and $\hat{J}(\gamma)$ remains almost 0. Let us stress that without the term $\lambda_\gamma \| \gamma \|_{H^2(0,T;\mathbb{R}^2)}$ in the cost functional (2.82) we could not achieve convergence.

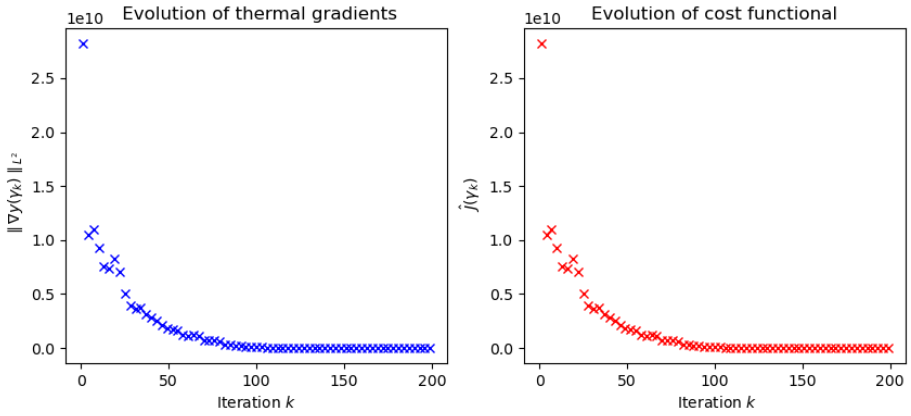


Figure 2.14: Evolution of thermal gradients and cost functional

2.4.6 Discretization of the constraint optimization problem

Box constraints

In order to keep the path inside $\Omega = [0,1] \times [0,1]$ we add the following box constraints on the points $(\alpha_1, \cdots, \alpha_N)$ of γ.

$$\begin{aligned} 0 \leq \alpha_k^1 \leq 1 \text{ for } k = 1, \cdots, N, \\ 0 \leq \alpha_k^2 \leq 1 \text{ for } k = 1, \cdots, N, \end{aligned} \tag{2.100}$$

To handle these box constraints we will apply the projected gradient algorithm where the projection is defined as follows

$$\begin{array}{rccc} \mathbb{P}_{[0,1]^{2N}} : & \mathbb{R}^{2N} & \longrightarrow & [0,1]^{2N} \\ & (\alpha_1, \cdots, \alpha_N) & \longmapsto & (c_1, \cdots, c_N) \end{array}$$

for $\alpha_j = \left(\alpha_j^1, \alpha_j^2\right) \in \mathbb{R}^2$, $c_j = \left(c_j^1, c_j^2\right) \in [0,1]^2$ is given by

$$\begin{aligned} c_j &= \begin{cases} \alpha_j \text{ if } 0 \leq \alpha_j \leq 1, \\ 0 \text{ if } \alpha_j < 0, \\ 1 \text{ if } \alpha_j > 1, \end{cases} \\ &= \max(0, \min(\alpha_j, 1)). \end{aligned} \tag{2.101}$$

Remark 7. The constraint (2.100) is not enough to justify that the curve remains inside Ω. In fact the derivatives can let the curves go out of the domain. But as a first approximation the constraints (2.100) seem sufficient to force the curve to stay inside the domain. In subsection 2.5.1 we propose an improvement of this constraints by using the fact that any Bézier curve is fully contained within the convex hull of its control polygon.

Algorithm 6: Box constrained optimization

0. Initial path γ^0, $\epsilon = 0.1$;
1. Computation of an approximated solution of the heat equation and the adjoint equations along the path γ_0;
2. Computation of the objective function $\hat{J}$ and $\nabla_{H_2}\hat{J}$ along the path γ_0;

while $\| \gamma_k - \mathbb{P}_{[0,1]^{2N}}(\gamma_k + \sigma_k d_k) \|_{\mathbb{R}^{2\mathbb{N}}} \geq \epsilon$ **do**

3. Set $d_k = -\nabla\hat{J}(\gamma_k)$;
4. Choose $\sigma_k > 0$ such that $\hat{J}(\gamma_k + \sigma_k d_k) < \hat{J}(\gamma_k)$;
5. Compute $\mathbb{P}_{[0,1]^{2N}}(\gamma_k + \sigma_k d_k)$ using formula (2.101) and set $\gamma_{k+1} = \mathbb{P}_{[0,1]^{2N}}(\gamma_k + \sigma_k d_k)$;
6. Update $\hat{J}$ and $\nabla_{H_2}\hat{J}$;
7. $k \leftarrow k + 1$;

end

8. return γ_k;

Inequality constraints and perspectives

To make the laser path cover the area of the domain Ω we add the following geometrical constraint:

$$\begin{aligned} c(\gamma) &:= 2R\int_0^T \| \gamma'(t) \| dt - | \Omega | \geq 0 \\ &= 2R\int_0^T \| \sum_{k=1}^{N} \mathbb{1}_{[t_{k-1},t_k]}(\begin{pmatrix} \alpha_{k-1}^1 \\ \alpha_{k-1}^2 \end{pmatrix} H^{0'}(t^{(k)}(t)) + \Delta_0 \begin{pmatrix} \beta_{k-1}^1 \\ \beta_{k-1}^2 \end{pmatrix} \hat{H}^{0'}(t^{(k)}(t)) \\ &+ \begin{pmatrix} \alpha_k^1 \\ \alpha_k^2 \end{pmatrix} H^{1'}(t^{(k)}(t)) + \Delta_0 \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} \hat{H}^{1'}(t^{(k)}(t))) \| dt - | \Omega | \geq 0. \end{aligned} \tag{2.102}$$

To solve this optimization problem with inequalities constraint we will use the augmented Lagrangian method [42]. We recall from [42] the Lagrangien:

$$\mathcal{L}(\gamma, \lambda_k; \mu_0) := \hat{J}^{\delta}(\gamma) + \sum_{j=1}^{4}\sum_{k=1}^{N} \psi(c_j^k(\gamma), \lambda_i^k; \mu_0) + \psi(c_5(\gamma), \lambda_i^{N+1}; \mu_0), \tag{2.103}$$

where $c_j^k(\gamma)$ represents the five constraints for all $k = 0, \cdots, N$. Let us define for $j \in [1, 5]$, $k = 1, \cdots, N$

$$\psi(c_j^k(\gamma), \lambda_i^k; \mu_k) = \begin{cases} \lambda_i^k c_j^k(\gamma) + \frac{1}{2\mu_0}(c_j^k(\gamma))^2, & if \quad -c_i^j(\gamma) - \mu_0\lambda_i^k \leq 0 \\ \frac{\mu_0}{2}(\lambda_i^k)^2, & otherwise. \end{cases} \tag{2.104}$$

We want to solve the following optimization problem:

$$\min_{\gamma} \mathcal{L}(\gamma, \lambda_k; \mu_k). \tag{2.105}$$

We are currently working on the inequality constraints and its related numerical tests are in progress and will be available soon.

2.5 Discussions and outlook

The prospects for laser path optimization are numerous. We mention in what follows some ideas that we will study in a further work.

2.5.1 Simplified computation of optimal trajectories using Bézier curve parametrization

In order to guarantee that the path stay inside the domain, we propose to use Bézier curve parametrization (see remark 7). We will go back to the three dimensional settings to describe the discrete control space.

We suppose that

$$\Gamma_1 = [a, b] \times [c, d]\,,$$

$-\infty < a < b < +\infty$, $-\infty < c < d < +\infty$ be a rectangle in the plane $\mathbb{R}^2$. Let us consider

$$\Gamma_{1,-\epsilon} = [a + \epsilon, b - \epsilon] \times [c + \epsilon, d - \epsilon]$$

formed by the points of Γ_1 whose distance to the boundary $\partial\Gamma_1$ of Γ_1 is larger or equal to ϵ, $\epsilon \in]0, \min(\frac{b-a}{2}, \frac{d-c}{2})[$.

Let us consider the uniform subdivision $t_0 = 0 < t_1 < \cdots < t_N = T$ of the time interval $[0, T]$ with uniform step size $\Delta_0 = \frac{T}{N}$. The cubic Hermite parametrized path given by (2.90) can be equivalently written as a cubic Bézier curve [22] on $[t_{k-1}, t_k]$ for all $k = 1, \cdots, N$:

$$\gamma(t) = B_0(t^{(k)}(t))b_{3k-3} + B_1(t^k(t))b_{3k-2} + B_2(t^{(k)}(t))b_{3k-1} + B_3(t^{(k)}(t))b_{3k}, \qquad (2.106)$$

where $t^{(k)}(t)$ is given by (2.98) and

- $B_0(t^{(k)}(t)) = (1 - t^{(k)}(t))^3$,
- $B_1(t^{(k)}(t)) = 3t^{(k)}(t)(1 - t^{(k)}(t))^2$,
- $B_2(t^{(k)}(t)) = 3t^{(k)}(t)^2(1 - t^{(k)}(t))$,
- $B_3(t^{(k)}(t)) = t^{(k)}(t)^3$,

are the Bernstein basis of degree 3.

Namely, we can express b_{3k-3}, b_{3k-2}, b_{3k-1}, b_{3k} in term of the points and the derivatives at t_{k-1} and t_k. For each time interval $[t_{k-1}, t_k]$, $k = 1, \cdots, N$, we have

- $\gamma(t_{k-1}) = b_{3k-3}$,
- $\gamma(t_k) = b_{3k}$,
- $\gamma'(t_{k-1}) = \frac{3}{t_k - t_{k-1}}(b_{3k-2} - b_{3k-3})$ thus $b_{3k-2} = \gamma(t_{k-1}) + \frac{t_k - t_{k-1}}{3}\gamma'(t_{k-1})$,
- $\gamma'(t_k) = \frac{3}{t_k - t_{k-1}}(b_{3k} - b_{3k-1})$ thus $b_{3k-1} = \gamma(t_k) - \frac{t_k - t_{k-1}}{3}\gamma'(t_k)$.

Replacing b_{3k-3}, b_{3k-2}, b_{3k-1}, b_{3k} in (2.106) by these latest formulas we obtain the cubic Hermite parametrization of γ given by (2.90). b_{3k-3}, b_{3k-2}, b_{3k-1}, b_{3k} are the control points of the curve $\gamma(t)$ for every $t \in [t_{k-1}, t_k]$ [22, pp.71-75] (see figure 2.15).

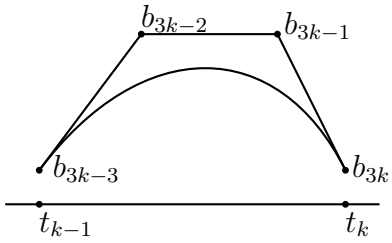


Figure 2.15: Bézier curve on $[t_{k-1}, t_k]$

Corollary 19. *For all $t \in [t_{k-1}, t_k]$, $\gamma(t)$ belongs to the closed convex hull of the four points b_{3k-3}, b_{3k-2}, b_{3k-1} and b_{3k}.*

Proof. For all $t \in [t_{k-1}, t_k]$, $t^k(t) \in [0, 1]$, the four coefficients $B_0(t^k(t))$, $B_1(t^k(t))$, $B_2(t^k(t))$, $B_3(t^k(t))$ are non-negative. Moreover, the sum of these four coefficients is equal to 1. Thus on each time interval $[t_{k-1}, t_k]$, γ is a convex combination of b_{3k-3}, b_{3k-2}, b_{3k-1} and b_{3k}. □

The discrete set of admissible controls is given by

$$U_{ad}^{discret} = \{(b_0, \dots, b_{3N}); b_0, \dots, b_{3N} \in \Gamma_{1,-\epsilon}\} = \Gamma_{1,-\epsilon}^{3N+1} \subset (\mathbb{R}^2)^{3N+1}, \tag{2.107}$$

under the form of box constraints. Therefore, in order to keep the curve inside the domain it will be interesting to use cubic Bézier curve parametrization. Instead of putting constraints on the points and their associated derivatives, we put them on its control polygone. By corrollary 19 the path will remain inside Γ_1.

2.5.2 Power control in SLM

In the previous sections we have presented optimization models controlling the laser path in additive manufacturing. In this section we will present preliminaries optimal control models about power optimization in SLM. In the litterature we can find some new engineering work discussing the effect of laser power on the fabricated part and the material [14, 39]. In the following we will present three ways to control the power in SLM. The first idea is to only control the laser power by considering it as a L^2-control, the second one is to control the power by only manipulating the laser path based on the theoritical results presented in sections 2.2 and 2.3 and the third one is to combine laser path and laser power optimization based on the latest two ideas.

The ideas presented in this section results from a joined work with Grégoire Allaire and Mathilde Boissier, from the Centre de Mathématiques appliquées (CMAP), in Palaiseau, France. In [8], numerical approaches were conducted to optimize the power, force its bang-bang properties and control its number of variations, also a coupled power and laser path optimization model is given. In [9, 8] a second numerical approach for path optimization is presented.

$\mathbf{P} \in \mathbf{L^2}(\mathbf{0}, \mathbf{T}, [\mathbf{0}, \mathbf{P_{max}}])$

Our aim is to find an optimal laser power minimizing temperature gradient in SLM. In what follows we use the same theoretical setting as in section 2.2. Therefore the state

equation is given by (2.3) with

$$g_\gamma(x,t) = \alpha \frac{2P(t)}{\pi R^2} \exp\left(-2\frac{|\, x-\gamma(t)\,|^2}{R^2}\right), \text{ for all } (x,t) \in \Sigma_1, \tag{2.108}$$

where

$$P : t \in [0,t_f] \to \mathbb{R} \tag{2.109}$$

is the laser power and $\gamma \in H^1(0,T;\mathbb{R}^2)$ is a given laser path which could also be the optimal control computed by solving (OCP^θ). We define the following cost functional

$$\begin{aligned} J(y,P) :=& \frac{1}{2}\int_0^T \int_\Omega |\, \nabla y(x,t)\,|^2 \, dx\, dt + \frac{\lambda_Q}{2}\int_0^T \int_\Omega |\, y(x,t) - y_Q(x,t)\,|^2 \, dx\, dt \\ &+ \frac{\lambda_P}{2} \parallel P \parallel^2_{L^2(0,T;\mathbb{R})}, \end{aligned} \tag{2.110}$$

where $\lambda_Q \geq 0$ and $\lambda_P > 0$ are constants, while $y_Q \in L^2(Q)$ is a given function. The optimal control problem is

$$\min_{P \in U_P} J(y(P),P), \tag{2.111}$$

where U_P is the set of admissible controls

$$U_P := \{P \in L^2(0,T;\mathbb{R}) \text{such that } 0 \leq P(t) \leq P_{max}\}.$$

Theorem 20 (Existence of an optimal control)**.** *Supposing $U_P \neq \emptyset$, then the optimal control problem (2.111) admits at least one optimal control $\bar{P} \in U_P$.*

Proof. The proof is based on the same technics used in the theorem 4. □

Theorem 21. *If $P \in U_P$ is an optimal control of (2.111) with associated state $y(\bar{P})$, and $p \in W(0,T)$ the corresponding adjoint state that solves (2.23), then the variational inequality*

$$\begin{aligned} &DJ(y(\bar{P}),\bar{P}) \cdot \delta P = \lambda_P \int_0^T \bar{P}(t) \cdot (\delta \bar{P})(t) dt \\ &+2ac_R \int_0^T (\delta \bar{P})(t) \int_{\Sigma_1} \alpha \frac{2\bar{P}(t)}{\pi R^2} \exp\left(-2\frac{|\, x-\gamma(t)\,|^2}{R^2}\right) p(x,t) dS(x) dt \geq 0 \end{aligned} \tag{2.112}$$

holds for all $\delta \bar{P} \in U_{ad}$.

Corollary 22. *One can check that*

$$\bar{P}(t) = \begin{cases} P_{max} & \text{if } \lambda_P \bar{P}(t) + \int_{\Sigma_1} \alpha \frac{2\bar{P}(t)}{\pi R^2} \exp\left(-2\frac{|x-\gamma(t)|^2}{R^2}\right) p(x,t) dS(x) < 0 \\ \in [0, P_{max}] & \text{if } \lambda_P \bar{P}(t) + \int_{\Sigma_1} \alpha \frac{2\bar{P}(t)}{\pi R^2} \exp\left(-2\frac{|x-\gamma(t)|^2}{R^2}\right) p(x,t) dS(x) = 0 \\ 0 & \text{if } \lambda_P \bar{P}(t) + \int_{\Sigma_1} \alpha \frac{2\bar{P}(t)}{\pi R^2} \exp\left(-2\frac{|x-\gamma(t)|^2}{R^2}\right) p(x,t) dS(x) > 0 \end{cases}$$

Controlling the power by controlling the path

An alternative approach to avoid laser power to induce important thermal distortions when the acceleration of the path is important is to replace (2.108) by

$$g_\gamma(x,t) = \alpha \frac{2(P * \varphi_\epsilon)(|\ \gamma''(t)\ |^2)}{\pi R^2} \exp\left(-2\frac{|\ x-\gamma(t)\ |^2}{R^2}\right), \text{ for all } (x,t) \in \Sigma_1, \tag{2.113}$$

where for all $t \in [0, t_f]$

$$P(|\ \gamma''(t)\ |^2) = \begin{cases} P_{max} & \text{if } |\ \gamma''(t)\ |^2 < s \\ 0 & \text{if } |\ \gamma''(t)\ |^2 \geq s. \end{cases}$$

P_{max} is a positive constant that represents laser power and $(\varphi_\epsilon)_{\epsilon\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ is a sequence of mollifiers [10, p. 108] and s is a threshhold. One can check that

$$(P * \varphi_\epsilon)(|\ \gamma''(t)\ |^2) = \begin{cases} P_{max} & \text{if } |\ \gamma''(t)\ |^2 < s-\epsilon \\ \in\ ]0, P_{max}[& \text{if } s-\epsilon \leq |\ \gamma''(t)\ |^2 < s+\epsilon. \\ 0 & \text{if } |\ \gamma''(t)\ |^2 \geq s+\epsilon. \end{cases}$$

In this way we have only one variable of optimization which is γ. Here we link the power and the curvature of γ. Indeed in additive manufacturing machines, speed and curvature are linked so if we want to scan homogeneously, it can be interesting to vary the power as a function of the curvature. The optimization problem is the following:

$$\min_{\gamma\in U_{ad}} J(y(\gamma),\gamma), \tag{2.114}$$

with J is given by (2.6) and U_{ad} defined by (2.7). The study of this problem can be done using the same steps presented in section 2.2 and will be investigated in a further work.

Controlling the path and the power

We consider the optimal control problem

$$\begin{aligned} \min_{P\in U_P, \gamma\in U_{ad}} J(y,P,\gamma) \coloneqq \frac{1}{2}\int_0^T \int_\Omega |\ \nabla y(x,t)\ |^2\, dx\, dt + \frac{\lambda_Q}{2}\int_0^T \int_\Omega |\ y(x,t) - y_Q(x,t)\ |^2\, dx\, dt \\ + \frac{\lambda_P}{2} \parallel P \parallel^2_{L^2(0,T;\mathbb{R})} + \lambda_\gamma \parallel \gamma \parallel_{H^1(0,T,\mathbb{R}^2)} \end{aligned} \tag{2.115}$$

subject to the state equation (2.3) with g_γ given by (2.108) and to the two set of admissible controls U_P and U_{ad}. The study of this problem is based on same technics presented in [53, Chapter 3].

The disadvantages of these three models is that in the industrial applications the power cannot for the moment really vary continuously.

Chapter 3

Existence of local solutions for the coupled Heat-Maxwell's equations with temperature dependent permittivity

3.1 Introduction

We are interested in heat diffusion equations with volumic heat source induced by electromagnetic waves. A typical heat-diffusion system involves a laser beam source irradiating locally a three dimensional medium Ω. This technique is used in laser melting process for additive metallic manufacturing [62, 6]. In chapter 2, we have simply considered the heat diffusion equation in Ω with a inhomogeneous Robin boundary condition on some part of its boundary irradiated the absorbed part of a laser beam energy. In chapter 2, we wanted to control the temperature and its gradient inside Ω during the processing time interval $[0, T]$, T fixed. In this chapter, to model accurately the laser interaction with the medium Ω, we consider in Ω the coupling between the heat diffusion equation and Maxwell's equations. We also take into account, the temperature dependence of the electric permittivity of the medium inside Ω. In [1], Maxwell's equations have also been considered to model accurately the interaction of the laser beam with biological tissues preferably to Beer's law or to the radiative transfer equation. In [59], time-harmonic electric and magnetic fields of some fixed frequency are considered, but the derived equations [59, Eq.(2.1)-(2.5)] lead to a contradiction as explained in [59, Remark 2.1]. The temperature solution of [59, (2.2)-(2.4)-(2.5)] which appears in the permittivity coefficient is not even necessary periodic in time, thus the electric and magnetic fields could not be time-harmonic. Therefore the choice of time-harmonic Maxwell's equations in [59] is not appropriate.

Let us now describe our model. Let us fix some $T > 0$, we consider in the space-time cylinder $Q = \Omega \times]0, T[$, the 3-dimensional parabolic initial-boundary value problem:

$$\begin{cases} \partial_t y - \operatorname{div}(\alpha \nabla y) = S(y) & \text{in } Q, \\ -\alpha \frac{\partial y}{\partial \nu} = h(y - y_b) & \text{on } \Sigma, \\ y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases} \tag{3.1}$$

Here, y denotes the temperature, α the thermal diffusivity constant, n the outward unit normal vectorfield along the boundary Γ of Ω, $h > 0$ the heat transfer coefficient and y_b the temperature of the surrounding medium (air). By $\Sigma = \Gamma \times]0, T[$, we denote

the lateral boundary of the space-time cylinder Q. The heat source function $S(y)$ in (3.1), represents the volumic power absorbed by the medium Ω from the electromagnetic field generated in Ω by an external source e.g. a laser beam. $S(y)(x,t)$ is defined as the multiplication of the absorption coefficient μ_a depending on $x \in \Omega$ and on the temperature $y(x,t)$, with the electric field intensity weighted around x:

$$S(y)(x,t) \coloneqq \mu_a(x, y(x,t)) \mid (\mathbf{E}(y) * \varphi_a)(x,t) \mid^2, \quad \text{for all } (x,t) \in Q. \tag{3.2}$$

$\varphi_a \in C_c^1(\mathbb{R}^3)$ represents a weight function which is supposed to be at least of class C^1 on $\mathbb{R}^3$ and with compact support. In formula (3.2), $\mathbf{E}$ represents the electric field in Ω solution of the following Maxwell's equations:

$$\begin{cases} \partial_t(\epsilon(\cdot,y)\mathbf{E}) - \operatorname{curl}\mathbf{H} + \sigma\mathbf{E} = 0 & \text{in } Q, \\ \partial_t(\mu\mathbf{H}) + \operatorname{curl}\mathbf{E} = 0 & \text{in } Q, \\ \operatorname{div}(\mu\mathbf{H}) = 0 & \text{in } Q, \\ \mathbf{E} \times n = \mathbf{E}_{ext} \times n & \text{on } \Sigma, \\ \mathbf{H} \cdot n = 0 & \text{on } \Sigma, \\ \mathbf{E}(\cdot,0) = \mathbf{E}_0,\ \mathbf{H}(\cdot,0) = \mathbf{H}_0 & \text{in } \Omega. \end{cases} \tag{3.3}$$

In these equations, $\mathbf{H}$ denotes the magnetic component of the electromagnetic field $(\mathbf{E}, \mathbf{H})$ in Ω, σ the electrical conductivity, μ the magnetic permeability, $\varepsilon(x, y(x,t))$ the electric permittivity dependant on the space and the temperature y. $\mathbf{E}_{ext}$ denotes the electric field irradiating the boundary of Ω due to an external source. The first equation in the Maxwell system (3.3) is coupled to the heat diffusion initial boundary value problem (3.1) by the dependence of the permittivity ε with respect to the temperature y, and the heat diffusion initial boundary value problem (3.1) is coupled to the Maxwell system (3.3) by the right-hand side into the heat equation (3.1), the heat source term $S(y)(x,t)$ (3.2) depending on the electric field $\mathbf{E}(y)$. Our hypotheses on the coefficients appearing in (3.1), (3.2), and (3.3) will be precised further.

Our purpose is to establish the existence of local solutions to this coupled problem. First, we fix the temperature distribution, we study the Maxwell system by using the theory of evolution systems (section 3.2), the difficulty being due to the dependence of the permittivity with respect to the temperature and consequently to time. Next, we study the coupled problem (section 3.3) by introducing a fixed point problem in the closed convex set $K(0;R) := \bar{B}(0;R) \cap \{z \in \bar{B}(0;R); z(0) = 0\}$ of the Banach space $C^1([0,T]; C^1(\bar{\Omega}))$ and proving that the hypotheses of Schauder's theorem are verified for R sufficiently large. The construction of the fixed point problem is nontrivial as we need $K(0;R)$ to be stable.

3.2 Maxwell's equations with temperature dependent permittivity

In this section, we fix the distribution of temperature appearing in the permittivity coefficient of the Maxwell system (3.3) decoupling in this way Problem (3.3) from the Heat initial boundary value problem (3.1). The resulting problem is thus a Maxwell initial boundary value problem with time dependent permittivity through the distribution of temperature.

3.2.1 Preliminaries

Throughout this chapter, Ω is always assumed to be a bounded domain in $\mathbb{R}^3$ and its boundary Γ to be of class C^2. We set $\mathbf{L}^2(\Omega) := L^2(\Omega)^3$ and $\mathbf{H}^1(\Omega) := H^1(\Omega)^3$. We begin by defining the following standard functional spaces:

$$\mathbf{H}(\mathrm{curl}\,,\Omega) = \{\varphi \in \mathbf{L}^2(\Omega); \mathrm{curl}\,\varphi \in \mathbf{L}^2(\Omega)\},$$

$$\mathbf{H}(\mathrm{div}\,,\Omega) = \{\psi \in \mathbf{L}^2(\Omega); \mathrm{div}\,\psi \in \mathbf{L}^2(\Omega)\},$$

$$\mathbf{H}_0(\mathrm{curl}\,,\Omega) = \{\varphi \in \mathbf{L}^2(\Omega); \mathrm{curl}\,\varphi \in \mathbf{L}^2(\Omega), \varphi \times n = 0 \text{ on } \Gamma\},$$
$$\mathbf{H}_0(\mathrm{div}\,,\Omega) = \{\psi \in \mathbf{L}^2(\Omega); \mathrm{div}\,\psi \in L^2(\Omega), \psi \cdot n = 0 \text{ on } \Gamma\},$$
$$\mathbf{J}_n(\Omega,\mu) = \{\psi \in \mathbf{L}^2(\Omega); \mathrm{div}\,(\mu\psi) = 0, \psi \cdot n = 0 \text{ on } \Gamma\}.$$
$$\mathbf{J}^1_n(\Omega,\mu) = \mathbf{H}(\mathrm{curl}\,,\Omega) \cap \mathbf{J}_n(\Omega,\mu),$$

We make the following assumptions on the coefficients and data of the Maxwell problem (3.3):

(H1) The function $(x,y) \mapsto \epsilon(x,y)$ is real valued, positive, continuous on $\bar{\Omega} \times \mathbb{R}$ with first order partial derivatives with respect to the variables x_i $(i = 1,2,3)$ and y, also continuous on $\bar{\Omega} \times \mathbb{R}$. Also, there exist positive constants ϵ_1, ϵ_0 such that:

$$0 < \epsilon_0 \leq \epsilon(x,y) \leq \epsilon_1, \text{ for all } (x,y) \in \Omega \times \mathbb{R}.$$

(H2) $\sigma \in L^\infty(\Omega)$ and the function $\mu \in W^{1,\infty}(\Omega)$. There are positive constants μ_0 and μ_1 such that:

$$0 < \mu_0 \leq \mu(x) \leq \mu_1, \text{ for all } x \in \Omega.$$

(H3) $(\mathbf{E}_0, \mathbf{H}_0) \in \mathbf{H}(\mathrm{curl}\,,\Omega) \times \mathbf{J}^1_n(\Omega,\mu)$ and $\mathbf{E}_{ext} \in C^2([0,T]; \mathbf{H}^1(\mathbb{R}^3 \setminus \bar{\Omega}))$ such that $\mathrm{curl}\,\mathbf{E}_{ext} \cdot n_{|\Sigma} = 0$, $\mathbf{E}_{ext}(.,0) \times n_{|\Gamma} = \mathbf{E}_0 \times n_{|\Gamma}$.

Remark 8. By identifying $\mathbf{E}_{ext}$ with the time-dependent differential form of degree 1 on $\mathbb{R}^3 \setminus \bar{\Omega}$: $\mathbf{E}_{ext,1}dx^1 + \mathbf{E}_{ext,2}dx^2 + \mathbf{E}_{ext,3}dx^3$, the hypothesis $\mathrm{curl}\,\mathbf{E}_{ext} \cdot n_{|\Sigma} = 0$ on $\mathbf{E}_{ext}$ in (H3), amounts to assume that at any time $t \in [0,T]$, the exterior derivative on Γ of its trace on Γ is equal to 0 [49, p.136].

Remark 9.
- By assumption (H2) on μ and the standard formula

$$\mathrm{div}\,(\mu\psi) = \mu \mathrm{div}\,\psi + \nabla\mu \cdot \psi = 0 \text{ for all } \psi \in \mathbf{J}_n(\Omega,\mu), \tag{3.4}$$

we have $\mathrm{div}\,\psi = -\frac{1}{\mu}\nabla\mu \cdot \psi \in L^2(\Omega)$, and therefore $\mathbf{J}_n(\Omega,\mu) \subset \mathbf{H}_0(\mathrm{div}\,,\Omega)$.

- Consequently: $\mathbf{J}^1_n(\Omega,\mu) \subset \mathbf{H}_0(\mathrm{div}\,,\Omega) \cap \mathbf{H}(\mathrm{curl}\,,\Omega) \hookrightarrow \mathbf{H}^1(\Omega)$ [24, p.44].

Proposition 23. *$\mathbf{J}_n(\Omega,\mu)$ is a closed subspace of $\mathbf{L}^2(\Omega)$.*

Proof. Let us consider a sequence $(\psi_k)_{k\in\mathbb{N}} \subset \mathbf{J}_n(\Omega,\mu)$ converging to some $\psi \in \mathbf{L}^2(\Omega)$. $\mathrm{div}\,(\mu\psi_k) = 0$ in Ω implies that

$$\int_\Omega \mu\psi_k \cdot \nabla\varphi dx = 0 \text{ for all } \varphi \in \mathcal{D}(\Omega) := C^\infty_c(\Omega).$$

Since $\psi_k \to \psi$ in $\mathbf{L}^2(\Omega)$, as $k \to +\infty$, we have also that

$$\int_\Omega \mu\psi \cdot \nabla\varphi dx = 0 \text{ for all } \varphi \in \mathcal{D}(\Omega),$$

which implies that $\operatorname{div}(\mu\psi) = 0$ in the weak sense. By assumption (H2) and the standard formula

$$\operatorname{div}(\mu\psi) = \mu \operatorname{div}\psi + \nabla\mu \cdot \psi,$$

we get

$$\mu \operatorname{div}\psi = -\underbrace{\nabla\mu}_{\in \mathbf{L}^\infty(\Omega)} \cdot \psi \in L^2(\Omega).$$

Thus $\operatorname{div}\psi \in L^2(\Omega)$ and thus $\psi \in \mathbf{H}(\operatorname{div};\Omega)$. Consequently the normal trace $\psi \cdot n_{|\Gamma}$ of ψ has sense, and $\psi \cdot n_{|\Gamma} \in H^{-1/2}(\Gamma)$. As $(\psi_k)_{k\in\mathbb{N}}$ converges to ψ in $\mathbf{L}^2(\Omega)$, it follows that:

$$\operatorname{div}\psi_k = -\frac{1}{\mu}\nabla\mu \cdot \psi_k \to -\frac{1}{\mu}\nabla\mu \cdot \psi = \operatorname{div}\psi \text{ in } L^2(\Omega),\ k \to +\infty.$$

Therefore $\psi_k \to \psi$ in $\mathbf{H}(\operatorname{div};\Omega)$ so that $\psi \cdot n_{|\Gamma} = 0$ since $\psi_k \cdot n_{|\Gamma} = 0$, for all $k \in \mathbb{N}$. We have therefore proved that $\operatorname{div}(\mu\psi) = 0$ and that $\psi \cdot n_{|\Gamma} = 0$ on Γ. Thus $\psi \in \mathbf{J}_n(\Omega,\mu)$. □

We will need the following lemma in the proof of Proposition 25.

Lemma 24. *If $\varphi_1, \varphi_2 \in \mathbf{H}(curl,\Omega)$ such that $\varphi_1 \times n_{|\Gamma} = \varphi_2 \times n_{|\Gamma}$, then $curl\,\varphi_1 \cdot n_{|\Gamma} = curl\,\varphi_2 \cdot n_{|\Gamma}$. In particular, if $\varphi \in \mathbf{H}_0(curl,\Omega)$, then $curl\,\varphi \cdot n_{|\Gamma} = 0$.*

Proof. Let $\psi \in \mathbf{H}(\operatorname{curl},\Omega)$. Using Green's formulas, firstly for the divergence operator and afer for the *curl* operator [24, p.34], we obtain that for all $v \in C^\infty(\bar\Omega)$:

$$\begin{aligned}
&\int_\Gamma \operatorname{curl}\psi(x) \cdot n(x)\, v(x) dS(x) = \int_\Omega \operatorname{div}\,(v\ \operatorname{curl}\psi)(x)\, dx \\
&= \int_\Omega v(x)(\operatorname{div}\ \operatorname{curl}\psi)(x)dx + \int_\Omega \operatorname{curl}\psi(x) \cdot \nabla v(x)dx \\
&= \int_\Omega \operatorname{curl}\psi(x) \cdot \nabla v(x)dx = \int_\Omega \psi(x) \cdot \operatorname{curl}\ \nabla v(x)dx \\
&+ \int_\Gamma \psi \times n(x)\nabla v(x)dS(x) = \int_\Gamma (\psi \times n)(x)\nabla v(x)dS(x).
\end{aligned} \tag{3.5}$$

Let us precise that the boundary integrals appearing in (3.5) must be considered as bracket of dualities

$$\int_\Gamma \operatorname{curl}\psi(x) \cdot n(x)\, v(x)dS(x) :=< (v\ \operatorname{curl}\psi) \cdot n, 1_\Gamma >_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}$$

and

$$\int_\Gamma (\psi \times n)(x)\nabla v(x)dS(x) :=< \psi \times n, \nabla v >_{\mathbf{H}^{-1/2}(\Gamma),\mathbf{H}^{1/2}(\Gamma)} .$$

Thus for all $v \in C^\infty(\bar\Omega)$

$$\int_\Gamma \operatorname{curl}\psi(x) \cdot n(x)\, v(x)dS(x) = \int_\Gamma (\psi \times n)(x) \cdot \nabla v(x)dS(x). \tag{3.6}$$

Applying (3.6) to φ_1 and φ_2, and using our hypothesis that $\varphi_1 \times n_{|\Gamma} = \varphi_2 \times n_{|\Gamma}$, we obtain that for all $v \in C^\infty(\bar{\Omega})$

$$\int_\Gamma \operatorname{curl} \varphi_1(x) \cdot n(x)\, v(x) dS(x) = \int_\Gamma \operatorname{curl} \varphi_2(x) \cdot n(x)\, v(x) dS(x). \tag{3.7}$$

Consequently $\operatorname{curl} \varphi_1 \cdot n_{|\Gamma} = \operatorname{curl} \varphi_2 \cdot n_{|\Gamma}$. □

In the following we fix the distribution of temperature y to

$$z \in C^1([0,T]; C^1(\bar{\Omega})), \tag{3.8}$$

in the permittivity coefficient of the Maxwell-Ampère equation $(4.7)_{(i)}$. In doing so, we obtain the following linear initial nonhomogeneous boundary value problem for the Maxwell's equations:

$$\begin{cases} \partial_t \mathbf{E} - \frac{1}{\epsilon(\cdot,z)} \operatorname{curl} \mathbf{H} + \frac{1}{\epsilon(\cdot,z)} \sigma \mathbf{E} + \frac{\partial_z \epsilon(\cdot,z) \partial_t z}{\epsilon(\cdot,z)} \mathbf{E} = 0 & \text{in } Q, \\ \partial_t \mathbf{H} + \frac{1}{\mu} \operatorname{curl} \mathbf{E} = 0 & \text{in } Q, \\ \operatorname{div}(\mu \mathbf{H}) = 0 & \text{in } Q, \\ \mathbf{E} \times n = \mathbf{E}_{ext} \times n & \text{on } \Sigma, \\ \mathbf{H} \cdot n = 0 & \text{on } \Sigma, \\ \mathbf{E}(\cdot,0) = \mathbf{E}_0,\ \mathbf{H}(\cdot,0) = \mathbf{H}_0 & \text{in } \Omega. \end{cases} \tag{3.9}$$

Let us note in the first equation, that the coefficients are time dependent due to the presence of z; moreover in the last coefficient appears also $\frac{\partial z}{\partial t}$. To reduce problem (3.9) to a linear initial homogeneous boundary value problem, we must construct an appropriate extension of $\mathbf{E}_{ext}$ to Ω.

Proposition 25. *Under assumption (H3), there exists a vectorial function* $\mathbf{W} \in C^2([0,T]; \mathbf{H}^1(\Omega))$ *such that*

$$\begin{cases} \mathbf{W} \times n = \mathbf{E}_{ext} \times n & on \ \ \Sigma, \\ curl\ \mathbf{W} \cdot n = 0 & on \ \ \Sigma, \\ \mathbf{W}(.,0) \times n = \mathbf{E}_0 \times n & on \ \ \Gamma. \end{cases} \tag{3.10}$$

Proof. As $Tr(\mathbf{E}_{ext}) \in C^2([0,T]; \mathbf{H}^{1/2}(\Gamma))$ ($Tr(.)$ denotes the trace operator), there exists $\mathbf{W} \in C^2([0,T]; \mathbf{H}^1(\Omega))$ such that for all $t \in [0,T]$

$$\begin{cases} W_1(.,t)_{|\Gamma} = E_{ext_1}(.,t)_{|\Gamma}\ , \\ W_2(.,t)_{|\Gamma} = E_{ext_2}(.,t)_{|\Gamma}\ , \\ W_3(.,t)_{|\Gamma} = E_{ext_3}(.,t)_{|\Gamma}\ . \end{cases} \tag{3.11}$$

Since the boundary of Ω is of class C^2, $n_i \in C^1(\Gamma),\ i = 1,2,3$ and

$$\begin{aligned} \mathbf{W} \times n &= \begin{pmatrix} n_3 W_2 - n_2 W_3 \\ n_1 W_3 - n_3 W_1 \\ n_2 W_1 - n_1 W_2 \end{pmatrix} \\ &= \mathbf{E}_{ext} \times n, \ \text{ on } \Sigma. \end{aligned} \tag{3.12}$$

By hypothesis (H3): $\mathbf{E}_{ext}(.,0)\times n_{|\Gamma} = \mathbf{E}_0\times n_{|\Gamma}$ so that $\mathbf{W}(.,0)\times n_{|\Gamma} = \mathbf{E}_0\times n_{|\Gamma}$. By formula (3.6):

$$\int_\Gamma \operatorname{curl}\mathbf{W}(x,t)\cdot n(x)\, v(x)dS(x) = \int_\Gamma (\mathbf{W}\times n)(x,t)\cdot\nabla v(x)dS(x), \tag{3.13}$$

for all $v\in C^\infty(\mathbb{R}^3)$. By similar reasonings on $\mathbb{R}^3\setminus\bar\Omega$ as those made on Ω to establish formula (3.6), we have also that for all $v\in C^\infty(\mathbb{R}^3)$:

$$\int_\Gamma \operatorname{curl}\mathbf{E}_{ext}(x,t)\cdot n(x)\, v(x)dS(x) = \int_\Gamma (\mathbf{E}_{ext}\times n)(x,t)\cdot\nabla v(x)dS(x). \tag{3.14}$$

Thus by (3.12), (3.13) and (3.14), we have that $\operatorname{curl}\mathbf{W}(.,t)\cdot n = \operatorname{curl}\mathbf{E}_{ext}(.,t)\cdot n$, on Γ. By hypothesis (H3) on $\mathbf{E}_{ext}$, $\operatorname{curl}\mathbf{E}_{ext}(.,t)\cdot n = 0$ on Γ. Thus $\operatorname{curl}\mathbf{W}(.,t)\cdot n = 0$ on Γ, for all $t\in[0,T]$, so that $(3.10)_{(ii)}$ is also true. □

Introducing the new variable $\mathcal{E} := \mathbf{E}-\mathbf{W}$, problem (3.9) is reduced to the following problem with homogeneous boundary conditions:

$$\begin{cases} \partial_t\mathcal{E} - \hat\epsilon(\cdot,z)\operatorname{curl}\mathbf{H} + \hat\epsilon(\cdot,z)(\sigma+\partial_z\epsilon(\cdot,z)\partial_t z)\mathcal{E} = & \\ -\partial_t\mathbf{W} - \hat\epsilon(\cdot,z)(\sigma+\partial_z\epsilon(\cdot,z)\partial_t z)\mathbf{W} & \text{in } Q,\\ \partial_t\mathbf{H} + \hat\mu\operatorname{curl}\mathcal{E} = -\hat\mu\operatorname{curl}\mathbf{W} & \text{in } Q,\\ \mathcal{E}\times n = 0 & \text{on } \Sigma,\\ \mathbf{H}\cdot n = 0 & \text{on } \Sigma,\\ \operatorname{div}(\mu\mathbf{H}) = 0 & \text{in } Q,\\ \mathcal{E}(\cdot,0) = \mathbf{E}_0 - \mathbf{W}(.,0),\ \mathbf{H}(\cdot,0) = \mathbf{H}_0 & \text{in } \Omega, \end{cases} \tag{3.15}$$

where

$$\hat\epsilon(x,z(x,t)) := \frac{1}{\epsilon(x,z(x,t))},\quad \text{and}\quad \hat\mu(x) := \frac{1}{\mu(x)},\ \text{for all } x\in\Omega. \tag{3.16}$$

Let us set:

$$\begin{aligned}\mathbf{G}(t) &= (\mathbf{G}_1(t),\mathbf{G}_2(t))\\ &:= (-\partial_t\mathbf{W}(\cdot,t) - \hat\epsilon(\cdot,z(\cdot,t))(\sigma+\partial_z\epsilon(\cdot,z(\cdot,t))\partial_t z(\cdot,t))\mathbf{W}(\cdot,t), -\hat\mu\operatorname{curl}\mathbf{W}(\cdot,t)),\end{aligned} \tag{3.17}$$

for all $t\in[0,T]$.

Let us introduce the Hilbert space

$$\mathcal{H} = \mathbf{L}^2(\Omega)\times\mathbf{J}_n(\Omega,\mu). \tag{3.18}$$

Remark 10. The condition $\operatorname{div}(\mu\mathbf{H}(.,t)) = 0$ and the boundary condition $\mathbf{H}(.,t)\cdot n = 0$ in the Maxwell's equations (3.9) follow from the requirement that $(\mathbf{E}(t),\mathbf{H}(t))$ belongs to $\mathcal{H}$.

Unless otherwise specified, we consider on $\mathcal{H}$ the scalar product induced by $\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$. The corresponding norm will be denoted by $\|\cdot\|_{\mathcal{H}}$ or $\|\cdot\|_{0,\Omega}$. By Proposition 23, $\mathcal{H}$ is a closed subspace of $\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$.

Proposition 26. *1.* $\mathbf{G}(t)\in\mathcal{H}$, *for a.e.* $t\in[0,T]$.

2. *Supposing that the permittivity $\epsilon(x,z)$ does not depend on z but only on x in a neighborhood $\bar{\Omega}\setminus K$ of the boundary of Ω, K compact set of Ω, replacing $\mathbf{W}$ by $\eta\mathbf{W}$ where η is a C^∞ function on $\bar{\Omega}$ equals to 1 in a neighborhood of Γ and 0 on a neighborhood of K, thus still satisfying the three conditions (3.10), we obtain a right-hand side $\mathbf{G}\in C^1([0,T];\mathcal{H})$ and independent of z.*

Proof. 1. From formula (3.17) and $\mathbf{W}\in C^2([0,T];\mathbf{H}^1(\Omega))$ follows that $\mathbf{G}_1(t)\in\mathbf{L}^2(\Omega)$ for a.e. $t\in[0,T]$. By $(3.10)_{(ii)}$ follows that $\mathbf{G}_2(t)\in\mathbf{J}_n(\Omega,\mu)$ for a.e. $t\in[0,T]$.
2. Replacing in equations $(3.15)_{(i)}$ and $(3.15)_{(ii)}$, $\mathbf{W}$ by $\eta(.)\mathbf{W}$, we obtain:

$$\mathbf{G}_1(t)=-\eta(.)\frac{\partial\mathbf{W}}{\partial t}(\cdot,t)-\sigma(\eta\,\hat{\epsilon})(\cdot)\mathbf{W}(\cdot,t)\text{ and }\mathbf{G}_2(t)=-\hat{\mu}\,\mathrm{curl}\,(\eta(.)\mathbf{W}(\cdot,t))$$

which are clearly independent of z. By Proposition 25, $\frac{d\mathbf{W}}{dt}$ belongs to $C^1([0,T];\mathbf{H}^1(\Omega))$ and thus a fortiori to $C^1([0,T];\mathbf{L}^2(\Omega))$. Using hypotheses (H1) and (H2), it follows that $\mathbf{G}_1\in C^1([0,T];\mathbf{L}^2(\Omega))$. By $\mathbf{W}\in C^2([0,T];\mathbf{H}^1(\Omega))$ and hypotheses (H2), it is clear that $\mathbf{G}_2\in C^2([0,T];\mathbf{L}^2(\Omega))$. A fortiori $\mathbf{G}_2\in C^1([0,T];\mathbf{L}^2(\Omega))$ and as $\mathbf{G}_2(t)$ belongs to the closed subspace $\mathbf{J}_n(\Omega,\mu)$ of $\mathbf{L}^2(\Omega)$ for a.e. $t\in[0,T]$, $\mathbf{G}_2\in C^1([0,T];\mathbf{J}_n(\Omega,\mu))$.

In conclusion $\mathbf{G}=(\mathbf{G}_1,\mathbf{G}_2)\in C^1([0,T];\mathcal{H})$ and is independent of z. □

Let us observe also that $\mathcal{E}(.,0)=\mathbf{E}_0-\mathbf{W}(.,0)\in\mathbf{H}_0(\mathrm{curl}\,,\Omega)$. Renaming $\mathcal{E}$ by $\mathbf{E}$, we will study thus in the following, the well posedness of the Maxwell initial value problem with homogeneous boundary conditions:

$$\begin{cases}\partial_t\mathbf{E}-\hat{\epsilon}(\cdot,z)\,\mathrm{curl}\,\mathbf{H}+\hat{\epsilon}(\cdot,z)(\sigma+\partial_z\epsilon(\cdot,z)\partial_t z)\mathbf{E}=\mathbf{G}_1(t) & \text{in } Q,\\ \partial_t\mathbf{H}+\hat{\mu}\,\mathrm{curl}\,\mathbf{E}=\mathbf{G}_2(t) & \text{in } Q,\\ \mathbf{E}\times n=0 & \text{on } \Sigma,\\ \mathbf{H}\cdot n=0 & \text{on } \Sigma,\\ \mathrm{div}\,(\mu\mathbf{H})=0 & \text{in } Q,\\ \mathbf{E}(\cdot,0)=\mathbf{E}_0,\ \mathbf{H}(\cdot,0)=\mathbf{H}_0 & \text{in } \Omega,\end{cases}\tag{3.19}$$

with a right-hand side $\mathbf{G}=(\mathbf{G}_1,\mathbf{G}_2)$ belonging to $C^1([0,T];\mathcal{H})$ and an initial condition $(\mathbf{E}_0,\mathbf{H}_0)\in\mathbf{H}_0(\mathrm{curl}\,,\Omega)\times\mathbf{J}^1_n(\Omega,\mu)$.

In our study, we will need also to endow

$$\mathcal{H}=\mathbf{L}^2(\Omega)\times\mathbf{J}_n(\Omega,\mu),\tag{3.20}$$

with other scalar products than the scalar product induced by $\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$ on $\mathcal{H}$.

We will also consider on $\mathcal{H}$ the scalar product with weights $\epsilon:=\epsilon(\cdot,z(\cdot,t))$ and $\mu:=\mu(\cdot)$:

$$\left(\begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix},\begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix}\right)_{\mathcal{H}_t}=\int_\Omega\{\epsilon(x,z(x,t))\varphi_1(x)\cdot\varphi_2(x)+\mu(x)\psi_1(x)\cdot\psi_2(x)\}dx,\tag{3.21}$$

for all (φ_1,φ_2), $(\psi_1,\psi_2)\in\mathcal{H}$. When $\mathcal{H}$ is endowed with this scalar product it will be denoted by $\mathcal{H}_t$.

Remark 11. Due to assumptions (H1)-(H2), the norms $\|\cdot\|_{\mathcal{H}_t}$ and $\|\cdot\|_{\mathcal{H}}$ are equivalent:

$$\begin{aligned} min(\epsilon_0,\mu_0)\left[\int_\Omega (|\varphi(x)|^2+|\psi(x)|^2)dx\right] &\le \int_\Omega (\epsilon(x,z(x,t))|\varphi(x)|^2+\mu(x)|\psi(x)|^2)dx \\ &\le max(\epsilon_1,\mu_1)\left[\int_\Omega (|\varphi(x)|^2+|\psi(x)|^2)dx\right] \end{aligned} \tag{3.22}$$

for all $(\varphi,\psi)\in\mathcal{H}$.

To rewrite the initial value problem (3.19) in the form of an abstract Cauchy problem, we now define two families of operators in the real Hilbert space $\mathcal{H}$. For $t\in[0,T]$, we firstly define the unbounded operator $\mathcal{A}(t)$ in $\mathcal{H}$. Its domain $D(\mathcal{A}(t))$ being independent of t, we denote it $D(\mathcal{A})$, where

$$D(\mathcal{A}) = \mathbf{H}_0(\mathrm{curl}\,,\Omega)\times\mathbf{J}^1_n(\Omega,\mu). \tag{3.23}$$

For all $\phi=(\varphi,\psi)\in D(\mathcal{A})$, we set

$$\mathcal{A}(t)\phi = \{\hat{\epsilon}(\cdot,z(\cdot,t))\,\mathrm{curl}\,\psi, -\hat{\mu}\,\mathrm{curl}\,\varphi\} \quad \in\mathcal{H}. \tag{3.24}$$

We also define the bounded operator in $\mathcal{H}$:

$$\mathcal{M}(t)\phi = \{-\hat{\epsilon}(\cdot,z(\cdot,t))(\sigma+\partial_z\epsilon(\cdot,z(\cdot,t))\partial_t z)\varphi, 0\}. \tag{3.25}$$

Having introduced these operators $\mathcal{A}(t)$ and $\mathcal{M}(t)$, the initial value problem (3.19) may be rewritten in the form of the following abstract Cauchy problem: given $\mathbf{G}=(\mathbf{G}_1,\mathbf{G}_2)$ belonging to $C([0,T];D(\mathcal{A}))$ or $C^1([0,T];\mathcal{H})$ and an initial condition $(\mathbf{E}_0,\mathbf{H}_0)\in D(\mathcal{A})$, find $(\mathbf{E},\mathbf{H})\in C([0,T];D(\mathcal{A}))\cap C^1([0,T];\mathcal{H})$ such that for all $t\in[0,T]$

$$\begin{cases} \frac{d}{dt}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}(t) = (\mathcal{A}(t)+\mathcal{M}(t))\begin{pmatrix}\mathbf{E}(t)\\ \mathbf{H}(t)\end{pmatrix}+\mathbf{G}(t), \\ \begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}(0) = \begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix}. \end{cases} \tag{3.26}$$

Proposition 27. *1. The common domain $\mathcal{D}(\mathcal{A})$ of the linear operators $\mathcal{A}(t)$ is dense in $\mathcal{H}$.*

2. $\mathcal{A}(t)$ is a closed linear operator in $\mathcal{H}$.

3. $D(\mathcal{A})$ endowed with the graph norm is a separable Hilbert space.

4. For every $t\in[0,T]$, $\mathcal{A}(t)$ and $-\mathcal{A}(t)$ are maximal dissipative operators in $\mathcal{H}_t$ i.e

$$(\mathcal{A}(t)\phi,\phi)_{\mathcal{H}_t} = 0 \quad \textit{for all } \phi\in D(\mathcal{A}), \tag{3.27}$$

and the range of the linear operator $(\mathcal{A}(t)\pm\lambda I)$ is equal to $\mathcal{H}$ for all $\lambda>0$. In particular, for all $t\in[0,T]$, $\mathcal{A}(t)$ generates a C_0-group of unitary operators in $\mathcal{H}_t$ [44, (6.2) pp. 22-23].

Proof. 1. $\mathbf{J}^1_n(\Omega,\mu)$ is dense in $\mathbf{J}_n(\Omega,\mu)$ [40, Lemma 2.3]. The density of $\mathbf{H}_0(curl,\Omega)$ in $\mathbf{L}^2(\Omega)$ is due to the fact that

$$\mathcal{D}(\Omega)^3\subset\mathbf{H}_0(curl,\Omega)\subset\mathbf{L}^2(\Omega) \text{ and } \mathcal{D}(\Omega)^3 \text{ is dense in } \mathbf{L}^2(\Omega),$$

where $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$. It follows that $D(\mathcal{A})$ is dense in $\mathcal{H}$.

2. Let us fix $t \in [0,T]$. If $(\phi_k := (\varphi_k, \psi_k))_{k\in\mathbb{N}} \subset D(\mathcal{A})$ such that $\phi_k \to \phi$ in $\mathcal{H}$ and $\mathcal{A}(t)\phi_k \to \Psi$ in $\mathcal{H}$, we have then

$$\varphi_k \to \varphi,\ \psi_k \to \psi \text{ in } \mathbf{L}^2(\Omega),$$

and $(\operatorname{curl}\psi_k)_{k\in\mathbb{N}}$, $(\operatorname{curl}\varphi_k)_{k\in\mathbb{N}}$ converge in $\mathbf{L}^2(\Omega)$. But $\operatorname{curl}\psi_k \to \operatorname{curl}\psi$ in $\mathcal{D}'(\Omega)^3$, $\operatorname{curl}\varphi_k \to \operatorname{curl}\varphi$ in $\mathcal{D}'(\Omega)^3$, so that $\operatorname{curl}\psi_k \to \operatorname{curl}\psi$ in $\mathbf{L}^2(\Omega)$ and $\operatorname{curl}\varphi_k \to \operatorname{curl}\varphi$ in $\mathbf{L}^2(\Omega)$. Also $\operatorname{div}(\mu\psi_k)(=0) \to \operatorname{div}(\mu\psi)$ in $\mathcal{D}'(\Omega)$. Therefore $\operatorname{curl}\varphi \in \mathbf{L}^2(\Omega)$, $\operatorname{curl}\psi \in \mathbf{L}^2(\Omega)$, $\operatorname{div}(\mu\psi) = 0$ and

$$\varphi_k \to \varphi \text{ in } \mathbf{H}(\operatorname{curl},\Omega),$$

$$\psi_k \to \psi \text{ in } \mathbf{H}(\operatorname{div},\Omega) \cap \mathbf{H}(\operatorname{curl},\Omega).$$

This implies that

$$\varphi_k \times n(=0) \to \varphi \times n \text{ in } H^{-1/2}(\Gamma)^3$$

and

$$\psi_k \cdot n(=0) \to \psi \cdot n \text{ in } H^{-1/2}(\Gamma).$$

Therefore $\varphi \times n = 0$ and $\psi \cdot n = 0$ on Γ. We conclude that $\phi \in D(\mathcal{A})$ and $\Psi = \mathcal{A}(t)\phi$. Consequently, $\mathcal{A}(t)$ is a closed operator in $\mathcal{H}$.

3. $D(\mathcal{A})$ endowed with the graph norm may be seen as a subspace of $\mathbf{H}_0(curl;\Omega)\times\mathbf{H}^1(\Omega)$ [24, p.54] and is thus separable [20, (3.10.9)].

4. The proof is done in [40, Lemma 3.1].

□

Corollary 28. *1. For each $t \in [0,T]$, $\mathcal{A}(t)$ is an infinitesimal generator of a C_0-semigroup $(P_t(s))_{s\geq 0}$ of contractions on $\mathcal{H}_t$.*

2. For each $t \in [0,T]$, $\mathcal{A}(t) + \mathcal{M}(t)$ is the infinitesimal generator of a C_0 semigroup $(T_t(s))_{s\geq 0}$ on $\mathcal{H}_t$ satisfying

$$\| T_t(s) \|_{\mathcal{L}(\mathcal{H}_t)} \leq e^{\,s\|\mathcal{M}(t)\|_{\mathcal{L}(\mathcal{H}_t)}}\ ,\ \textit{for all } s \geq 0. \tag{3.28}$$

Proof. 1. From Proposition 27 follows that $\mathcal{A}(t)$ is an infinitesimal generator of a C_0-semigroup $(P_t(s))_{s\geq 0}$ of contractions on $\mathcal{H}_t$.

2. As $\mathcal{M}(t)$ is a bounded linear operator on $\mathcal{H}_t$, by Theorem 1.1 in [44, p. 76], $\mathcal{A}(t) + \mathcal{M}(t)$ is also the infinitesimal generator of a C_0 semigroup on $\mathcal{H}_t$ satisfying moreover the growth bound (3.28).

□

Corollary 29. *$\mathcal{A}^*(t) = -\mathcal{A}(t)$ in $\mathcal{H}$ endowed with the scalar product $(\cdot,\cdot)_{\mathcal{H}_t}$, for all $t \in [0,T]$. In particular $D(\mathcal{A}^*(t)) = D(\mathcal{A})$ for all $t \in [0,T]$.*

Proof. The proof is a consequence of Stone's Theorem [44, p. 41] that we recall below:

Theorem 30 (Stone's theorem)**.** *A is the infinitesimal generator of a C_0-group of unitary operators on a Hilbert space H if and only if $A^* = -A$.*

By Proposition 27 and [44, (6.2) p. 23], $\mathcal{A}(t)$ satisfies the hypothesis of Stone's Theorem in the real Hilbert space $\mathcal{H}$ endowed with the scalar product $(\cdot,\cdot)_{\mathcal{H}_t}$. Thus $\mathcal{A}(t)^* = -\mathcal{A}(t)$, for all $t \in [0,T]$. □

3.2.2 Well posedness of Maxwell's equations using evolution systems theory

In appendix B, we have proved the existence of a weak solution to the linear Maxwell' system by using Galerkin's method like in [21], but we are not able to prove its uniqueness like in [21] due to the dependence of the permittivity coefficient with respect to the temperature and thus a fortiori with respect to time. Also, in our coupled nonlinear problem with the heat equation, to prove the existence of local solutions, we will need information on the dependence of the solution to the Maxwell' system with tespect to the temperature.

Consequently, in this subsection, we will use rather the method of evolution systems in the hyperbolic case [44, Chapter 5, pp.126-149] to establish the well posedness of our initial boundary value problem (3.26). We must thus verify hypothesis (H_1) in [44, p.135] i.e. that $\{\mathcal{A}(t)+\mathcal{M}(t)\}_{t\in[0,T]}$ [44, p.130] is a stable family of infinitesimal generators [44, p.130], hypotheses $(H_2)^+$ in [44, p.142] and (H_3) in [44, p.135]:

$(H_2)^+$ There is a family $\{Q(t)\}_{t\in[0,T]}$ of isomorphisms of $Y := D(\mathcal{A})$ onto $\mathcal{H}$ such that for every Y, $Q(t)v$ is continuously differentiable in $\mathcal{H}$ on $[0,T]$ and

$$Q(t)\mathcal{A}(t)Q(t)^{-1} = \mathcal{A}(t) + B(t)$$

where $B(t)$, $0 \le t \le T$, is a strongly continuous family of bounded operators on $\mathcal{H}$.

(H_3) For $t \in [0,T]$, $\mathcal{A}(t)$ is a bounded operator from Y into $\mathcal{H}$ and $t \to \mathcal{A}(t)$ is continuous in the $\mathcal{L}(Y,\mathcal{H})$ (linear continuous form from Y to $\mathcal{H}$).

Firstly, concerning the stability of the family of infinitesimal generators $\{\mathcal{A}(t)\}_{t\in[0,T]}$, we have the following proposition:

Proposition 31. *The family $\{\mathcal{A}(t)\}_{t\in[0,T]}$ of infinitesimal generators of the C_0 semigroups $(P_t(s))_{s\ge 0}$ is stable on $\mathcal{H}$. More precisely, there is a constant $M \ge 1$ such that:*

$$\| \prod_{j=1}^{k} P_{t_j}(s_j) \|_{\mathcal{L}(\mathcal{H})} \le M, \text{ for all } s_j \ge 0, \quad j = 1, \cdots, k \tag{3.29}$$

and any finite sequence $0 \le t_1 \le t_2 \le \cdots \le t_k \le T$, $k = 1, 2, \cdots$.

Remark 12. For any finite sequence $0 \le t_1 \le t_2 \le \cdots \le t_k$ of real numbers, products like $\prod_{j=1}^{k} P_{t_j}(s_j)$ means $P_{t_k}(s_k)P_{t_{k-1}}(s_{k-1})\cdots P_{t_2}(s_2)P_{t_1}(s_1)$. We say that they are "time-ordered" [44, p.130]. This is important because in general such products are non-commutative.

The proof of Proposition 31 requires the following lemma:

Lemma 32. *Let* $q(t) = \int_\Omega (\epsilon(x, z(x,t))|\varphi(x)|^2 + \mu(x)|\psi(x)|^2)dx$, *for all* $(\varphi, \psi) \in \mathcal{H}$. *Then there exists* $c_0 > 0$ (3.32) *such that*

$$\sqrt{q(t)} \leq e^{\frac{c_0}{2}(t-s)}\sqrt{q(s)}, \quad \text{for all } t \geq s. \tag{3.30}$$

Proof. By assumption (H1) and (3.8), we have,

$$\begin{aligned} \frac{dq}{dt}(t) &= \int_\Omega \frac{dz}{dt}(x,t)\epsilon_z(x, z(x,t))|\varphi(x)|^2 dx \\ &\leq \|\frac{dz}{dt}\|_{\infty,\bar{Q}}\|\epsilon_z\|_{\infty,\overline{\Omega}\times\mathbb{R}} \left\{ \int_\Omega |\varphi(x)|^2 + |\psi(x)|^2 dx \right\} \\ &\leq c_0\, q(t), \end{aligned} \tag{3.31}$$

with

$$c_0 = \frac{1}{\min(\epsilon_0,\ \mu_0)}\|\frac{dz}{dt}\|_{\infty,\bar{Q}}\|\epsilon_z\|_{\infty,\overline{\Omega}\times\mathbb{R}}. \tag{3.32}$$

Then

$$\frac{d}{dt}[\ln q(t)] \leq c_0.$$

If we integrate from s to t in the above inequality, we obtain

$$q(t) \leq q(s)e^{c_0(t-s)},$$

and therefore the requested estimate is proved. □

Proof. (**of Proposition 31**) First of all note that $(P_t(s))_{s\geq 0}$ is a C_0 semigroup of contractions in $\mathcal{H}_t$ for all $t \in [0,T]$ i.e. that

$$\| P_t(s) \|_{\mathcal{L}(\mathcal{H}_t)} \leq 1 \text{ for all } s \geq 0.$$

Let us set $c_1 = (max(\epsilon_1, \mu_1))^{1/2}$. Then by Lemma 32:

$$\begin{aligned} \| \prod_{j=1}^{k} P_{t_j}(s_j)v \|_{\mathcal{H}} &:= \| P_{t_k}(s_k)\cdots P_{t_1}(s_1)v \|_{\mathcal{H}} \\ &\leq c_1 \| P_{t_k}(s_k)\cdots P_{t_1}(s_1)v \|_{\mathcal{H}_{t_k}} \\ &\leq c_1 \| P_{t_k}(s_k) \|_{\mathcal{L}(\mathcal{H}_{t_k})} \| P_{t_{k-1}}(s_{k-1})\cdots P_{t_1}(s_1)v \|_{\mathcal{H}_{t_k}} \\ &\leq c_1\, e^{\frac{c_0}{2}(t_k - t_{k-1})} \| P_{t_{k-1}}(s_{k-1})\cdots P_{t_1}(s_1)v \|_{\mathcal{H}_{t_{k-1}}} \\ &\leq c_1\, e^{\frac{c_0}{2}(t_k - t_{k-1})} \| P_{t_{k-1}}(s_{k-1}) \|_{\mathcal{L}(\mathcal{H}_{t_{k-1}})} \| P_{t_{k-2}}(s_{k-2})\cdots P_{t_1}(s_1)v \|_{\mathcal{H}_{t_{k-1}}} \\ &\leq c_1\, e^{\frac{c_0}{2}(t_k - t_{k-1})} e^{c_0(t_{k-1}-t_{k-2})} \| P_{t_{k-2}}(s_{k-2})\cdots P_{t_1}(s_1)v \|_{\mathcal{H}_{t_{k-2}}} \\ &\ \ \vdots \\ &\leq c_1 e^{\frac{c_0}{2}(t_k - t_1)} \| v \|_{\mathcal{H}_{t_1}} \\ &\leq c_1^2 e^{\frac{c_0}{2}T} \| v \|_{\mathcal{H}}\ . \end{aligned} \tag{3.33}$$

Since $\mathcal{A}(t)$, for all $t \in [0,T]$ is a C_0 semigroup of contractions in $\mathcal{H}_t$, then by the Hille-Yosida Theorem [44, p. 8], the resolvent set $\rho(\mathcal{A}(t))$ contains $\mathbb{R}_+^*$, i.e. $\rho(\mathcal{A}(t)) \supset]0, \infty[$ for all $t \in [0,T]$. Also by the last inequality in (3.33), inequality (2.4) of Theorem 2.2 of [44, p.131] is verified with $M = c_1^2 e^{\frac{c_0}{2}T}$ (3.32) and $\omega = 0$, proving the stability of $\{\mathcal{A}(t)\}_{t\in[0,T]}$. □

Since $\mathcal{M}(t)$ is a bounded linear operator for all $t \in [0,T]$, by Theorem 1.1 in [44, p. 76], $\mathcal{A}(t)+\mathcal{M}(t)$ is also the infinitesimal generator of a C_0 semigroup on the real Hilbert space $\mathcal{H}$. Concerning the stability of the family of infinitesimal generators $\{\mathcal{A}(t)+\mathcal{M}(t)\}_{t\in[0,T]}$, we have the following proposition:

Corollary 33. *$\{\mathcal{A}(t)+\mathcal{M}(t)\}_{t\in[0,T]}$ is a stable family of infinitesimal generators of C_0 semigroups $(T_t(s))_{s\geq 0}$ on $\mathcal{H}$ with stablity constants M and KM, where K is any constant such that $\| \mathcal{M}(t) \|_{\mathcal{L}(\mathcal{H})} \leq K$ for all $t \in [0,T]$.*

Proof. For each $t \in [0,T]$, let us denote by $(T_t(s))_{s\geq 0}$ the C_0 semigroup generated by the operator $\mathcal{A}(t)+\mathcal{M}(t)$. By the previous Lemma $\{\mathcal{A}(t)\}_{t\in[0,T]}$ is stable with stability constants M and 0. Then by Theorem 2.3 in [44, p. 132], $\{\mathcal{A}(t)+\mathcal{M}(t)\}_{t\in[0,T]}$ is a stable family of infinitesimal generators with stablity constants M and KM i.e

$$\| \prod_{j=1}^{k} R(\lambda_j; \mathcal{A}(t_j)+\mathcal{M}(t_j)) \|_{\mathcal{L}(\mathcal{H})} \leq \frac{M}{\prod_{j=1}^{k}(\lambda_j - KM)} \text{ for all } \lambda_j > KM \tag{3.34}$$

and

$$\| \prod_{j=1}^{k} T_{t_j}(s_j) \|_{\mathcal{L}(\mathcal{H})} \leq M e^{KM\sum_{j=1}^{k} s_j} \text{ for all } s_j \geq 0, \tag{3.35}$$

with any finite sequence $0 \leq t_1 \leq \cdots \leq t_k \leq T$, $k = 1, 2, 3, \cdots$. □

Thus the family $\{\mathcal{A}(t)+\mathcal{M}(t)\}_{t\in[0,T]}$ of infinitesimal generators on $\mathcal{H}$ verifies hypothesis (H_1) in [44, p.135].

In the following, according with the notations of [44, Chapter 5, pp.126-149], we will denote by Y the domain of the operator $\mathcal{A}(t)$ which is also the domain of the operator $\mathcal{A}(t)+\mathcal{M}(t)$. The domains of these operators do not depend on t. For this reason Y has also been denoted previously by $D(\mathcal{A})$ (4.12).

Proposition 34. *Y is $\mathcal{A}(t)+\mathcal{M}(t)$ admissible (see Definition 5.3 in [44, p.122] or [44, p.135]) for all $t \in [0,T]$ i.e. Y is an invariant subspace of $(T_t(s))_{s\geq 0}$. Moreover the C_0 semigroup generated by the operator $\mathcal{A}(t)+\mathcal{M}(t)$, and the restriction of $(T_t(s))_{s\geq 0}$ to Y is a C_0 semigroup in Y (i.e. it is a strongly continuous semigroup for the graph norm $\| \cdot \|_Y$).*

Proof. Since Y is the domain of $\mathcal{A}(t)+\mathcal{M}(t)$, then Y is stable by the semigroup $(T_t(s))_{s\geq 0}$ whose generator is $\mathcal{A}(t)+\mathcal{M}(t)$ for all $t \in [0,T]$. Let us now check that

$$T_t(s)_{|_Y} : Y \to Y \tag{3.36}$$

is a C_0 semigroup on Y endowed with the graph norm. Let $v \in Y$, we observe that

$$(\mathcal{A}(t)+\mathcal{M}(t))T_t(s)v = T_t(s)(\mathcal{A}(t)+\mathcal{M}(t))v, \tag{3.37}$$

then by (3.35)

$$\| (\mathcal{A}(t)+\mathcal{M}(t))T_t(s)v \|_{\mathcal{H}} \leq M e^{KMs}(\| \mathcal{A}(t)v \|_{\mathcal{H}} + K \| v \|_{\mathcal{H}}).$$

Due to the stability of $(T_t(s))_{s\geq 0}$ and to the last inequatlity we obtain

$$\begin{aligned}\| \mathcal{A}(t)T_t(s)v \|_{\mathcal{H}} &\leq Me^{KMs}(\| \mathcal{A}(t)v \|_{\mathcal{H}} + K \| v \|_{\mathcal{H}}) + K \| T_t(s)v \|_{\mathcal{H}} \\ &\leq Me^{KMs}(\| \mathcal{A}(t)v \|_{\mathcal{H}} + K \| v \|_{\mathcal{H}}) + KMe^{KMs} \| v \|_{\mathcal{H}} \\ &\leq 2KMe^{KMs} \| v \|_{\mathcal{H}} + Me^{KMs} \| \mathcal{A}(t)v \|_{\mathcal{H}} .\end{aligned}$$

This implies

$$\begin{aligned}\| T_t(s)v \|_Y &:= \| \mathcal{A}(t)T_t(s)v \|_{\mathcal{H}} + \| T_t(s)v \|_{\mathcal{H}} \\ &\leq M(2K+1)e^{KMs} \| v \|_{\mathcal{H}} + Me^{KMs} \| \mathcal{A}(t)v \|_{\mathcal{H}} \\ &\leq M(2K+1)e^{KMs}(\| v \|_{\mathcal{H}} + \| \mathcal{A}(t)v \|_{\mathcal{H}}).\end{aligned}$$

Therefore (A.6) follows. Furthermore recalling (3.37)

$$\begin{aligned}&\| \mathcal{A}(t)(T_t(s)v - v) \|_{\mathcal{H}} = \| T_t(s)(\mathcal{A}(t) + \mathcal{M}(t))v - \mathcal{M}(t)T_t(s)v - \mathcal{A}(t)v \|_{\mathcal{H}} \\ &= \| T_t(s)(\mathcal{A}(t) + \mathcal{M}(t))v - (\mathcal{A}(t) + \mathcal{M}(t))v + \mathcal{M}(t)(v - T_t(s)v) \|_{\mathcal{H}} \\ &= \| (T_t(s) - I)(\mathcal{A}(t) + \mathcal{M}(t))v \|_{\mathcal{H}} + \| \mathcal{M}(t) \|_{\mathcal{L}(\mathcal{H})} \| v - T_t(s)v \|_{\mathcal{H}} \\ &\to 0 \text{ as } s \to 0^+.\end{aligned} \tag{3.38}$$

Thus $\| T_t(s)v - v \|_Y \to 0$ as $s \to 0^+$. □

Proposition 34 is a part of hypothesis (H_2) in [44, p.135]. But we will now prove rather that $\{\mathcal{A}(t) + \mathcal{M}(t)\}_{t\in[0,T]}$ verifies the stronger hypothesis $(H_2)^+$ in [44, p.142]. In fact, as we already know that the stability hypothesis (H_1) in [44, p.135] is verified by the family $\{\mathcal{A}(t) + \mathcal{M}(t)\}_{t\in[0,T]}$, this will imply that hypothesis (H_2) in [44, p.135] is also verified consequently to [44, Lemma 4.4 p.142]). By showing that $\{\mathcal{A}(t) + \mathcal{M}(t)\}_{t\in[0,T]}$ verifies hypotheses (H_1), (H_3) [44, p.135], and hypothesis $(H_2)^+$ in [44, p.142] instead of simply hypotheses (H_1), (H_2), (H_3) [44, p.135], we will obtain that the generated evolution system $U(t,s)$, $0 \leq s \leq t \leq T$, by $\{\mathcal{A}(t) + \mathcal{M}(t)\}_{t\in[0,T]}$ satisfies properties (E_1), (E_2), (E_3) in [44, p.135]:

(E_1) $\| U(t,s) \| \leq M \exp\{\omega(t-s)\}$ for $0 \leq s \leq t \leq T$.

(E_2) $\frac{\partial^+}{\partial t}U(t,s)v \mid_{t=s} = \{\mathcal{A}(s) + \mathcal{M}(s)\}v$ for $v \in Y, 0 \leq s \leq T$.

(E_3) $\frac{\partial}{\partial s}U(t,s)v = -U(t,s)\{\mathcal{A}(s) + \mathcal{M}(s)\}v$ for $v \in Y, 0 \leq s \leq t \leq T$,

(E_4) $U(t,s)Y \subset Y$ for $0 \leq s \leq t \leq T$,

(E_5) for $v \in Y$, $U(t,s)v$ is continuous in Y for $0 \leq s \leq t \leq T$

where the derivative from the right in (E_2) and the derivative in (E_3) are in the strong sense in $\mathcal{H}$. These properties are essentials to solve our inhomogeneous initial value problem (3.26) [44, Theorem 5.2 p.146]. To achieve that purpose, we will need a stronger hypothesis on the conductivity σ than made previously in subsection 3.2.1. We suppose in the following that

(H4)

$$\sigma \in W^{1,\infty}(\Omega). \tag{3.39}$$

We define $Q(t) := I - \mathcal{A}(t)$ for all $t \in [0,T]$. $(Q(t))_{t\in[0,T]}$ is a family of isomorphisms of Y onto $\mathcal{H}$. We firstly check that $(H_2)^+$ in[44, p.142] is true :

Proposition 35. *For every $v = (\phi, \psi) \in Y$, the mapping $t \mapsto Q(t)v$ is continuously differentiable in $\mathcal{H}$ on $[0,T]$. Denoting by $\frac{d}{dt}Q(t)$ the mapping $Y \to \mathcal{H} : v \mapsto \frac{d}{dt}(Q(t)v)$, the mapping $t \in [0,T] \to \frac{d}{dt}Q(t) \in \mathcal{L}(Y,\mathcal{H})$ is also continuous.*

Proof. Let us first show that the mapping $v \in Y \mapsto \frac{d}{dt}Q(t)v \in \mathcal{H}$ is linear and bounded. By assumptions (H1) and (3.8), we have:

$$\begin{aligned} \| \frac{d}{dt}Q(t)v \|_{\mathcal{H}} =& \| \frac{\partial \hat{\epsilon}}{\partial z}(\cdot, z(\cdot,t))\frac{\partial z}{\partial t}(\cdot,t)\mathrm{curl}\,\psi \|_{\mathbf{L}^2(\Omega)} \\ \leq& \| \frac{\partial \hat{\epsilon}}{\partial z}(\cdot, z(\cdot,t))\frac{\partial z}{\partial t}(\cdot,t) \|_{\infty,\bar{\Omega}} \| v \|_Y . \end{aligned} \tag{3.40}$$

Furthermore, for all $v \in Y$, using assumptions (H1) and (3.8), the mapping $t \in [0,T] \mapsto \frac{d}{dt}Q(t)v \in \mathcal{H}$ is continuous:

$$\begin{aligned} & \| (\frac{d}{dt}Q(t)v)_{t=t_2} - (\frac{d}{dt}Q(t)v)_{t=t_1} \|_{\mathcal{H}} \\ \leq & \| \frac{\partial \hat{\epsilon}}{\partial z}(\cdot, z(\cdot,t_2))\frac{\partial z}{\partial t}(\cdot,t_2) - \frac{\partial \hat{\epsilon}}{\partial z}(\cdot, z(\cdot,t_1))\frac{\partial z}{\partial t}(\cdot,t_1) \|_{\infty,\bar{\Omega}} \| v \|_Y \\ \leq & O(|t_2 - t_1|) \| v \|_Y, \end{aligned} \tag{3.41}$$

for all $0 \leq t_1 \leq t_2 \leq T$. We have:

$$\| \frac{d}{dt}Q(t_2) - \frac{d}{dt}Q(t_1) \|_{\mathcal{L}(Y,\mathcal{H})} \leq O(|t_2 - t_1|) \to 0 \text{ as } | t_2 - t_1 | \to 0, \tag{3.42}$$

proving that the mapping $[0,T] \to \mathcal{L}(Y,\mathcal{H}) : t \mapsto \frac{d}{dt}Q(t)$ is also continuous. □

To prove hypothesis $(H_2)^+$ in [44, p.142] is true in our context, it remains to prove that

$$Q(t)(\mathcal{A}(t) + \mathcal{M}(t))Q(t)^{-1} = (\mathcal{A}(t) + \mathcal{M}(t)) + B(t)$$

where

$$B : [0,T] \to \mathcal{L}(\mathcal{H}) : t \mapsto B(t)$$

is a strongly continuous family of bounded linear operators on $\mathcal{H}$ to be defined later. To prove the following proposition, we need a stronger hypothesis on the permittivity $\epsilon(x,z)$ than we have made previously in subsection 3.2.1. In the following, we suppose moreover on the permittivity $\epsilon(x,z)$ that:

(H5) $\frac{\partial^2 \epsilon}{\partial z^2}$ and $\frac{\partial^2 \epsilon}{\partial x_k \partial z}$ $(k = 1,2,3)$ exist and are continuous and bounded.

Proposition 36. *For all $t \in [0,T]$, we have*

$$\mathcal{M}(t)Y \subset Y. \tag{3.43}$$

Proof. Let $\varphi \in \mathbf{H}_0(\mathrm{curl}\,,\Omega)$ we will prove that $\hat{\epsilon}(\cdot, z(\cdot,t))(\sigma + \partial_z \epsilon(\cdot, z(\cdot,t))\partial_t z)\varphi \in \mathbf{H}_0(\mathrm{curl}\,,\Omega)$. We use the standard formula

$$\begin{aligned} & \mathrm{curl}\,((\hat{\epsilon}(\cdot, z(\cdot,t))(\sigma + \partial_z \epsilon(\cdot, z(\cdot,t))\partial_t z(\cdot,t)))\varphi)(x) \\ & = \hat{\epsilon}(x, z(x,t))(\sigma + \partial_z \epsilon(x, z(x,t))\partial_t z(x,t))\mathrm{curl}\,\varphi(x) \\ & + \nabla_x(\hat{\epsilon}(\cdot, z(\cdot,t))(\sigma + \partial_z \epsilon(\cdot, z(\cdot,t))\partial_t z(\cdot,t)))(x) \times \varphi(x), \text{ for all } x \in \Omega. \end{aligned} \tag{3.44}$$

By assumptions (H1), (H5) and (H4) on ϵ, z, $\partial_t z$ and σ, we have:

$$\| (x,t) \mapsto \hat{\epsilon}(x,z(x,t))(\sigma + \partial_z \epsilon(x,z(x,t))\partial_t z(x,t)) \|_{\infty,\bar{Q}} < +\infty$$

and

$$\| (x,t) \mapsto \nabla_x(\hat{\epsilon}(\cdot,z(\cdot,t))(\sigma + \partial_z \epsilon(\cdot,z(\cdot,t))\partial_t z(\cdot,t)))(x) \|_{\infty,\bar{Q}} < +\infty.$$

Therefore

$$\operatorname{curl}\left((\hat{\epsilon}(\cdot,z(\cdot,t))(\sigma + \partial_z \epsilon(\cdot,z(\cdot,t))\partial_t z(\cdot,t)))\varphi\right) \in \mathbf{L}^2(\Omega).$$

Also $[\hat{\epsilon}(\cdot,z(\cdot,t))(\sigma + \partial_z \epsilon(\cdot,z(\cdot,t))\partial_t z(.,t))\varphi] \times \mathbf{n}|_\Gamma = \mathbf{0}$. This concludes the proof. □

It is now easy to verify that the domain of definition of the operator $Q(t)(\mathcal{A}(t) + \mathcal{M}(t))Q(t)^{-1}$ is Y. Thus $Q(t)(\mathcal{A}(t) + \mathcal{M}(t))Q(t)^{-1}$ and $\mathcal{A}(t) + \mathcal{M}(t)$ have the same domain of definition Y. Also the domain of definition of $Q(t)\mathcal{A}(t)Q(t)^{-1}$ is Y. Thus

$$\begin{aligned} Q(t)(\mathcal{A}(t) + \mathcal{M}(t))Q(t)^{-1} &= \mathcal{A}(t) + Q(t)\mathcal{M}(t)Q(t)^{-1} \\ &= (\mathcal{A}(t) + \mathcal{M}(t)) + Q(t)\mathcal{M}(t)Q(t)^{-1} - \mathcal{M}(t), \end{aligned} \tag{3.45}$$

for all $t \in [0,T]$.

To prove that hypothesis $(H_2)^+$ in [44, p.142] is true in our context, it remains thus to prove that the mapping

$$t \in [0,T] \mapsto B(t) \coloneqq Q(t)\mathcal{M}(t)Q(t)^{-1} - \mathcal{M}(t) \in \mathcal{L}(\mathcal{H}),$$

is a strongly continuous family of bounded operators on $\mathcal{H}$.

Lemma 37. *The mapping $[0,T] \to \mathcal{L}(Y,\mathcal{H}) : t \mapsto \mathcal{A}(t)$ is continuous. Consequently the mapping $[0,T] \to \mathcal{L}(Y,\mathcal{H}) : t \mapsto Q(t)$ is also continuous.*

Proof. Let $(\varphi,\psi) \in Y = \mathbf{H}_0(\operatorname{curl},\Omega) \times \mathbf{J}^1_n(\Omega,\mu)$. By the definition of the operator $\mathcal{A}(t)$, we have $\mathcal{A}(t)\,(\varphi,\psi) = (\hat{\epsilon}(.,z(.,t))\operatorname{curl}\psi, -\hat{\mu}\operatorname{curl}(\varphi)) \in \mathcal{H}$. Then for $t_1,t_2 \in [0,T]$,

$$(\mathcal{A}(t_2) - \mathcal{A}(t_1))\,(\varphi,\psi) = ((\hat{\epsilon}(.,z(.,t_2)) - \hat{\epsilon}(.,z(.,t_1)))\operatorname{curl}\psi, 0)\,.$$

Thus

$$\|(\mathcal{A}(t_2) - \mathcal{A}(t_1))\,(\varphi,\psi)\|_{\mathcal{H}} \le \|\hat{\epsilon}(.,z(.,t_2)) - \hat{\epsilon}(.,z(.,t_1))\|_{\infty,\Omega}\,\|(\varphi,\psi)\|_Y\,.$$

The function $t \mapsto \hat{\epsilon}(.,z(.,t))$ is uniformly continuous from the closed interval $[0,T]$ into $C(\bar{\Omega})$. Thus

$$\|(\mathcal{A}(t_2) - \mathcal{A}(t_1))\,(\varphi,\psi)\|_{\mathcal{H}} \le O(|t_2 - t_1|)\,\|(\varphi,\psi)\|_Y\,, \forall\,(\varphi,\psi) \in Y.$$

Consequently

$$\|(\mathcal{A}(t_2) - \mathcal{A}(t_1))\|_{\mathcal{L}(Y,\mathcal{H})} \le O(|t_2 - t_1|).$$

The second assertion follows immediately from the definition of $Q(t)$. □

Corollary 38. *The mapping $[0,T] \to \mathcal{L}(\mathcal{H},Y) : t \mapsto Q(t)^{-1}$ is also continuous.*

Proof. The set of linear invertible continuous mapping from Y into $\mathcal{H}$ is an open subset denoted by A of $\mathcal{L}(Y,\mathcal{H})$ and the mapping from A into $\mathcal{L}(\mathcal{H},Y)$ which sends $S\in A$ onto its inverse S^{-1} is continuous [20, (8.3.2)]. Thus the result follows by composition of continous mappings. □

For the mapping $t\mapsto\mathcal{M}(t)$, one has the following lemma:

Lemma 39. *The mapping $[0,T]\to\mathcal{L}(\mathcal{H}) : t\mapsto\mathcal{M}(t)$ is continuous. A fortiori, it is continuous as a mapping from $[0,T]\to\mathcal{L}(Y,\mathcal{H})$. It is also continuous as a mapping from $[0,T]\to\mathcal{L}(Y)$.*

Proof. Let$(\varphi,\psi)\in Y$. By the definition of the operator $\mathcal{M}(t)$, we have for $t_1,t_2\in[0,T]$:

$$\begin{aligned}\left(\mathcal{M}(t_2)-\mathcal{M}(t_1)\right)(\varphi,\psi) = (-[\hat{\epsilon}(.,z(.,t_2))(\sigma(.)+\frac{\partial\epsilon}{\partial z}(.,z(.,t_2))\frac{\partial z}{\partial t}(.,t_2))\\ -\hat{\epsilon}(.,z(.,t_1))(\sigma(.)+\frac{\partial\epsilon}{\partial z}(.,z(.,t_1))\frac{\partial z}{\partial t}(.,t_1))]\varphi,0).\end{aligned}\tag{3.46}$$

$z\in C^1([0,T],C^1(\bar{\Omega}))$ implies that the functions $t\mapsto z(.,t)$ and $t\mapsto\frac{\partial z}{\partial t}(.,t)$ are uniformly continuous from $[0,T]$ into $C(\bar{\Omega})$. Thus

$$\left\|\left(\mathcal{M}(t_2)-\mathcal{M}(t_1)\right)(\varphi,\psi)\right\|_{\mathcal{H}}\leq O(|t_2-t_1|)\left\|(\varphi,0)\right\|_{\mathcal{H}},\ \text{for all}\ (\varphi,\psi)\in\mathcal{H}.$$

Consequently

$$\left\|\left(\mathcal{M}(t_2)-\mathcal{M}(t_1)\right)\right\|_{\mathcal{L}(\mathcal{H})}\leq O(|t_2-t_1|).$$

From formula (3.44) and $z\in C^1([0,T],C^1(\bar{\Omega}))$, follows also

$$\left\|\left(\mathcal{M}(t_2)-\mathcal{M}(t_1)\right)\right\|_{\mathcal{L}(Y)}\leq O(|t_2-t_1|).$$

□

Corollary 40. *The mapping $B : [0,T]\to\mathcal{L}(\mathcal{H}) : t\mapsto Q(t)\mathcal{M}(t)Q(t)^{-1}-\mathcal{M}(t)$ is a strongly continuous family of bounded linear operators on $\mathcal{H}$.*

Proof. Let $v:=(\varphi,\psi)\in\mathcal{H}$. The mapping $[0,T]\to\mathcal{H} : t\mapsto\mathcal{M}(t)v$ is continuous. The mapping $[0,T]\to Y : t\mapsto Q(t)^{-1}v$ is also continuous. $t\mapsto\mathcal{M}(t)\in\mathcal{L}(Y)$ being continuous, the mapping $[0,T]\to Y : t\mapsto\mathcal{M}(t)Q(t)^{-1}v$ is also continuous. The mapping

$$[0,T]\to\mathcal{L}(Y,\mathcal{H}) : t\mapsto Q(t)$$

being also continuous, the mapping $[0,T]\to\mathcal{H} : t\mapsto Q(t)\mathcal{M}(t)Q(t)^{-1}v$ is also continuous. Thus, the mapping

$$[0,T]\to\mathcal{H} : t\mapsto Q(t)\mathcal{M}(t)Q(t)^{-1}v-\mathcal{M}(t)v$$

is also continuous. □

As a consequence, hypothesis $(H_2)^+$ in [44, p.142] is verified. Also, from Lemma 37 and Lemma 39 follows, that the mapping $[0,T]\to\mathcal{L}(Y,\mathcal{H}) : t\mapsto\mathcal{A}(t)+\mathcal{M}(t)$ is continuous. Thus, hypothesis (H_3) in [44, p.135] is also verified. Hypothesis (H_1) in [44, p.135] follows from Corollary 33. Consequently, by Theorem 4.6 in [44, p.143], there exists a unique evolution system $U(t,s), 0\leq s\leq t\leq T$ in $\mathcal{H}$ satisfying properties $(E_1)-(E_5)$ listed in

[44]. Also, by [44, Theorem 5.2, pp.146-147] for every $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2) \in C([0,T];Y)$ and every $(E_0, H_0) \in Y$, the initial value problem

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(t) = (\mathcal{A}_z(t) + \mathcal{M}_z(t))\begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} + \mathbf{G}(t) \\ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(0) = \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \end{cases} \tag{3.47}$$

possesses a unique Y-valued solution given by the formula:

$$(\mathbf{E}, \mathbf{H})(t) = U_z(t,0)(\mathbf{E}_0\mathbf{H}_0) + \int_0^t U_z(t,s)\mathbf{G}(s)ds. \tag{3.48}$$

We have considered a right-hand side $\mathbf{G} \in C([0,T];Y)$, but we will prove in Proposition 41, that this is true also for a right-hand side $\mathbf{G} \in C^1([0,T];\mathcal{H})$. By a Y-valued solution, we mean that

$$(\mathbf{E}, \mathbf{H}) \in C([0,T];Y) \cap C^1([0,T];\mathcal{H}),$$

verifies

$$\frac{d}{dt}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(t) = (\mathcal{A}_z(t) + \mathcal{M}_z(t))\begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} + \mathbf{G}(t)$$

at every point $t \in [0,T]$ and $(\mathbf{E}, \mathbf{H})(0) = (\mathbf{E}_0, \mathbf{H}_0) \in Y$. We have added the subscript z to $\mathcal{A}(t)$, $\mathcal{M}(t)$ and $U(t,s)$ to underline their dependence with respect to the distribution of temperature z through the permittivity coefficient $\epsilon(\cdot, z(\cdot,t))$ in the Maxwell equations (3.19) :

$$\begin{cases} \partial_t\mathbf{E} = \hat{\epsilon}(\cdot,z)\operatorname{curl}\mathbf{H} - \hat{\epsilon}(\cdot,z)(\sigma + \partial_z\epsilon(\cdot,z)\partial_t z)\mathbf{E} + \mathbf{G}_1(t) & \text{in } Q, \\ \partial_t\mathbf{H} = -\hat{\mu}\operatorname{curl}\mathbf{E} + \mathbf{G}_2(t) & \text{in } Q, \\ \mathbf{E}\times n = 0 & \text{on } \Sigma, \\ \mathbf{H}\cdot n = 0 & \text{on } \Sigma \\ \operatorname{div}(\mu\mathbf{H}) = 0 & \text{in } Q \\ \mathbf{E}(\cdot,0) = \mathbf{E}_0,\ \mathbf{H}(\cdot,0) = \mathbf{H}_0 & \text{in } \Omega. \end{cases} \tag{3.49}$$

Thus, the unbounded operator $\mathcal{A}_z(t)$ in the space $\mathcal{H}$ with domain of definition $Y \equiv D(\mathcal{A})$ is defined by formula (3.24) and the linear bounded operator $\mathcal{M}_z(t)$ in the space $\mathcal{H}$, by formula (3.25). In formula (3.48), $(U_z(t,s))_{0\le s\le t\le T}$ denotes the evolution system generated by the family of operators $(\mathcal{A}_z(t) + \mathcal{M}_z(t))_{0\le t\le T}$ [44, p.135].

3.2.3 The inhomogeneous Cauchy problem with a r.h.s. $\mathbf{G} \in C^1([0,T];\mathcal{H})$

Previously we have considered in the abstract Cauchy problem for Maxwell's equations (3.47), a right-hand side (r.h.s.) $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2)$ belonging to $C([0,T];Y)$. We want now to consider the case of a right-hand side $\mathbf{G} \in C^1([0,T];\mathcal{H})$:

Theorem 41. *For every* $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2) \in C^1([0,T];\mathcal{H})$, *and every*

$$(\mathbf{E}_0\mathbf{H}_0) \in Y,$$

the initial value problem

$$\begin{cases} \frac{d}{dt}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}(t) = (\mathcal{A}_z(t)+\mathcal{M}_z(t))\begin{pmatrix}\mathbf{E}(t)\\ \mathbf{H}(t)\end{pmatrix} + \mathbf{G}(t) \\ \begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}(0) = \begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix} \end{cases} \tag{3.50}$$

possesses a unique Y-valued solution given by the formula:

$$\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}(t) = U_z(t,0)\begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix} + \int_0^t U_z(t,r)\mathbf{G}(r)dr. \tag{3.51}$$

As explained at the end of the previous subsection, the subscript z indicates that the quantity under consideration depends on z. In particular $Q_z(t) = I - \mathcal{A}_z(t)$. In the present subsection, when indicated, the subscript z is rather superflous, as the distribution of temperature z is fixed in this subsection. But it will be no more the case in section 3.3. Before going into the proof of Theorem 41, we give some lemmas

Lemma 42. *For every $f \in C([0,T];\mathcal{H})$, the mapping*

$$h_z : [0,T] \to Y : t \mapsto Q_z(t)^{-1}f(t) \tag{3.52}$$

is continuous.

Proof. Let $t_0 \in [0,T]$,

$$\begin{aligned} &\|h_z(t) - h_z(t_0)\|_Y = \left\|Q_z(t)^{-1}f(t) - Q_z(t_0)^{-1}f(t_0)\right\|_Y \\ &\leq \left\|Q_z(t)^{-1}f(t) - Q_z(t)^{-1}f(t_0)\right\|_Y + \left\|Q_z(t)^{-1}f(t_0) - Q_z(t_0)^{-1}f(t_0)\right\|_Y \\ &\leq \left(\sup_{0\leq s\leq T}\left\|Q_z(s)^{-1}\right\|_{\mathcal{L}(\mathcal{H};Y)}\right)\|f(t)-f(t_0)\|_{\mathcal{H}} + \left\|Q_z(t)^{-1} - Q_z(t_0)^{-1}\right\|_{\mathcal{L}(\mathcal{H};Y)}\|f(t_0)\|_{\mathcal{H}} \\ &\leq \left\|Q_z(\cdot)^{-1}\right\|_{C([0,T];\mathcal{L}(\mathcal{H};Y))}\|f(t)-f(t_0)\|_{\mathcal{H}} + \left\|Q_z(t)^{-1} - Q_z(t_0)^{-1}\right\|_{\mathcal{L}(\mathcal{H};Y)}\|f(t_0)\|_{\mathcal{H}}. \end{aligned} \tag{3.53}$$

f being continuous, $\| f(t) - f(t_0) \|_{\mathcal{H}} \to 0$ as $t \to t_0$. By Corollary 38, the mapping $[0,T] \to \mathcal{L}(\mathcal{H},Y) : t \mapsto Q_z(t)^{-1}$ is continuous, and thus $\| Q_z(t)^{-1} - Q_z(t_0)^{-1} \|_{\mathcal{L}(\mathcal{H};Y)} \to 0$ as $t \to t_0$. Thus by the previous inequality $h_z(t) \to h_z(t_0)$ as $t \to t_0$. Therefore h_z is continuous at point t_0. t_0 being arbitrary in $[0,T]$, the lemma follows. □

Lemma 43. *For every $f \in C^1([0,T];\mathcal{H})$, the mapping*

$$h_z : [0,T] \to Y : t \mapsto Q_z(t)^{-1}f(t) \tag{3.54}$$

is continuously differentiable. Its derivative is given by the formula:

$$h_z'(t) = -Q_z(t)^{-1}\dot{Q}_z(t)Q_z(t)^{-1}f(t) + Q_z(t)^{-1}f'(t) = Q_z(t)^{-1}(f'(t) - \dot{Q}_z(t)Q_z(t)^{-1}f(t)),$$

for any $t \in [0,T]$, where $\dot{Q}_z(t) \coloneqq \frac{d}{dt}Q_z(t)$.

Proof. Let us write the mapping $h_z : [0,T] \to Y$ as the composition of the following mappings:

- $\mathcal{U} : [0,T] \to \mathcal{L}(Y,\mathcal{H}) \times \mathcal{H} : t \mapsto (Q_z(t), f(t))$,
- $\mathcal{V} : \mathcal{O} \times \mathcal{H} \to \mathcal{L}(\mathcal{H},Y) \times \mathcal{H} : (B,g) \mapsto (B^{-1},g)$ where

$$\mathcal{O} := \{B \in \mathcal{L}(Y,\mathcal{H}); \exists B^{-1} \in \mathcal{L}(\mathcal{H},Y)\},$$

- $\mathcal{W} : \mathcal{L}(\mathcal{H},Y) \times \mathcal{H} \to Y : (S,g) \mapsto Sg$.

We have for all $t \in [0,T]$: $h_z(t) = \mathcal{W} \circ \mathcal{V} \circ \mathcal{U}(t)$. Now:

1. The mapping $\mathcal{U}$ is continuously differentiable as it is easily seen that the mapping $[0,T] \to \mathcal{L}(Y,\mathcal{H}) : t \mapsto Q_z(t) = I - \mathcal{A}_z(t)$ is continuously differentiable by inspection of the explicit form of the operator $\mathcal{A}_z(t) : Y \to \mathcal{H} : (\varphi,\psi) \mapsto (\hat{\epsilon}(.,z(.,t))\operatorname{curl}\psi, -\hat{\mu}\operatorname{curl}\varphi)$.

2. The mapping $\mathcal{V}$ is continuously differentiable by [20, (8.3.2)] and its derivative at the point (B,g) is the linear continuous mapping

$$D\mathcal{V}(B,g) : \mathcal{L}(Y,\mathcal{H}) \times \mathcal{H} \to \mathcal{L}(\mathcal{H},Y) \times \mathcal{H} : (\delta B, \delta g) \mapsto (-B^{-1}\delta B\ B^{-1}, \delta g)$$

3. The mapping $\mathcal{W}$ is a bilinear continuous mapping. It is thus continuously differentiable by [20, (8.1.4)] and its derivative at the point (S,g) is the linear continuous mapping

$$D\mathcal{W}(S,g) : \mathcal{L}(\mathcal{H},Y) \times \mathcal{H} \to Y : (\delta S, \delta g) \mapsto (\delta S)g + S(\delta g).$$

By [20, (8.2.1), (8.4.1)], we have that

$$h_z'(t) = (D\mathcal{W}(Q_z(t)^{-1}, f(t)) \circ D\mathcal{V}(Q_z(t), f(t))).(\dot{Q}_z(t), f'(t)).$$

Replacing in that formula by the above expressions for the derivatives, we obtain the result. □

Lemma 44. *Let g be any continuous function from $[0,T]$ into Y, continuously differentiable as a function from $[0,T]$ into $\mathcal{H}$. Then for every $t \in [0,T]$, the mapping*

$$k_t : [0,t] \to Y : r \mapsto U_z(t,r)g(r)$$

is continuously differentiable. Its derivative is given by the formula:

$$\frac{dk_t}{dr}(r) = -U_z(t,r)A_z(r)g(r) + U_z(t,r)g'(r),\ (0 \le r \le t \le T)$$

where in the present context $A_z(r) := \mathcal{A}_z(r) + \mathcal{M}_z(r)$.

Proof. Let $r \in [0,t]$ and $\Delta r \in \mathbb{R} \setminus \{0\}$ such that $r + \Delta r \in [0,t]$,

$$\begin{aligned}\frac{U_z(t,r+\Delta r)g(r+\Delta r) - U_z(t,r)g(r)}{\Delta r} &= U_z(t,r+\Delta r)\frac{g(r+\Delta r)-g(r)}{\Delta r} \\ &+ \frac{U_z(t,r+\Delta r) - U_z(t,r)}{\Delta r}g(r).\end{aligned} \tag{3.55}$$

As $g(r) \in Y$, the second term in the right-hand side tends to $-U_z(t,r)A_z(r)g(r)$ [44, p.135] as $\Delta r \to 0$. For the first term in the right-hand side:

$$\begin{aligned}
&\left\| U_z(t,r+\Delta r)\frac{g(r+\Delta r)-g(r)}{\Delta r} - U_z(t,r)g'(r)\right\| \\
&\leq \left\| U_z(t,r+\Delta r)(\frac{g(r+\Delta r)-g(r)}{\Delta r} - g'(r))\right\| + \|(U_z(t,r+\Delta r)-U_z(t,r))g'(r)\| \quad (3.56)\\
&\leq Cst\left\|\frac{g(r+\Delta r)-g(r)}{\Delta r} - g'(r)\right\| + \|(U_z(t,r+\Delta r)-U_z(t,r))g'(r)\| \to 0
\end{aligned}$$

as $\Delta r \to 0$. Summarizing:

$$\frac{U_z(t,r+\Delta r)g(r+\Delta r) - U_z(t,r)g(r)}{\Delta r} \to U_z(t,r)g'(r) - U_z(t,r)A_z(r)g(r),$$

as $\Delta r \to 0$. □

We now give the proof of Theorem 41; we follow the proof of Theorem 5.3 of [44, p.147] but taking into account that in our context $Q_z(t) := I - \mathcal{A}_z(t)$ and not $I - \mathcal{A}_z(t) - \mathcal{M}_z(t)$:

Proof. (**of Theorem 41**) Let us set

$$w_z(t) := \int_0^t U_z(t,r)\mathbf{G}(r)dr.$$

By hypothesis $\mathbf{G} \in C^1([0,T];\mathcal{H})$. By Lemma 43, the function $r \mapsto Q_z(r)^{-1}\mathbf{G}(r)$ is continuously differentiable from $[0,T]$ into Y. By the preceding lemma applied with the function $r \mapsto Q_z(r)^{-1}\mathbf{G}(r)$, we get:

$$\frac{\partial}{\partial r}(U_z(t,r)Q_z(r)^{-1}\mathbf{G}(r)) = U_z(t,r)[-(\mathcal{A}_z(r)+\mathcal{M}_z(r))Q_z(r)^{-1}\mathbf{G}(r) + Q_z(r)^{-1}g_z(r)],$$

where by Lemma 43: $g_z(r) := \mathbf{G}'(r) - \dot{Q}_z(r)Q_z(r)^{-1}\mathbf{G}(r)$. Recalling that in our context $Q_z(t) := I - \mathcal{A}_z(t)$, we get:

$$\begin{aligned}
\frac{\partial}{\partial r}(U_z(t,r)Q_z(r)^{-1}\mathbf{G}(r)) &= U_z(t,r)\mathbf{G}(r) + U_z(t,r)Q_z(r)^{-1}(g_z(r)-\mathbf{G}(r)) \\
&\quad - U_z(t,r)\mathcal{M}_z(r)Q_z(r)^{-1}\mathbf{G}(r).
\end{aligned} \quad (3.57)$$

Let us integrate both sides of this equality from $r = 0$ to $r = t$. We obtain: $w_z(t) = Q_z(t)^{-1}\mathbf{G}(t) - v_z(t)$, where

$$v_z(t) := U_z(t,0)Q_z(0)^{-1}\mathbf{G}(0) + \int_0^t U_z(t,r)[Q_z(r)^{-1}(g_z(r)-\mathbf{G}(r)) - \mathcal{M}_z(r)Q_z(r)^{-1}\mathbf{G}(r)]dr.$$

Now, what can we say about $v_z(t)$: The function

$$r \in [0,T] \mapsto Q_z(r)^{-1}(g_z(r)-\mathbf{G}(r)) - \mathcal{M}_z(r)Q_z(r)^{-1}\mathbf{G}(r) \in Y$$

is continuous. Let us explain why. By Corollary 38, the mapping from

$$r[0,T] \mapsto Q_z(r)^{-1} \in \mathcal{L}(\mathcal{H},Y)$$

is continuous. Also, by Lemma 39, the mapping

$$r \in [0,T] \mapsto \mathcal{M}_z(r) \in \mathcal{L}(Y)$$

is continuous. Thus the mapping

$$r \in [0,T] \mapsto \mathcal{M}_z(r)Q_z(r)^{-1}\mathbf{G}(r) \in Y$$

is continuous. On the other hand, from the explicit expression of $\mathcal{A}_z(r)$, it follows easily that the mapping from

$$r \in [0,T] \mapsto \dot{Q}_z(r) \in \mathcal{L}(Y,\mathcal{H})$$

is continuous. As

$$g_z(r) := \mathbf{G}'(r) - \dot{Q}_z(r)Q_z(r)^{-1}\mathbf{G}(r),$$

it is now clear that g_z is a continuous mapping from $[0,T]$ into $\mathcal{H}$, so that the mapping

$$r \in [0,T] \mapsto Q_z(r)^{-1}(g_z(r) - \mathbf{G}(r)) \in Y$$

is continuous. Therefore, we obtain that the mapping

$$r \in [0,T] \mapsto Q_z(r)^{-1}(g_z(r) - \mathbf{G}(r)) - \mathcal{M}_z(r)Q_z(r)^{-1}\mathbf{G}(r) \in Y$$

is continuous. Also,

$$v_z(0) = Q_z(0)^{-1}\mathbf{G}(0) \in Y.$$

Thus by [44, Theorem 5.2. p.146], the mapping v_z from $[0,T]$ into Y is the Y-solution of the initial value problem:

$$\begin{cases} \frac{dv_z}{dt}(t) = (\mathcal{A}_z(t) + \mathcal{M}_z(t))v_z(t) + Q_z(t)^{-1}(g_z(t) - \mathbf{G}(t)) - \mathcal{M}_z(t)Q_z(t)^{-1}\mathbf{G}(t) \\ v_z(0) = Q_z(0)^{-1}\mathbf{G}(0) \end{cases} \tag{3.58}$$

We have $\frac{dw_z}{dt}(t) = \frac{d}{dt}Q_z^{-1}(t)\mathbf{G}(t) - \frac{dv_z}{dt}(t)$, so that:

$$\begin{aligned} \frac{dw_z}{dt}(t) &= \frac{d}{dt}Q_z(t)^{-1}\mathbf{G}(t) - \frac{dv_z}{dt}(t) = Q_z(t)^{-1}(\mathbf{G}'(t) - \dot{Q}_z(t)Q_z(t)^{-1}\mathbf{G}(t)) - \frac{dv_z}{dt}(t) \\ &= Q_z(t)^{-1}(\mathbf{G}'(t) - \dot{Q}_z(t)Q_z(t)^{-1}\mathbf{G}(t)) - (\mathcal{A}_z(t) + \mathcal{M}_z(t))v_z(t) \\ &\qquad - Q_z(t)^{-1}(g_z(t) - \mathbf{G}(t)) + \mathcal{M}_z(t)Q_z(t)^{-1}\mathbf{G}(t) \\ &= Q_z(t)^{-1}(\mathbf{G}'(t) - \dot{Q}_z(t)Q_z(t)^{-1}\mathbf{G}(t)) - (\mathcal{A}_z(t) + \mathcal{M}_z(t))v_z(t) - Q_z(t)^{-1}\mathbf{G}'(t) \\ &\qquad + Q_z(t)^{-1}\dot{Q}_z(t)Q_z(t)^{-1}\mathbf{G}(t) + Q_z(t)^{-1}\mathbf{G}(t) + \mathcal{M}_z(t)Q_z(t)^{-1}\mathbf{G}(t) \\ &= -(\mathcal{A}_z(t) + \mathcal{M}_z(t))v_z(t) + \mathcal{M}_z(t)Q_z(t)^{-1}\mathbf{G}(t) + Q_z(t)^{-1}\mathbf{G}(t) \\ &= (\mathcal{A}_z(t) + \mathcal{M}_z(t))w_z(t) - (\mathcal{A}_z(t) \\ &\qquad + \mathcal{M}_z(t))Q_z(t)^{-1}\mathbf{G}(t) + \mathcal{M}_z(t)Q_z(t)^{-1}\mathbf{G}(t) + Q_z(t)^{-1}\mathbf{G}(t) \\ &= (\mathcal{A}_z(t) + \mathcal{M}_z(t))w_z(t) - Q_z(t)^{-1}\mathbf{G}(t) + \mathbf{G}(t) + Q_z(t)^{-1}\mathbf{G}(t) \\ &= (\mathcal{A}_z(t) + \mathcal{M}_z(t))w_z(t) + \mathbf{G}(t). \end{aligned} \tag{3.59}$$

Consequently the mapping $[0,T] \to Y : t \mapsto w_z(t) + U_z(t,0)\begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}$ is the Y-solution of the initial value problem (3.50). □

3.3 Existence of a local solution to the coupled problem

In this section, we want to prove that the coupled problem between Maxwell's equations (3.49)≡ (3.19) with $z := y$, and the heat equation (3.1) with the heat source (3.2), has a local solution. By a local solution we mean a solution on a time interval $[0, t_f]$ with $t_f > 0$ and $t_f \leq T$. In that purpose, we will define a fixed point problem between Maxwell's equations (3.49) with $z \in \bar{B}(0; R) \subset C^1([0,T]; C^1(\bar{\Omega}))$ and the heat equation (3.1) but with the heat source $S(z)$. In the following subsections 3.3.1 and 3.3.2, we suppose that the right-hand side $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2)$ in the Maxwell system (3.50) belongs to $C([0,T]; Y)$.

Later, in subsection 3.3.3, we will consider the case of a right-hand side $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2)$ in the Maxwell system (3.50) in $C^1([0,T]; \mathcal{H})$.

3.3.1 Boundedness and continuity results

Firstly, we study the boundedness properties of the evolution systems $(U_z(t,s))_{0\leq s\leq t\leq T}$ generated by the family of operators $(\mathcal{A}_z(t) + \mathcal{M}_z(t))_{0\leq t\leq T}$ for $z \in \bar{B}(0;R) \subset C^1([0,T]; C^1(\bar{\Omega}))$, $R > 0$ fixed, of the $(\mathbf{E}_z, \mathbf{H}_z)$ in $C([0,T]; Y)$ and of their time derivatives $\left(\dot{\mathbf{E}}_z, \dot{\mathbf{H}}_z\right) := \frac{d}{dt}(\mathbf{E}_z, \mathbf{H}_z)$ in $C([0,T]; \mathcal{H})$. Then, we study the continuity properties of the heat source term (3.2): $S(z) := \mu_a(.,z(.,.)) \mid (\mathbf{E}(z) * \varphi_a)(.,.) \mid^2$ and of its time derivative, as a function of z from the space $C^1([0,T]; C^1(\bar{\Omega}))$ into the space $L^p(0,T; C(\bar{\Omega})^3)$, for any $p \in]1, +\infty[$. These results will be needed to be allowed to apply Schauder's fixed point theorem to prove the existence of a local solution to our coupled problem (3.1)-(3.2): see Propositions 68, 69.

Proposition 45. *Let $\varphi_a \in C_c^1(\mathbb{R}^3)$, then for all $1 < p < \infty$:*

$$\mathbf{E} \in L^p(0,T; \mathbf{L}^2(\Omega)) \mapsto \mathbf{E} * \varphi_a \in L^p(0,T; C^1(\bar{\Omega})^3)$$

is linear continuous, and

$$\| \mathbf{E} * \varphi_a \|_{L^p(0,T;C^1(\bar{\Omega})^3)} \lesssim \| \mathbf{E} \|_{L^p(0,T;\mathbf{L}^2(\Omega))} \| \varphi_a \|_{C^1(\bar{\Omega})} .$$

.

Proof. Let us first prove that for all $\varphi_a \in C_c^1(\mathbb{R}^3)$, $\mathbf{E} \in \mathbf{L}^2(\Omega) \mapsto \mathbf{E} * \varphi_a \in C^1(\bar{\Omega})^3$ is linear continuous. By Cauchy-Schwarz's inequality, we have $\forall x \in \mathbb{R}^3$:

$$\begin{aligned} \mid (\mathbf{E} * \varphi_a)(x) \mid &= \mid \int_\Omega \mathbf{E}(y)\varphi_a(x-y)dy \mid \\ &\leq \left(\int_\Omega \mid \mathbf{E}(y) \mid^2 dy\right)^{\frac{1}{2}} \left(\int_\Omega \mid \varphi_a(x-y) \mid^2 dy\right)^{\frac{1}{2}} \\ &\leq \left(\int_\Omega \mid \mathbf{E}(y) \mid^2 dy\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \mid \varphi_a(x-y) \mid^2 dy\right)^{\frac{1}{2}} \\ &\leq \| \mathbf{E} \|_{2,\Omega} \| \varphi_a \|_{2,\mathbb{R}^3} . \end{aligned} \tag{3.60}$$

Thus,

$$\| \mathbf{E} * \varphi_a \|_{\infty,\mathbb{R}^3} \leq \| \mathbf{E} \|_{2,\Omega} \| \varphi_a \|_{2,\mathbb{R}^3} .$$

If we take $(\psi_n)_{n\in\mathbb{N}} \subset C_c(\Omega)^3$ such that $\psi_n \to \mathbf{E}$ in $\mathbf{L}^2(\Omega)$ as $n \to \infty$, then by (3.60) $\psi_n * \varphi_a$ converges uniformly to $\mathbf{E} * \varphi_a$ on $\mathbb{R}^3$. From $(\psi_n * \varphi_a)_{|_{\bar\Omega}} \in C(\bar\Omega)^3$, it then follows that $\mathbf{E} * \varphi_a \in C(\bar\Omega)^3$.

It is clear that $\partial_{x_k}(\mathbf{E} * \varphi_a)(x) = (\mathbf{E} * \partial_{x_k}\varphi_a)(x)$ for all $x \in \mathbb{R}^3$, $k = 1, 2, 3$. Then also by the same argument $\partial_{x_k}(\mathbf{E} * \varphi_a) \in C(\bar\Omega)^3$. Therefore $\mathbf{E} * \varphi_a \in C^1(\bar\Omega)^3$.

Now, let us consider $\mathbf{E} \in L^p(0, T; \mathbf{L}^2(\Omega))$. Then, for almost every $t \in [0, T]$, $\mathbf{E}(\cdot, t) \in \mathbf{L}^2(\Omega)$, $\mathbf{E}(\cdot, t) * \varphi_a \in C^1(\bar\Omega)^3$ and

$$\| \mathbf{E}(\cdot, t) * \varphi_a \|_{C^1(\bar\Omega)^3} \lesssim \| \mathbf{E}(\cdot, t) \|_{2,\Omega} \| \varphi_a \|_{C^1(\bar\Omega)} .$$

Hence

$$\| \mathbf{E} * \varphi_a \|_{L^p(0,T;C^1(\bar\Omega)^3)} \lesssim \left(\int_0^T \| \mathbf{E}(\cdot, t) \|^p_{\mathbf{L}^2(\Omega)} \, dt \right)^{\frac{1}{p}} \| \varphi_a \|_{C^1(\bar\Omega)} .$$

□

Proposition 46. *For $z \in \bar B(0; R) \subset C^1([0, T]; C^1(\bar\Omega))$, the families of evolution operators $(U_z(t, s))_{0\le s\le t\le T}$, are uniformly bounded in $\mathcal{L}(\mathcal{H})$ by a constant $C(R)$ independent of z, depending only on R.*

Proof. We know by Corollary 33, that $\{\mathcal{A}_z(t) + \mathcal{M}_z(t)\}_{t\in[0,T]}$ is a stable family of infinitesimal generators with stablity constants M and KM, so that by Theorem 3.1 of [44, p.135]

$$\| U_z(t, s) \|_{\mathcal{L}(\mathcal{H})} \le M e^{KM(t-s)} \le M e^{KMT}.$$

Looking to the proof of Corollary 33, it appears that $M = c_1^2 e^{(c_0/2)T}$ (3.33) with $c_1^2 = max(\epsilon_1, \mu_1)$, $c_0 = \frac{1}{min(\epsilon_0,\mu_0)} \| \frac{dz}{dt} \|_{\infty,Q} \| \frac{\partial\epsilon}{\partial z} \|_{\infty,\bar\Omega\times\mathbb{R}}$ (3.32) and $K = \sup_{0\le t\le T} \|\mathcal{M}_z(t)\|_{\mathcal{L}(\mathcal{H})}$. Thus, clearly $M \le C(R)$. In view of the definition of $\mathcal{M}_z(t)$ (3.25) and hypotheses (H1), (H2) of our paper (see subsection 3.2.1), it is clear that $\| \mathcal{M}_z(t) \|_{\mathcal{L}(\mathcal{H})} \le C(R)$ where $C(R)$ denotes a constant depending only on R. Thus $\| U_z(t, s) \|_{\mathcal{L}(\mathcal{H})} \le C(R)$. □

Using the previous proposition, it follows immediately from formula (3.48), the following $\mathcal{H}$-estimate on the solution of the initial boundary value problem (3.47) for z in the ball $\bar B(0; R) \subset C^1([0, T]; C^1(\bar\Omega))$.

Corollary 47. *For every fixed*

$$(\mathbf{E}_0, \mathbf{H}_0) \in Y,$$

and every fixed $G = (\mathbf{G}_1, \mathbf{G}_2) \in C([0, T]; Y)$, the Y-valued solution

$$(\mathbf{E}_z, \mathbf{H}_z) : [0, T] \to Y : t \mapsto (\mathbf{E}_z, \mathbf{H}_z)(t)$$

of the initial boundary value problem (3.47) with $z \in \bar B(0; R) \subset C^1([0, T]; C^1(\bar\Omega))$, defined by formula (3.48) satisfies the inequality:

$$\|(\mathbf{E}_z, \mathbf{H}_z)(t)\|_{\mathcal{H}} \le C(R)(\|(\mathbf{E}_0, \mathbf{H}_0)\|_{\mathcal{H}} + \sup_{0\le s\le T} \|\mathbf{G}(s)\|_{\mathcal{H}} \; t) \tag{3.61}$$

$$\le C(R)(\|(\mathbf{E}_0, \mathbf{H}_0)\|_{\mathcal{H}} + \sup_{0\le s\le T} \|\mathbf{G}(s)\|_{\mathcal{H}} \; T). \tag{3.62}$$

In particular the solution $(\mathbf{E}_z, \mathbf{H}_z) \in C([0, T]; \mathcal{H})$ and is uniformly bounded in this space by a constant $C(R)$ depending only on the radius of the ball $\bar B(0; R) \subset C^1([0, T]; C^1(\bar\Omega))$.

But, we need also a Y-estimate on the solution of the initial boundary value problem (3.47) for z in the ball $\bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$. Firstly, we need to establish that $\| U_z(t,s)|_Y \|_{\mathcal{L}(Y)} \leq C(R)$. We have the following propostion similar to Propostion 46, but in the space Y:

Proposition 48. *For $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$, the families of evolution operators $(U_z(t,s)|_Y)_{0\leq s\leq t\leq T}$, are uniformly bounded in $\mathcal{L}(Y)$ by a constant $Cst(R)$ independent of z, depending only on R in fact.*

Proof. By Theorem 4.6 of [44, (4.15) p. 143], we have:

$$U_z(t,s) = Q_z(t)^{-1}W_z(t,s)Q_z(s), \tag{3.63}$$

where $W_z(t,s)$ is the unique solution of the integral equation:

$$W_z(t,s)x = U_z(t,s)x + \int_s^t W_z(t,r)[B_z(r) + C_z(r)]U_z(r,s)x\ dr, \forall x \in \mathcal{H}.$$

$C_z(r) := \dot{Q}_z(r)Q_z(r)^{-1}$. $B_z(r)$ has been defined in Corollary 40. We have written $B_z(r)$ (resp. $C_z(r)$) instead of $B(r)$ (resp. $C(r)$) to underline the dependence with respect to $z(.,r)$ of $B(r)$ (resp. $C(r)$). It is clear from the definition of the operator $\mathcal{A}_z(s)$, from $Q_z(s) := I - \mathcal{A}_z(s)$, and hypothesis (H1), that $\|Q_z(s)\|_{\mathcal{L}(Y,\mathcal{H})} \leq Cst(R)$ independent of z and s. From Ascoli's theorem [20, (7.5.7)], it follows that the embedding of $\bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$ into the space $C(\bar{Q})$ is relatively compact. This implies that $\{z(.,s); z \in \bar{B}(0;R), s \in [0,T]\}$ is relatively compact in $C(\bar{\Omega})$. The mapping

$$\xi \in C(\bar{\Omega} \mapsto Q_\xi \in \mathcal{L}(Y,\mathcal{H})$$

is continuous so that $\{Q_z(s) \in \mathcal{L}(Y,\mathcal{H}); z \in \bar{B}(0;R), s \in [0,T]\}$ is relatively compact in G the open set of linear invertible mappings $\mathbf{G} \subset \mathcal{L}(Y,\mathcal{H})$. The mapping mappings

$$U \in \mathbf{G} \subset \mathcal{L}(Y,\mathcal{H}) \mapsto U^{-1} \in \mathcal{L}(\mathcal{H},Y)$$

being continuous, it follows also that $\{Q_z(s)^{-1} \in \mathcal{L}(\mathcal{H},Y); z \in \bar{B}(0;R), s \in [0,T]\}$ is a relatively compact subset in $\mathcal{L}(\mathcal{H},Y)$ so that $\|Q_z(s)^{-1}\|_{\mathcal{L}(\mathcal{H},Y)} \leq Cst(R)$ independent of z and s. From $\{z(.,s); z \in \bar{B}(0;R), s \in [0,T]\}$ relatively compact in $C(\bar{\Omega})$, hypothesis (H1), and $\| \dot{z}(.,s) \| \leq Cst(R)$, we have also that $\left\|\dot{Q}_z(s)\right\|_{\mathcal{L}(Y,\mathcal{H})} \leq Cst(R)$ independent of z. Consequently

$$\| C_z(s) \|_{\mathcal{L}(\mathcal{H})} \leq Cst(R).$$

Now let us recall that

$$B_z(t) \coloneqq Q_z(t)\mathcal{M}_z(t)Q_z(t)^{-1} - \mathcal{M}_z(t) \in \mathcal{L}(\mathcal{H}).$$

Looking to formula (3.44) in the proof of Proposition 36, we deduce that

$$\| \mathcal{M}_z(t) \|_{\mathcal{L}(Y)} \leq Cst(R).$$

It follows that

$$\| B_z(t) \|_{\mathcal{L}(\mathcal{H})} \leq Cst(R)$$

and

$$\| B_z(t) + C_z(t) \|_{\mathcal{L}(\mathcal{H})} \leq Cst(R).$$

From Proposition 46 and

$$\| B_z(t) + C_z(t) \|_{\mathcal{L}(\mathcal{H})} \leq Cst(R)$$

follows by inequality (4.11) in [44, p.142], that

$$\| W_z(t) \|_{\mathcal{L}(\mathcal{H})} \leq Cst(R).$$

From formula (3.63),

$$\|Q_z(s)\|_{\mathcal{L}(Y,\mathcal{H})} \leq Cst(R)$$

and

$$\left\|Q_z(s)^{-1}\right\|_{\mathcal{L}(\mathcal{H},Y)} \leq Cst(R),$$

now follows

$$\| U_z(t,s)|_Y \|_{\mathcal{L}(Y)} \leq Cst(R).$$

□

We have now a similar estimate to (3.62) but in the space Y on the solution of the initial boundary value problem (3.47) for z in the ball $\bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$. Its proof follows immediately from the previous proposition and formula (3.48):

Corollary 49. *For every fixed*

$$(\mathbf{E}_0, \mathbf{H}_0) \in Y,$$

and every fixed $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2) \in C([0,T];Y)$*, the* Y*-valued solution* $(\mathbf{E}_z, \mathbf{H}_z) : [0,T] \to Y : t \mapsto (\mathbf{E}_z, \mathbf{H}_z)(t)$ *of the initial value problem (3.47) with* $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$*, defined by formula (3.48) satisfies the inequality:*

$$\|(\mathbf{E}_z, \mathbf{H}_z)(t)\|_Y \leq C(R)(\|(\mathbf{E}_0, \mathbf{H}_0)\|_Y + \sup_{0\leq s\leq T} \|(\mathbf{G}(s))\|_Y \ t) \tag{3.64}$$

$$\leq C(R)(\|(\mathbf{E}_0, \mathbf{H}_0)\|_Y + \sup_{0\leq s\leq T} \|(\mathbf{G}(s))\|_Y \ T). \tag{3.65}$$

In particular the solution $(\mathbf{E}_z, \mathbf{H}_z) \in C([0,T];Y)$ *and is uniformly bounded in this space by a constant* $C(R)$ *depending only on the radius of the ball* $\bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$.

By the equation of evolution (3.47), $\| \mathcal{A}_z(t) \|_{\mathcal{L}(Y,\mathcal{H})} \leq C(R)$, $\| \mathcal{M}_z(t) \|_{\mathcal{L}(Y,\mathcal{H})} \leq C(R)$ and estimate (3.65), we have the following $\mathcal{H}$-estimate, on the time derivative of the solution to the initial boundary value problem (3.47):

Corollary 50. *For every fixed*

$$(\mathbf{E}_0, \mathbf{H}_0) \in Y,$$

and every fixed $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2) \in C([0,T];Y)$*, the time-derivative of the* Y*-valued solution* $(\mathbf{E}_z, \mathbf{H}_z) : [0,T] \to Y : t \mapsto (\mathbf{E}_z, \mathbf{H}_z)(t)$ *of the initial value problem (3.47) with* $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$*, defined by formula (3.48) satisfies the inequality:*

$$\left\|\left(\dot{\mathbf{E}}_z, \dot{\mathbf{H}}_z\right)(t)\right\|_{\mathcal{H}} \leq C(R)(\|(\mathbf{E}_0, \mathbf{H}_0)\|_Y + \sup_{0\leq s\leq T} \|\mathbf{G}(s)\|_Y \ t) + \| \mathbf{G}(t) \|_{\mathcal{H}} \tag{3.66}$$

$$\leq C(R)(\|(\mathbf{E}_0, \mathbf{H}_0)\|_Y + \sup_{0\leq s\leq T} \|\mathbf{G}(s)\|_Y \ T) + \| \mathbf{G}(t) \|_{\mathcal{H}} \tag{3.67}$$

$\left(\dot{\mathbf{E}}_z, \dot{\mathbf{H}}_z\right) \in C([0,T];\mathcal{H})$ *and is uniformly bounded in this space by a constant* $C(R)$ *depending only on the radius of the ball* $\bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$.

In particular, from Corollary 49 and Corollary 50 follow the inequalities:

$$\| (\mathbf{E}_z, \mathbf{H}_z) \|_{C([0,T];Y)} \leq C(R) \tag{3.68}$$

and

$$\| \left(\dot{\mathbf{E}}_z, \dot{\mathbf{H}}_z\right) \|_{C([0,T];\mathcal{H})} \leq C(R). \tag{3.69}$$

But the spaces $C([0,T];Y)$ and the space $C([0,T];\mathcal{H})$ are not reflexive. So we will consider rather $(\mathbf{E}_z, \mathbf{H}_z)$ and $\left(\dot{\mathbf{E}}_z, \dot{\mathbf{H}}_z\right)$ in the spaces $L^p(0,T;Y)$ $(1 < p < +\infty)$ and $L^p(0,T;\mathcal{H})$ $(1 < p < +\infty)$ respectively, spaces which are reflexive. As a first continuity result, we want to prove that:

Proposition 51. *If $(z_n)_{n\in\mathbb{N}} \subset C^1([0,T];C^1(\bar{\Omega}))$ converges to z in $C^1([0,T];C^1(\bar{\Omega}))$ then $(\mathbf{E}_{z_n}, \mathbf{H}_{z_n})_{n\in\mathbb{N}}$ converges weakly to $(\mathbf{E}_z, \mathbf{H}_z)$ in $L^p(0,T;Y)$ and $\left(\dot{\mathbf{E}}_{z_n}, \dot{\mathbf{H}}_{z_n}\right)_{n\in\mathbb{N}}$ converges weakly to $\left(\dot{\mathbf{E}}_z, \dot{\mathbf{H}}_z\right)$ in $L^p(0,T;\mathcal{H})$, for any $p \in]1,+\infty[$.*

To prove this continuity result, we need to introduce the notion of regular solution to the initial boundary value problem (3.47) and we will prove the uniqueness of regular solutions. In particular, we will show that every Y-solution is a regular solution in that sense.

Definition 52. We will say that $U = (\mathbf{E}, \mathbf{H}) \in L^2(0,T;Y)$ such that $\frac{\mathbf{d}}{\mathbf{dt}}(\mathbf{E}, \mathbf{H}) \in L^2(0,T;\mathcal{H})$ is a regular solution to the initial boundary value problem (3.47) with $\mathbf{G} \in L^2(0,T;\mathcal{H})$ and

$$U_0 = (\mathbf{E}_0, \mathbf{H}_0) \in \mathcal{H}, \tag{3.70}$$

if and only if

$$\begin{aligned} &\int_0^T [-(U(t), \frac{\partial \phi(t)}{\partial t})_{\mathcal{H}_t} + (U(t), \mathcal{A}_z(t)\phi(t))_{\mathcal{H}_t} - (\mathcal{B}_z(t)U(t), \phi(t))_{\mathcal{H}_t} \\ &- (\mathcal{M}_z(t)U(t), \phi(t))_{\mathcal{H}_t}]dt = \int_0^T (\mathbf{G}(t), \phi(t))_{\mathcal{H}_t} dt + (U_0, \phi(0))_{\mathcal{H}_0} \end{aligned} \tag{3.71}$$

for all

$$\phi \in L^2(0,T;Y) \text{ such that } \frac{d\phi}{dt} \in L^2(0,T;\mathcal{H}) \text{ and } \phi(T) = 0, \tag{3.72}$$

where

$$\mathcal{B}_z(t)U(t) := \{\hat{\epsilon}(\cdot, z(\cdot,t))\partial_z \epsilon(\cdot, z(\cdot,t))\partial_t z(\cdot,t)\mathbf{E}(t), 0\}. \tag{3.73}$$

Remark 13.

1. Let us remark that $U := (\mathbf{E}, \mathbf{H}) \in L^2(0,T;Y)$ and $\frac{\mathbf{d}}{\mathbf{dt}}(\mathbf{E}, \mathbf{H}) \in L^2(0,T;\mathcal{H})$, implies that $(\mathbf{E}, \mathbf{H}) \in C([0,T];\mathcal{H})$. In particular $U(0)$ has sense and belongs to $\mathcal{H}$.

2. Let us observe that

$$(\mathcal{B}_z(t) + \mathcal{M}_z(t))U(t) = (-\hat{\epsilon}(\cdot, z(\cdot,t))\sigma\mathbf{E}, \mathbf{0}). \tag{3.74}$$

Proposition 53. *Let $U := (\mathbf{E}, \mathbf{H})$ be a regular solution to the initial boundary value problem (3.47) in the sense of the preceding definition. Then for every $y \in Y$, the function '$t \mapsto (U(t), y)_{\mathcal{H}_t}$' $\in H^1(0,T)$ and verifies for a.e. $t \in]0,T[$:*

$$\frac{d}{dt}(U(t), y)_{\mathcal{H}_t} = -(U(t), \mathcal{A}_z(t)y)_{\mathcal{H}_t} + ((\mathcal{B}_z(t) + \mathcal{M}_z(t))U(t), y)_{\mathcal{H}_t} + (\mathbf{G}(t), y)_{\mathcal{H}_t}. \tag{3.75}$$

Moreover $U(0) = U_0$.

Proof. Let us take $\phi(t) = \theta(t)y$ in (3.71) where $\theta \in \mathcal{D}([0,T[)$ and $y \in Y$. We obtain:

$$\int_0^T [-(U(t),y)_{\mathcal{H}_t}\theta'(t) + (U(t),\mathcal{A}_z(t)y)_{\mathcal{H}_t}\theta(t) - ((\mathcal{B}_z(t)+\mathcal{M}_z(t))U(t),y)_{\mathcal{H}_t}\theta(t)]dt = \int_0^T (\mathbf{G}(t),y)_{\mathcal{H}_t}\theta(t)dt + (U_0,y)_{\mathcal{H}_0}\theta(0). \tag{3.76}$$

Now, the function

$$t \mapsto -(U(t),\mathcal{A}_z(t)y)_{\mathcal{H}_t} + ((\mathcal{B}_z(t)+\mathcal{M}_z(t))U(t),y)_{\mathcal{H}_t} + (\mathbf{G}(t),y)_{\mathcal{H}_t}$$

belongs to $L^2(0,T)$. If we restrict us in (3.76) to functions in $\mathcal{D}(]0,T[)$, the last term in the right-hand side of (3.76) disappears and (3.76) impies that for every $y \in Y$, the function '$t \mapsto (U(t),y)_{\mathcal{H}_t}$' $\in H^1(0,T)$ and verifies for a.e. $t \in]0,T[$ (3.75).

Now, let us suppose that our function θ in (3.76) belongs to $\mathcal{D}([0,T[)$, and knowing that '$t \mapsto (U(t),y)_{\mathcal{H}_t}$' $\in H^1(0,T)$, let us integrate by parts the first term in the left-hand side in (3.76). We obtain:

$$\int_0^T [\tfrac{d}{dt}(U(t),y)_{\mathcal{H}_t}]\theta(t)dt + (U(0),y)_{\mathcal{H}_0}\theta(0) =$$
$$\int_0^T [-(U(t),\mathcal{A}_z(t)y)_{\mathcal{H}_t} + ((\mathcal{B}_z(t)+\mathcal{M}_z(t))U(t),y)_{\mathcal{H}_t} + (G(t),y)_{\mathcal{H}_t}]\theta(t)dt + (U_0,y)_{\mathcal{H}_0}\theta(0).$$

Using (3.75), after simplifying it remains: $(U(0),y)_{\mathcal{H}_0} = (U_0,y)_{\mathcal{H}_0}$. By the density of Y into $\mathcal{H}$, we obtain $U(0) = U_0$. □

Proposition 54. *Every regular solution to the initial boundary value problem (3.47) with initial condition $U_0 = (\mathbf{E}_0,\mathbf{H}_0) \in \mathcal{H}$ is unique.*

Proof. Let $U := (\mathbf{E},\mathbf{H}) = U_1 - U_2$ where U_1 and U_2 are two regular solutions for the initial boundary value problem (3.47). Thus, in view of the previous definition and proposition, $U := (\mathbf{E},\mathbf{H})$ verifies $U(0) = 0$ and:

$$\int_0^T [-(U(t),\frac{\partial\phi(t)}{\partial t})_{\mathcal{H}_t} + (U(t),\mathcal{A}_z(t)\phi(t))_{\mathcal{H}_t} - (\mathcal{B}_z(t)U(t),\phi(t))_{\mathcal{H}_t} - (\mathcal{M}_z(t)U(t),\phi(t))_{\mathcal{H}_t}]dt = 0, \tag{3.77}$$

for all

$$\phi \in L^2(0,T;Y) \text{ such that } \frac{d\phi}{dt} \in L^2(0,T;\mathcal{H}) \text{ and } \phi(T) = 0. \tag{3.78}$$

Our aim is to prove that $U = 0$. Let us choose for test function ϕ in (3.77), the function

$$\phi : t \mapsto \phi(t) = (T-t)^n U(t). \tag{3.79}$$

We obtain:

$$\int_0^T [-(U(t),\frac{\partial(T-t)^n U(t)}{\partial t})_{\mathcal{H}_t} + (U(t),\mathcal{A}_z(t)(T-t)^n U(t))_{\mathcal{H}_t} - (\mathcal{B}_z(t)U(t),(T-t)^n U(t))_{\mathcal{H}_t} - (\mathcal{M}_z(t)U(t),(T-t)^n U(t))_{\mathcal{H}_t}]dt = 0. \tag{3.80}$$

Let us first notice that:

$$\begin{aligned}
&\int_0^T -(U(t), \frac{\partial (T-t)^n U(t)}{\partial t})_{\mathcal{H}_t} dt \ = \\
&- \int_Q \left[\epsilon(x, z(x,t)) \mathbf{E}(x,t) \frac{\partial}{\partial t}((T-t)^n \mathbf{E}(x,t)) + \mu(x) \mathbf{H}(x,t) \frac{\partial}{\partial t}((T-t)^n \mathbf{H}(x,t)) \right] dxdt \\
&= -\frac{1}{2} \int_Q \left[\epsilon(x, z(x,t)) (\frac{\partial}{\partial t} \mid \mathbf{E}(x,t) \mid^2)(T-t)^n + \mu(x) \frac{\partial}{\partial t} \mid \mathbf{H}(x,t) \mid^2 (T-t)^n \right] dxdt \\
&+ \int_Q \left[\epsilon(x, z(x,t)) \mid \mathbf{E}(x,t) \mid^2 n(T-t)^{n-1} + \mu(x) \mid \mathbf{H}(x,t) \mid^2 n(T-t)^{n-1} \right] dxdt \\
&= \frac{1}{2} \int_Q \left[\epsilon(x, z(x,t)) \mid \mathbf{E}(x,t) \mid^2 n(T-t)^{n-1} + \mu(x) \mid \mathbf{H}(x,t) \mid^2 n(T-t)^{n-1} \right] dxdt \\
&+ \frac{1}{2} \int_Q \frac{\partial \epsilon}{\partial z}(x, z(x,t)) \frac{\partial z}{\partial t}(x,t) \mid \mathbf{E}(x,t) \mid^2 (T-t)^n dxdt,
\end{aligned} \tag{3.81}$$

having performed at the end two integration by parts with respect to the time t. By assumption (H1) on ϵ and (3.8) we have the bound on the last term in (3.81):

$$\begin{aligned}
&\mid \frac{1}{2} \int_Q \frac{\partial \epsilon}{\partial z}(x, z(x,t)) \frac{\partial z}{\partial t}(x,t) \mid \mathbf{E}(x,t) \mid^2 (T-t)^n dxdt \mid \\
&\leq \frac{1}{2} \parallel \partial_z \epsilon \parallel_{\infty, \bar{\Omega} \times z(\bar{Q})} \parallel \partial_t z \parallel_{\infty, Q} \int_Q \mid \mathbf{E}(x,t) \mid^2 (T-t)^n dxdt \\
&\leq \frac{T \parallel \partial_z \epsilon \parallel_{\infty, \bar{\Omega} \times z(\bar{Q})} \parallel \partial_t z \parallel_{\infty, Q}}{2n\epsilon_0} \int_Q \epsilon(x, z(x,t)) \mid \mathbf{E}(x,t) \mid^2 n(T-t)^{n-1} dxdt.
\end{aligned} \tag{3.82}$$

On the other hand by Green's formula for the **curl** operator [24, Theorem 2.11], the fact that $\mathbf{E}(.,t) \in \mathbf{H}_0(\mathrm{curl}\,;\Omega)$, $\mathbf{H}(.,t) \in \mathbf{H}(\mathrm{curl}\,;\Omega)$ and the density of $\mathcal{D}(\bar{\Omega})^3$ in $\mathbf{H}(\mathrm{curl}\,;\Omega)$ [24, p.34]:

$$\begin{aligned}
(U(t), \mathcal{A}_z(t)(T-t)^n U(t))_{\mathcal{H}_t} = &\int_\Omega \mathbf{E}(x,t) \cdot \mathrm{curl}\,((T-t)^n \mathbf{H}(x,t)) dx \\
&- \int_\Omega \mathbf{H}(x,t) \cdot \mathrm{curl}\,((T-t)^n \mathbf{E}(x,t)) dx = 0
\end{aligned} \tag{3.83}$$

for all $t \in [0,T]$. Thus $\int_0^T (U(t), \mathcal{A}_z(t)(T-t)^n U(t))_{\mathcal{H}_t} dt = 0$. By (3.74), assumption (H1) on ϵ and assumption (H2) on σ, we have the following bound:

$$\begin{aligned}
&\mid \int_0^T ((\mathcal{M}_z(t) + \mathcal{B}_z(t)) U(t), (T-t)^n U(t))_{\mathcal{H}_t} dt \mid \leq \\
&\int_Q \mid \sigma(x) \mid \mid \mathbf{E}(x,t \mid^2 (T-t)^n dxdt \leq \\
&\frac{2 \parallel \sigma \parallel_{\infty, \Omega} T}{n} \int_\Omega \frac{1}{2} \mid \mathbf{E}(x,t) \mid^2 n(T-t)^{n-1} dxdt \leq \\
&\frac{2 \parallel \sigma \parallel_{\infty, \Omega} T}{n\epsilon_0} \int_\Omega \frac{1}{2} \epsilon(x, z(x,t)) \mid \mathbf{E}(x,t) \mid^2 n(T-t)^{n-1} dxdt.
\end{aligned} \tag{3.84}$$

From (3.80), (3.81), (3.82), (3.83) and (3.84) follows that

$$
\begin{aligned}
0 &= \int_0^T \left[-(U(t), \frac{\partial}{\partial t}(T-t)^n U(t)) + (U(t), \mathcal{A}_z(t)(T-t)^n U(t))_{\mathcal{H}_t} dt \right] \\
&- \int_0^T \left[((\mathcal{M}_z(t) + \mathcal{B}_z(t))U(t), (T-t)^n U(t))_{\mathcal{H}_t} \right] dt \\
&\geq \frac{1}{2}(1 - c\frac{T}{n}) \int_Q \left[\epsilon(x, z(x,t)) \mid \mathbf{E}(x,t) \mid^2 n(T-t)^n + \mu(x) \mid \mathbf{H}(x,t) \mid^2 n(T-t)^{n-1} \right] dxdt \\
&\geq \frac{1}{2} \int_Q \left[\epsilon(x, z(x,t)) \mid \mathbf{E}(x,t) \mid^2 n(T-t)^{n-1} + \mu(x) \mid \mathbf{H}(x,t) \mid^2 n(T-t)^{n-1} \right] dxdt,
\end{aligned}
\tag{3.85}
$$

for n sufficiently large which implies that $U = (\mathbf{E}, \mathbf{H}) = 0$ i.e. $U_1 = U_2$. □

Proposition 55. *Every Y-valued solution to the initial boundary value problem (3.47) with $(\mathbf{E}_0, \mathbf{H}_0) \in Y$ is a regular solution.*

Proof. Let $(\mathbf{E}, \mathbf{H}) \in C([0,T]; Y) \cap C^1([0,T]; \mathcal{H})$ a Y-valued solution to the initial boundary value problem (3.47). If we multiply (3.47) by $\phi = (\varphi, \psi) \in L^2(0,T;Y)$ $(Y = D(\mathcal{A}))$ such that $\frac{d\phi}{dt} = (\frac{d\varphi}{dt}, \frac{d\psi}{dt}) \in L^2(0,T;\mathcal{H})$ and $(\varphi(T), \psi(T)) = 0$, we get:

$$
\begin{aligned}
&\int_Q \left[\epsilon(x, z(x,t)) \frac{d\mathbf{E}}{dt}(x,t) \cdot \varphi(x,t) - \operatorname{curl} \mathbf{H}(x,t) \cdot \varphi(x,t) + \mu(x) \frac{d\mathbf{H}}{dt}(x,t) \cdot \psi(x,t) \right] dxdt \\
&+ \int_Q \left[\operatorname{curl} \mathbf{E}(x,t) \cdot \psi(x,t) + (\sigma + \frac{\partial \epsilon}{\partial z}(x, z(x,t)) \frac{\partial z}{\partial t}(x,t)) \mathbf{E}(x,t) \cdot \varphi(x,t) \right] dxdt \\
&= \int_Q \left[\epsilon(x, z(x,t)) \mathbf{G}_1(x,t) \cdot \varphi(x,t) + \mu(x) \mathbf{G}_2(x,t) \cdot \psi(x,t) \right] dxdt,
\end{aligned}
\tag{3.86}
$$

Integrating by parts in t on $[0,T]$ and using the fact that a fortiori $(\varphi, \psi) \in H^1(0,T;\mathcal{H})$, we obtain after simplification:

$$
\begin{aligned}
&- \int_Q \epsilon(x, z(x,t)) \mathbf{E}(x,t) \frac{\partial \varphi}{\partial t}(x,t) dxdt - \int_Q \mu(x) \mathbf{H}(x,t) \frac{\partial \psi}{\partial t}(x,t) dxdt \\
&+ \int_Q \operatorname{curl} \mathbf{E}(x,t) \cdot \psi(x,t) dxdt - \int_Q \operatorname{curl} \mathbf{H}(x,t) \cdot \varphi(x,t) dxdt \\
&+ \int_Q \sigma \mathbf{E}(x,t) \cdot \varphi(x,t) dxdt \\
&= \int_Q \left[\epsilon(x, z(x,t)) \mathbf{G}_1(x,t) \cdot \varphi(x,t) + \mu(x) \mathbf{G}_2(x,t) \cdot \psi(x,t) \right] dxdt \\
&+ \int_\Omega \left[\epsilon(x, z(x,0)) \mathbf{E}(x,0) \cdot \varphi(x,0) + \mu(x) \mathbf{H}(x,0) \cdot \psi(x,0) \right] dx.
\end{aligned}
\tag{3.87}
$$

Using now Green's formula for the **curl** operator [24, Theorem 2.11], the fact that $\mathbf{E}(.,t) \in \mathbf{H}_0(\operatorname{curl}; \Omega)$, $\varphi(.,t) \in \mathbf{H}_0(\operatorname{curl}; \Omega)$ and the density of $\mathcal{D}(\bar{\Omega})^3$ in $\mathbf{H}(\operatorname{curl}; \Omega)$ [24, p.34], we

obtain from the previous equation:

$$
\begin{aligned}
&-\int_Q \epsilon(x,z(x,t))\mathbf{E}(x,t)\frac{\partial\varphi}{\partial t}(x,t)dxdt - \int_Q \mu(x)\mathbf{H}(x,t)\frac{\partial\psi}{\partial t}(x,t)dxdt \\
&+\int_Q \mathbf{E}(x,t)\cdot \operatorname{curl}\psi(x,t)dxdt - \int_Q \mathbf{H}(x,t)\cdot\operatorname{curl}\varphi(x,t)dxdt \\
&+\int_Q \sigma\mathbf{E}(x,t)\cdot\varphi(x,t)dxdt \\
&=\int_Q \left[\epsilon(x,z(x,t))\mathbf{G}_1(x,t)\cdot\varphi(x,t)+\mu(x)\mathbf{G}_2(x,t)\cdot\psi(x,t)\right]dxdt \\
&+\int_\Omega \left[\epsilon(x,z(x,0))\mathbf{E}(x,0)\cdot\varphi(x,0)+\mu(x)\mathbf{H}(x,0)\cdot\psi(x,0)\right]dx,
\end{aligned}
\tag{3.88}
$$

for all $\phi=(\varphi,\psi)\in L^2(0,T;Y)$ such that $\frac{d\phi}{dt}=(\frac{d\varphi}{dt},\frac{d\psi}{dt})\in L^2(0,T;\mathcal{H})$ and $(\varphi(T),\psi(T))=0$. Comparing (3.88) with (3.71), taking into account (3.74), it follows that $U=(\mathbf{E},\mathbf{H})$ is a regular solution. □

We now give the proof of Proposition 51:

Proof. Being convergent, the sequence $(z_n)_{n\in\mathbb{N}}$ is contained in a certain ball of radius R in $C^1([0,T];C^1(\bar{\Omega}))$. Consequently, by Corollary 49, the sequence $((\mathbf{E}_{z_n},\mathbf{H}_{z_n}))_{n\in\mathbb{N}}$ is bounded in $L^p(0,T;Y)$ and by Corollary 50, the sequence of its time-derivatives $((\dot{\mathbf{E}}_{z_n},\dot{\mathbf{H}}_{z_n}))_{n\in\mathbb{N}}$ is bounded in $L^p(0,T;\mathcal{H})$. There exists thus a subsequence $((\mathbf{E}_{z_{n_k}},\mathbf{H}_{z_{n_k}}))_{k\in\mathbb{N}}$ weakly convergent in the space $L^p(0,T;Y)$ to some element $(\mathbf{E},\mathbf{H})\in L^p(0,T;Y)$ such that the sequence of its time derivatives $((\dot{\mathbf{E}}_{z_{n_k}},\dot{\mathbf{H}}_{z_{n_k}}))_{k\in\mathbb{N}}$ converges weakly to some element $\left(\check{\mathbf{E}},\check{\mathbf{H}}\right)\in L^p(0,T;\mathcal{H})$, for every $1<p<+\infty$ (it is easy to see considering an increasing sequence of real numbers $p_n\in]1,+\infty[$ tending to $+\infty$ and using the Cantor diagonal process of extraction of a subsequence, that the limits of the two sequences $((\mathbf{E}_{z_{n_k}},\mathbf{H}_{z_{n_k}}))_{k\in\mathbb{N}}$ and $((\dot{\mathbf{E}}_{z_{n_k}},\dot{\mathbf{H}}_{z_{n_k}}))_{k\in\mathbb{N}}$ are independent of $p\in]1,+\infty[$). Firstly, we must prove that $(\check{\mathbf{E}},\check{\mathbf{H}})=\frac{d}{dt}(\mathbf{E},\mathbf{H})$ in the sense of distributions on $]0,T[$ with range in $\mathcal{H}$. Let $\varphi\in\mathcal{D}(]0,T[)$:

$$
\begin{aligned}
<\frac{d}{dt}(\mathbf{E}_{z_{n_k}},\mathbf{H}_{z_{n_k}}),\varphi> &= -<(E_{z_{n_k}},H_{z_{n_k}}),\varphi'> \\
&= -\left(\int_0^T \mathbf{E}_{z_{n_k}}(t)\varphi'(t)\,dt,\int_0^T \mathbf{H}_{z_{n_k}}(t)\varphi'(t)\,dt\right).
\end{aligned}
\tag{3.89}
$$

Let us observe that the left-hand side in the previous equality belongs to $\mathcal{H}$ and the

right-hand side in $Y \subset \mathcal{H}$. Let $(\varphi^*, \psi^*) \in \mathcal{H}^* \hookrightarrow Y^*$. On one side:

$$
\begin{aligned}
&- < \left(\int_0^T \mathbf{E}_{z_{n_k}}(t)\varphi'(t)\ dt, \int_0^T \mathbf{H}_{z_{n_k}}(t)\varphi'(t)\ dt \right), (\varphi^*, \psi^*) >_{\mathcal{H},\mathcal{H}^*} \\
&= - \int_0^T < \mathbf{E}_{z_{n_k}}(t), \varphi^* > \varphi'(t)\ dt + \int_0^T < \mathbf{H}_{z_{n_k}}(t), \psi^* > \varphi'(t)\ dt \\
&= - \int_0^T < \left(\mathbf{E}_{z_{n_k}}, \mathbf{H}_{z_{n_k}}\right)(t), (\varphi^*, \psi^*) >_{Y,Y^*} \varphi'(t)\ dt \\
&\to - \int_0^T < (\mathbf{E}, \mathbf{H})(t), (\varphi^*, \psi^*) >_{Y,Y^*} \varphi'(t)\ dt \quad \text{as } k \to +\infty, \\
&= - < \int_0^T (\mathbf{E}, \mathbf{H})(t)\varphi'(t)\ dt, (\varphi^*, \psi^*) >_{Y,Y^*}, \\
&= - < \int_0^T (\mathbf{E}, \mathbf{H})(t)\varphi'(t)\ dt, (\varphi^*, \psi^*) >_{\mathcal{H},\mathcal{H}^*},
\end{aligned}
$$

and on the other side

$$
\begin{aligned}
&< \int_0^T \left(\dot{\mathbf{E}}_{z_{n_k}}, \dot{\mathbf{H}}_{z_{n_k}}\right)(t)\varphi(t)\ dt, (\varphi^*, \psi^*) >_{\mathcal{H},\mathcal{H}^*} \\
&= \int_0^T < \left(\dot{\mathbf{E}}_{z_{n_k}}, \dot{\mathbf{H}}_{z_{n_k}}\right)(t), (\varphi^*, \psi^*) >_{\mathcal{H},\mathcal{H}^*} \varphi(t)\ dt \\
&\to \int_0^T < \left(\check{\mathbf{E}}, \check{\mathbf{H}}\right), (\varphi^*, \psi^*) >_{\mathcal{H},\mathcal{H}^*} \varphi(t)\ dt \quad \text{as } k \to +\infty. \\
&=< \int_0^T \left(\check{\mathbf{E}}, \check{\mathbf{H}}\right)(t)\varphi(t)\ dt, (\varphi^*, \psi^*) >_{\mathcal{H},\mathcal{H}^*}\ .
\end{aligned}
$$

Thus $< \int_0^T \left(\check{\mathbf{E}}, \check{\mathbf{H}}\right)(t)\varphi(t)\ dt, (\varphi^*, \psi^*) >_{\mathcal{H},\mathcal{H}^*} = - < \int_0^T (\mathbf{E}, \mathbf{H})(t)\varphi'(t)\ dt, (\varphi^*, \psi^*) >_{\mathcal{H},\mathcal{H}^*}$, for all $(\varphi^*, \psi^*) \in \mathcal{H}^*$. Therefore:

$$
\int_0^T \left(\check{\mathbf{E}}, \check{\mathbf{H}}\right)(t)\varphi(t)\ dt = - \int_0^T (\mathbf{E}, \mathbf{H})(t)\varphi'(t)\ dt, \ \forall \varphi \in \mathcal{D}(]0, T[).
$$

We have thus proved that $\left(\check{\mathbf{E}}, \check{\mathbf{H}}\right) = \frac{d}{dt}(\mathbf{E}, \mathbf{H})$ in the sense of distributions.

Now, we must prove that $(\mathbf{E}, \mathbf{H}) = (\mathbf{E}_z, \mathbf{H}_z)$. It suffices for that to prove that $(\mathbf{E}, \mathbf{H})$ is a regular solution of the initial boundary value problem (3.47) i.e. satisfies (3.71) so that by uniqueness, we will have $(\mathbf{E}, \mathbf{H}) = (\mathbf{E}_z, \mathbf{H}_z)$. What we have mentioned previously in the proof is valid for any $p \in]1, +\infty[$; thus we may choose $p = 2$. For all $k \in \mathbb{N}$, $(\mathbf{E}_{n_k}, \mathbf{H}_{n_k})$, $(\mathbf{E}, \mathbf{H}) \in L^2(0, T; Y)$ and $\left(\dot{\mathbf{E}}_{n_k}, \dot{\mathbf{H}}_{n_k}\right)$, $\left(\dot{\mathbf{E}}, \dot{\mathbf{H}}\right) \in L^2(0, T; \mathcal{H})$. $U_{n_k} := (\mathbf{E}_{n_k}, \mathbf{H}_{n_k})$ satisfies the variational equation:

$$
\begin{aligned}
&\int_0^T [-(U_{n_k}(t), \frac{\partial \phi(t)}{\partial t})_{\mathcal{H}_{z_{n_k}(.,t)}} + (U_{n_k}(t), \mathcal{A}_{z_{n_k}}(t)\phi(t))_{\mathcal{H}_{z_{n_k}(.,t)}} - (\mathcal{B}_{z_{n_k}}(t)U_{n_k}(t), \phi(t))_{\mathcal{H}_{z_{n_k}(.,t)}} \\
&- (\mathcal{M}_{z_{n_k}}(t)U_{n_k}(t), \phi(t))_{\mathcal{H}_{z_{n_k}(.,t)}}]dt = \int_0^T (\mathbf{G}(t), \phi(t))_{\mathcal{H}_{z_{n_k}(.,t)}} dt + (U_0, \phi(0))_{\mathcal{H}_{z_{n_k}(.,0)}}
\end{aligned}
\tag{3.90}
$$

for each

$$
\phi \in L^2(0, T; Y) \text{ such that } \frac{d\phi}{dt} \in L^2(0, T; \mathcal{H}) \text{ and } \phi(T) = 0. \tag{3.91}
$$

We have written $\mathcal{H}_{z_{n_k}(.,t)}$ (resp. $\mathcal{H}_{z_{n_k}(.,0)}$) instead of $\mathcal{H}_t$ (resp.$\mathcal{H}_0$) in equation (3.90) to avoid confusion as this scalar product (3.21) on $\mathcal{H}$ depends in fact of $z_{n_k}(.,t)$ (resp. $z_{n_k}(.,0)$), so that we must be more precise than previously in our notations. We must now pass to the limit as $k \to +\infty$ in equation (3.90). In that purpose let us firstly remark, using the definition of the $\mathcal{H}_{z_{n_k}(.,t)}$-scalar product (3.21), that equation (3.90) is equivalent to

$$\begin{aligned}
&-\int_0^T\int_\Omega(\epsilon(x,z_{n_k}(x,t))\mathbf{E}_{z_{n_k}}(x,t).\frac{\partial\varphi}{\partial t}(x,t)+\mu(x)\mathbf{H}_{z_{n_k}}(x,t).\frac{\partial\psi}{\partial t}(x,t))dx\ dt\\
&+\int_0^T\int_\Omega(\mathbf{E}_{z_{n_k}}(x,t).\operatorname{curl}(\psi)(x,t)-\mathbf{H}_{z_{n_k}}(x,t).\operatorname{curl}(\varphi)(x,t))dx\ dt\\
&+\int_0^T\int_\Omega\sigma(x)\mathbf{E}_{z_{n_k}}(x,t).\varphi(x,t)dx\ dt\\
&=\int_0^T\int_\Omega\{\epsilon(x,z_{n_k}(x,t))\mathbf{G}_1(x,t).\varphi(x,t)+\mu(x)\mathbf{G}_2(x,t).\psi(x,t)\}dx\ dt\\
&+\int_\Omega\epsilon(x,z_{n_k}(x,0))\mathbf{E}_0(x).\varphi(x,0)dx+\int_\Omega\mu(x)\mathbf{H}_0(x).\psi(x,0)dx
\end{aligned}\tag{3.92}$$

for all $\phi=(\varphi,\psi)\in L^2(0,T;Y)$ such that $\frac{d\phi}{dt}=(\frac{d\varphi}{dt},\frac{d\psi}{dt})\in L^2(0,T;\mathcal{H})$ and $\phi(T)=0$. Concerning the first term in the left-hand side of equation (3.92), we have the following inequality:

$$\begin{aligned}
&\left|\int_0^T\int_\Omega\epsilon(x,z_{n_k}(x,t))\mathbf{E}_{z_{n_k}}(x,t).\frac{\partial\varphi}{\partial t}(x,t)dx\ dt-\int_0^T\int_\Omega\epsilon(x,z(x,t))\mathbf{E}(x,t).\frac{\partial\varphi}{\partial t}(x,t)dx\ dt\right|\\
&\le\left|\int_0^T\int_\Omega(\epsilon(x,z_{n_k}(x,t))-\epsilon(x,z(x,t)))\mathbf{E}_{z_{n_k}}(x,t).\frac{\partial\varphi}{\partial t}(x,t)dx\ dt\right|\\
&+\left|\int_0^T\int_\Omega\epsilon(x,z(x,t))(\mathbf{E}_{z_{n_k}}(x,t)-\mathbf{E}(x,t)).\frac{\partial\varphi}{\partial t}(x,t)dx\ dt\right|\\
&\le C\sup_{(x,t)\in\bar{Q}}|\epsilon(x,z_{n_k}(x,t))-\epsilon(x,z(x,t))|\left\|\frac{d\varphi}{dt}\right\|_{L^2(0,T;L^2(\Omega)^3)}\\
&+\left|<\mathbf{E}_{z_{n_k}}-\mathbf{E},\epsilon(.,z(.,.))\frac{d\varphi}{dt}>_{L^2(0,T;L^2(\Omega)^3),L^2(0,T;L^2(\Omega)^3)}\right|,
\end{aligned}\tag{3.93}$$

as $\left\|\mathbf{E}_{z_{n_k}}\right\|_{L^2(0,T;L^2(\Omega)^3)}\le C$ a constant independent of k, the sequence $(\mathbf{E}_{z_{n_k}})_{k\in\mathbb{N}}$ being weakly convergent in $L^2(0,T;L^2(\Omega)^3)$ and thus bounded. The sequence of functions $(z_{n_k}(.,.))_{k\in\mathbb{N}}$ converging uniformly on $\bar{Q}$ to $z(.,.)$ and the permittivity function $\epsilon(.,.)$ being supposed to be a continuous function (hypothesis (H1)), the first term in the right hand side of inequality (3.93) tends to 0 as $k\to+\infty$. The sequence $(\mathbf{E}_{z_{n_k}})_{k\in\mathbb{N}}$ being weakly convergent in $L^2(0,T;L^2(\Omega)^3)$ and $\epsilon(.,z(.,.))\frac{d\varphi}{dt}$ belonging to $L^2(0,T;L^2(\Omega)^3)$, the second term in the right-hand side of inequality (3.93) tends to 0 as $k\to+\infty$. Thus, the lefthand side of inequality (3.93) tends to 0 as $k\to+\infty$ so that the first term in the left hand side of equation (3.92):

$$-\int_0^T\epsilon(x,z_{n_k}(x,t))\mathbf{E}_{z_{n_k}}(x,t).\frac{\partial\varphi}{\partial t}(x,t)dx\ dt\to-\int_0^T\epsilon(x,z(x,t))\mathbf{E}(x,t).\frac{\partial\varphi}{\partial t}(x,t)dx\ dt$$

as $k \to +\infty$. Let us now consider the second term in the left hand side of equation (3.92):

$$\int_0^T \int_\Omega \mu(x)\mathbf{H}_{z_{n_k}}(x,t).\frac{\partial \psi}{\partial t}(x,t)dx\ dt.$$

As $\frac{d\psi}{dt} \in L^2(0,T;L^2(\Omega)^3)$ and $\mu \in L^\infty(\Omega)$, $\mu\frac{d\psi}{dt}$ belongs also to $L^2(0,T;L^2(\Omega)^3)$. The sequence $(\mathbf{H}_{z_{n_k}})_{k\in\mathbb{N}}$ converging weakly to $\mathbf{H}$ in $L^2(0,T;L^2(\Omega)^3)$, it follows that:

$$-\int_0^T\int_\Omega \mu(x)\mathbf{H}_{z_{n_k}}(x,t).\tfrac{\partial\psi}{\partial t}(x,t)dx\ dt \ \to\ -\int_0^T\int_\Omega \mu(x)\mathbf{H}(x,t).\tfrac{\partial\psi}{\partial t}(x,t)dx\ dt \quad \text{as } k\to+\infty.$$

It is the same kind of argument for the third and fourth term in the left hand side of equation (3.92). $\sigma \in L^\infty(\Omega)$ by hypothesis (H2), it is also straightforward to pass to the limit in the last term of the left hand side of equation (3.92). Passing to the limit in the first term of the right-hand side of equation (3.92) results from the Lebesgue dominated convergence theorem and the convergence of the uniformly bounded sequence of functions $(\epsilon(.,z_{n_k}(.,.)))_{k\in\mathbb{N}}$ to $\epsilon(.,z(.,.))$ uniformly on $\bar{Q}$ to $z(.,.)$ (punctly would suffice). To pass to the limit in the third term of the right-hand side of equation (3.92), we remark that from $\phi \in H^1(0,T;\mathcal{H})$ follows that $\varphi(.,0) \in L^2(\Omega)^3$. Also, $\mathbf{E}_0 \in L^2(\Omega)^3$. On the other hand, our hypothesis (H1) on the permittivity ϵ and $z_{n_k}(.,0) \to z(.,0)$ uniformly on Ω as $k \to +\infty$, implies that $\epsilon(.,z_{n_k}(.,0)) \to \epsilon(.,z(.,0))$ uniformly on Ω. Thus by the Lebesgue dominated convergence theorem $\int_\Omega \epsilon(x,z_{n_k}(x,0))\mathbf{E}_0(x).\varphi(x,0)dx$ tends to $\int_\Omega \epsilon(x,z(x,0))\mathbf{E}_0(x).\varphi(x,0)dx$ as $k \to +\infty$. Passing to the limit in equation (3.92), we obtain:

$$\begin{aligned}
&-\int_0^T\int_\Omega (\epsilon(x,z(x,t))\mathbf{E}.\frac{\partial\varphi}{\partial t}(x,t) + \mu(x)\mathbf{H}(x,t).\frac{\partial\psi}{\partial t}(x,t))dx\ dt\\
&+\int_0^T\int_\Omega (\mathbf{E}(x,t).\operatorname{curl}(\psi)(x,t) - \mathbf{H}(x,t).\operatorname{curl}(\varphi)(x,t))dx\ dt\\
&+\int_0^T\int_\Omega \sigma(x)\mathbf{E}(x,t).\varphi(x,t)dx\ dt\\
&=\int_0^T\int_\Omega \{\epsilon(x,z(x,t))\mathbf{G}_1(x,t).\varphi(x,t) + \mu(x)\mathbf{G}_2(x,t).\psi(x,t)\}dx\ dt\\
&+\int_\Omega \epsilon(x,z(x,0))\mathbf{E}_0(x).\varphi(x,0)dx + \int_\Omega \mu(x)\mathbf{H}_0(x).\psi(x,0)dx.
\end{aligned} \tag{3.94}$$

$(\mathbf{E},\mathbf{H})$ is thus the unique regular solution of the variational equation (3.94). But $(\mathbf{E}_z,\mathbf{H}_z)$ being the Y-solution of (3.47) is also a regular solution of the variational equation (3.94). So, by uniqueness, we have: $(\mathbf{E},\mathbf{H}) = (\mathbf{E}_z,\mathbf{H}_z)$. Thus, the subsquence $\left(\mathbf{E}_{z_{n_k}},\mathbf{H}_{z_{n_k}}\right)_{k\in\mathbb{N}}$ converges weakly to $(\mathbf{E}_z,\mathbf{H}_z)$ in $L^p(0,T;Y)$ and $\left(\dot{\mathbf{E}}_{z_{n_k}},\dot{\mathbf{H}}_{z_{n_k}}\right)_{k\in\mathbb{N}}$ converges weakly to $\left(\dot{\mathbf{E}}_z,\dot{\mathbf{H}}_z\right)$ in $L^p(0,T;\mathcal{H})$. But any subsequence of the sequence $(\mathbf{E}_{z_n},\mathbf{H}_{z_n})_{n\in\mathbb{N}}$ possesses a further subsequence with that property. Thus the sequence itself has this property. This concludes the proof of Proposition 51. □

We have now the following corollary to Proposition 51.

Corollary 56. *If* $(z_n)_{n\in\mathbb{N}} \subset C^1([0,T];C^1(\bar{\Omega}))$ *converges to* z *in* $C^1([0,T];C^1(\bar{\Omega}))$, *then* $(\mathbf{E}_{z_n} * \varphi_a)_{n\in\mathbb{N}}$ *(resp.* $(\dot{\mathbf{E}}_{z_n} * \varphi_a)_{n\in\mathbb{N}}$*) converges weakly to* $\mathbf{E}_z * \varphi_a$ *(resp.* $\dot{\mathbf{E}}_z * \varphi_a$*) in* $L^p(0,T;C^1(\bar{\Omega})^3)$ *for any* $p \in]1,+\infty[$. *Moreover* $(\mathbf{E}_{z_n} * \varphi_a)_{n\in\mathbb{N}}$ *converges strongly to* $\mathbf{E}_z * \varphi_a$ *in* $L^p(0,T;C(\bar{\Omega})^3)$ *for any* $p \in]1,+\infty[$.

Proof. From Proposition 51 follows that the sequences $(\mathbf{E}_{z_n})_{n\in\mathbb{N}}$ (resp. $(\dot{\mathbf{E}}_{z_n})_{n\in\mathbb{N}}$) converges weakly to $\mathbf{E}_z$ (resp. $\dot{\mathbf{E}}_z$) in $L^p(0,T;L^2(\Omega)^3)$. The mapping

$$L^p(0,T;L^2(\Omega)^3) \to L^p(0,T;C^1(\bar{\Omega})^3) : J \mapsto J * \varphi_a$$

being linear and continuous, we have also that $(\mathbf{E}_{z_n} * \varphi_a)_{n\in\mathbb{N}}$ (resp. $(\dot{\mathbf{E}}_{z_n} * \varphi_a)_{n\in\mathbb{N}}$) converges weakly to $\mathbf{E}_z * \varphi_a$ (resp. $\dot{\mathbf{E}}_z * \varphi_a$) in $L^p(0,T;C^1(\bar{\Omega})^3)$. Now, the embedding from

$$L^p(0,T;C^1(\bar{\Omega})^3) \hookrightarrow L^p(0,T;W^{1,q}(\Omega)^3)$$

being continuous for any $q \in]1,+\infty[$, so that $(\mathbf{E}_{z_n} * \varphi_a)_{n\in\mathbb{N}}$ converges also weakly to $\mathbf{E}_z * \varphi_a$ in $L^p(0,T;W^{1,q}(\Omega)^3)$. The embedding from $L^p(0,T;C^1(\bar{\Omega})^3) \hookrightarrow L^p(0,T;L^2(\Omega)^3)$ being continuous, $(\dot{\mathbf{E}}_{z_n} * \varphi_a)_{n\in\mathbb{N}}$ converges weakly to $\dot{\mathbf{E}}_z * \varphi_a$ in $L^p(0,T;L^2(\Omega)^3)$. The embedding from $W^{1,q}(\Omega)^3 \hookrightarrow C(\bar{\Omega})^3$ is compact for $q > 3$, and the embedding from $C(\bar{\Omega})^3 \hookrightarrow L^2(\Omega)^3$ is continuous. Moreover, the Banach spaces $W^{1,q}(\Omega)^3$ and $L^2(\Omega)^3$ are reflexive Banach spaces. Thus, we are in the setting of the Lions-Aubin compacity Lemma [47, p.106], which tells us that the embedding

$$\{f \in L^p(0,T;W^{1,q}(\Omega)^3); f' \in L^p(0,T;L^2(\Omega)^3)\} \hookrightarrow L^p(0,T;C(\bar{\Omega})^3)$$

is compact for $q > 3$, so that $(\mathbf{E}_{z_n} * \varphi_a)_{n\in\mathbb{N}}$ strongly converges to $\mathbf{E}_z * \varphi_a$ in the space $L^p(0,T;C(\bar{\Omega})^3)$. □

We now study the continuity with respect to z of the heat source term (3.2):

$$S(z)(x,t) := \mu_a(x,z(x,t)) \mid (\mathbf{E}(z) * \varphi_a)(x,t) \mid^2, \quad \forall (x,t) \in Q, \tag{3.95}$$

in the right hand side of our heat equation (3.1) and also of its time derivative $\frac{dS(z)}{dt}$. This heat source term may also be equivalently written in the form:

$$S(z)(x,t) := \mu_a(x,z(x,t))((\mathbf{E}(z) * \varphi_a)(x,t) \mid (\mathbf{E}(z) * \varphi_a)(x,t))_{\mathbb{R}^3}, \tag{3.96}$$

$\forall (x,t) \in Q$, where $(. \mid .)_{\mathbb{R}^3}$ denotes the scalar product in $\mathbb{R}^3$.

In the study of the continuity properties of $S(z)$ and $\frac{dS(z)}{dt}$ with respect to z, we will need the following hypothesis on the absorption coefficient $\mu_a(.,.)$:

(H6) We suppose that the absorption coefficient $\mu_a(.,.)$ in formula (3.95) for S belongs to the Banach space $C_b^1(\bar{\Omega} \times \mathbb{R})$.

This hypothesis is implicitely assumed in the following. Firstly, we prove the following continuity result for $S(z)$ in z:

Proposition 57. *If the sequence* $(z_n)_{n\in\mathbb{N}} \subset C^1([0,T];C^1(\bar{\Omega}))$ *converges to* z *in* $C^1([0,T];C^1(\bar{\Omega}))$, *then* $(S(z_n))_{n\in\mathbb{N}}$ *converges strongly to* $S(z)$ *in* $L^p(0,T;C(\bar{\Omega}))$, *for all* $p \in]1,+\infty[$.

Proof. Corollary 56 implies a fortiori that for any $p \in]1,+\infty[$, the sequence $(\mathbf{E}_{z_n} * \varphi_a)_{n\in\mathbb{N}}$ converges strongly to $\mathbf{E}_z * \varphi_a$ in the Banach space $L^{2p}(0,T;C(\bar{\Omega})^3)$ as $n \to +\infty$.

$$\begin{aligned}&\left|\left|\mathbf{E}_{z_n} * \varphi_a\right|^2(x,t) - \left|\mathbf{E}_z * \varphi_a\right|^2(x,t)\right| \\ &= \left|\left|\mathbf{E}_{z_n} * \varphi_a\right|(x,t) - \left|\mathbf{E}_z * \varphi_a\right|(x,t)\right| \ \times \ \left|\left|\mathbf{E}_{z_n} * \varphi_a\right|(x,t) + \left|\mathbf{E}_z * \varphi_a\right|(x,t)\right|.\end{aligned} \tag{3.97}$$

Thus:

$$\begin{aligned}&\left\|\left|\mathbf{E}_{z_n} * \varphi_a\right|^2(.,t) - \left|\mathbf{E}_z * \varphi_a\right|^2(.,t)\right\|_{\infty,\Omega} \\ &\leq \left\|\left|\mathbf{E}_{z_n} * \varphi_a\right|(.,t) - \left|\mathbf{E}_z * \varphi_a\right|(.,t)\right\|_{\infty,\Omega} \times \\ &(\left\|\left|\mathbf{E}_{z_n} * \varphi_a\right|(.,t)\right\|_{\infty,\Omega} + \left\|\left|\mathbf{E}_z * \varphi_a\right|(.,t)\right\|_{\infty,\Omega}).\end{aligned} \tag{3.98}$$

Consequently:

$$\begin{aligned}&\left(\int_0^T \left\|\left|\mathbf{E}_{z_n} * \varphi_a\right|^2(.,t) - \left|\mathbf{E}_z * \varphi_a\right|^2(.,t)\right\|_{\infty,\Omega}^p \ dt\right)^{1/p} \\ &\leq \left(\int_0^T \|(\mathbf{E}_{z_n} * \varphi_a)(.,t) - (\mathbf{E}_z * \varphi_a)(.,t)\|_{\infty,\Omega}^{2p} \ dt\right)^{1/2p} \\ &\times \left(\int_0^T (\left\|\left|\mathbf{E}_{z_n} * \varphi_a\right|(.,t)\right\|_{\infty,\Omega} + \left\|\left|\mathbf{E}_z * \varphi_a\right|(.,t)\right\|_{\infty,\Omega})^{2p} dt\right)^{1/2p} \\ &\leq \|\mathbf{E}_{z_n} * \varphi_a - \mathbf{E}_z * \varphi_a\|_{L^{2p}(0,T;C(\bar{\Omega})^3)} \\ &\times 2\left(\|\mathbf{E}_{z_n} * \varphi_a\|_{L^{2p}(0,T;C(\bar{\Omega})^3)} + \|\mathbf{E}_z * \varphi_a\|_{L^{2p}(0,T;C(\bar{\Omega})^3)}\right) \\ &\to 0 \text{ as } n \to +\infty.\end{aligned} \tag{3.99}$$

Thus $\left|\mathbf{E}_{z_n} * \varphi_a\right|^2 \to \left|\mathbf{E}_z * \varphi_a\right|^2$ in the space $L^p(0,T;C(\bar{\Omega}))$ as $n \to +\infty$. Using hypothesis (H6), we have also that the sequence $(\mu_a(.,z_n(.,.)))_{n\in\mathbb{N}}$ converges uniformly to $\mu_a(.,z(.,.))$ as $n \to +\infty$. So, it now follows immediately that the sequence $(S(z_n))_{n\in\mathbb{N}}$ converges strongly to $S(z)$ as $n \to +\infty$ in the space $L^p(0,T;C(\bar{\Omega}))$ as $n \to +\infty$. □

For the sequence of its time derivatives, we have the following result:

Proposition 58. *If the sequence $(z_n)_{n\in\mathbb{N}} \subset C^1([0,T];C^1(\bar{\Omega}))$ converges to z in $C^1([0,T];C^1(\bar{\Omega}))$, then the sequence of the time derivatives $(\frac{d}{dt}S(z_n))_{n\in\mathbb{N}}$ converges weakly to $\frac{d}{dt}S(z)$ in $L^p(0,T;C(\bar{\Omega}))$, $\forall p \in]1,+\infty[$.*

Firstly, we need to prove the following lemmas:

Lemma 59. *If $(z_n)_{n\in\mathbb{N}} \subset C^1([0,T];C^1(\bar{\Omega}))$ converges to z in $C^1([0,T];C^1(\bar{\Omega}))$, then for $i = 1,2,3$ and for all $s \in]1,+\infty[$, $\left(((\mathbf{E}_{z_n})_i * \varphi_a) \cdot \left(\frac{\partial(\mathbf{E}_{z_n})_i}{\partial t} * \varphi_a\right)\right)_{n\in\mathbb{N}}$ converges weakly to $((\mathbf{E}_z)_i * \varphi_a) \cdot \left(\frac{\partial(\mathbf{E}_z)_i}{\partial t} * \varphi_a\right)$ in the Banach space $L^s(0,T;C(\bar{\Omega}))$.*

Proof. In the following, to alleviate the notations, the $i-th$ component $(\mathbf{E}_{z_n})_i$ of $\mathbf{E}_{z_n}$ (resp. $(\mathbf{E}_z)_i$ of $\mathbf{E}_z$), will be denoted $\mathbf{E}_{z_n,i}$ (resp. $\mathbf{E}_{z,i}$) $(i = 1,2,3)$. The sequence $\left(\frac{\partial \mathbf{E}_{z_n,i}}{\partial t} * \varphi_a\right)_{n\in\mathbb{N}}$ being weakly convergent in $L^r(0,T;C(\bar{\Omega}))$, is bounded in $L^r(0,T;C(\bar{\Omega}))$

for any $r \in]1,+\infty[$. We know also that the sequence $(\mathbf{E}_{z_n,i} * \varphi_a)_{n\in\mathbb{N}}$ strongly converges in $L^p(0,T;C(\bar{\Omega}))$ to $\mathbf{E}_{z,i} * \varphi_a$, for all $p \in]1,+\infty[$. Thus:

$$\left\| (\mathbf{E}_{z_n,i} * \varphi_a - \mathbf{E}_{z,i} * \varphi_a) \left(\frac{\partial \mathbf{E}_{z_n,i}}{\partial t} * \varphi_a \right) \right\|_{L^s(0,T;C(\bar{\Omega}))}$$
$$\leq \|\mathbf{E}_{z_n,i} * \varphi_a - \mathbf{E}_{z,i} * \varphi_a\|_{L^p(0,T;C(\bar{\Omega}))} \left\| \frac{\partial \mathbf{E}_{z_n,i}}{\partial t} * \varphi_a \right\|_{L^r(0,T;C(\bar{\Omega}))}$$
$$\to 0 \text{ as } n \to +\infty \text{ for } \frac{1}{s} = \frac{1}{p} + \frac{1}{r} \text{ by the generalized Hölder's inequality.}$$

A fortiori

$$\left((\mathbf{E}_{z_n,i} * \varphi_a - \mathbf{E}_{z,i} * \varphi_a) \left(\frac{\partial \mathbf{E}_{z_n,i}}{\partial t} * \varphi_a \right) \right)_{n\in\mathbb{N}}$$

converges weakly to 0 as $n \to +\infty$ in $L^s(0,T;C(\bar{\Omega}))$ as $n \to +\infty$.

On the other hand, $\frac{\partial \mathbf{E}_{z_n,i}}{\partial t} * \varphi_a - \frac{\partial \mathbf{E}_{z,i}}{\partial t} * \varphi_a$ converges weakly to 0 as $n \to +\infty$ in $L^r(0,T;C(\bar{\Omega}))$ and we may view $\mathbf{E}_{z,i} * \varphi_a \in L^p(0,T;C(\bar{\Omega}))$ as the linear continuous operator:

$$L^r(0,T;C(\bar{\Omega})) \to L^s(0,T;C(\bar{\Omega})) : k \mapsto (\mathbf{E}_{z,i} * \varphi_a).k.$$

Thus the sequence

$$\left((\mathbf{E}_{z,i} * \varphi_a).(\frac{\partial \mathbf{E}_{z_n,i}}{\partial t} * \varphi_a - \frac{\partial \mathbf{E}_{z,i}}{\partial t} * \varphi_a) \right)_{n\in\mathbb{N}}$$

converges weakly to 0 as $n \to +\infty$ in $L^s(0,T;C(\bar{\Omega}))$. Adding both sequences, we obtain the result. □

Let us set

$$g_n := \sum_{i=1}^{i=3} (\mathbf{E}_{z_n,i} * \varphi_a) \left(\frac{\partial \mathbf{E}_{z_n,i}}{\partial t} * \varphi_a \right)$$

and

$$g := \sum_{i=1}^{i=3} (\mathbf{E}_{z,i} * \varphi_a) \left(\frac{\partial \mathbf{E}_{z,i}}{\partial t} * \varphi_a \right).$$

We have also that the sequence $(g_n)_{n\in\mathbb{N}}$ converges weakly to g as $n \to +\infty$ in $L^s(0,T;C(\bar{\Omega}))$.

Lemma 60. *The sequence $(\mu_a(.,z_n(.,.))g_n)_{n\in\mathbb{N}}$ converges weakly to $\mu_a(.,z(.,.))g$ as $n \to +\infty$ in $L^s(0,T;C(\bar{\Omega}))$, for all $s \in]1,+\infty[$.*

Proof. We notice that

$$\mu_a(.,z_n(.,.))g_n - \mu_a(.,z(.,.))g = [\mu_a(.,z_n(.,.)) - \mu_a(.,z(.,.))]\, g_n + \mu_a(.,z(.,.))\,[g_n - g].$$

By the preceding lemma, the sequence $(g_n - g)_{n\in\mathbb{N}}$ converges weakly to 0 in the Banach space $L^s(0,T;C(\bar{\Omega}))$. Now, $\mu_a(.,z(.,.)) \in C(\bar{Q})$, so that it defines a linear and continuous multiplication operator in $L^s(0,T;C(\bar{\Omega}))$. Consequently, $(\mu_a(.,z(.,.))\,[g_n - g])_{n\in\mathbb{N}}$ converges also weakly to 0 in $L^s(0,T;C(\bar{\Omega}))$. On the other hand, the weak convergence of $(g_n)_{n\in\mathbb{N}}$ in $L^s(0,T;C(\bar{\Omega}))$, implies that $(g_n)_{n\in\mathbb{N}}$ is bounded in $L^s(0,T;C(\bar{\Omega}))$. The sequence $(\mu_a(.,z_n(.,.)) - \mu_a(.,z(.,.)))_{n\in\mathbb{N}}$ converging uniformly to 0 in $C(\bar{Q})$, $([\mu_a(.,z_n(.,.)) - \mu_a(.,z(.,.))]g_n)_{n\in\mathbb{N}}$ converges strongly to 0 in $L^s(0,T;C(\bar{\Omega}))$, and thus a fortiori weakly. To sum up, the sequence $(\mu_a(.,z_n(.,.))g_n - \mu_a(.,z(.,.))g)_{n\in\mathbb{N}}$ converges weakly to 0 in $L^s(0,T;C(\bar{\Omega}))$. □

Lemma 61. *The sequence* $\left(\frac{\partial \mu_a(.,z_n(.,.))}{\partial u}\frac{\partial z_n}{\partial t}\left|\mathbf{E}_{z_n}*\varphi_a\right|^2\right)_{n\in\mathbb{N}}$ *strongly converges and thus a fortiori weakly to* $\frac{\partial \mu_a(.,z(.,.))}{\partial y}\ \frac{\partial z}{\partial t}\ \left|\mathbf{E}_z*\varphi_a\right|^2$ *in the Banach space* $L^p(0,T;C(\bar{\Omega}))$*, for all* $p\in]1,+\infty[$.

Proof. We have seen in the proof of Proposition 57 that the sequence $(|\mathbf{E}_{z_n}*\varphi_a|^2)_{n\in\mathbb{N}}$ strongly converges to $|\mathbf{E}_z*\varphi_a|^2$ in $L^p(0,T;C(\bar{\Omega}))$ as $n\to+\infty$, and thus is a fortiori bounded in $L^p(0,T;C(\bar{\Omega}))$. The sequence $(z_n)_{n\in\mathbb{N}}$ tends to z in $C^1([0,T];C^1(\bar{\Omega}))$ as $n\to+\infty$, it follows using hypothesis (H1), that the sequence $\left(\frac{\partial \mu_a(.,z_n(.,.))}{\partial y}\frac{\partial z_n}{\partial t}\right)_{n\in\mathbb{N}}$ tends to $\frac{\partial \mu_a(.,z(.,.))}{\partial y}\frac{\partial z}{\partial t}$ in $C(\bar{Q})$ as $n\to+\infty$. These two facts imply the result. □

Proof. We give now the proof **of Proposition 58**. We have:

$$\frac{d}{dt}S(z_n)=\frac{\partial \mu_a(.,z_n(.,.))}{\partial y}\frac{\partial z_n}{\partial t}\left|\mathbf{E}_{z_n}*\varphi_a\right|^2+2\mu_a(.,z_n(.,.))g_n. \tag{3.100}$$

By the preceding lemma, the first term in the righthand side of formula (3.100) converges weakly to $\frac{\partial \mu_a(.,z(.,.))}{\partial y}\frac{\partial z}{\partial t}\left|\mathbf{E}_z*\varphi_a\right|^2$ in $L^p(0,T;C(\bar{\Omega}))$. By Lemma 60, the second term $2\mu_a(.,z_n(.,.))g_n$ converges weakly to $2\mu_a(.,z(.,.))g$ in $L^p(0,T;C(\bar{\Omega}))$. Thus $(\frac{d}{dt}S(z_n))_{n\in\mathbb{N}}$ converges weakly to $\frac{d}{dt}S(z)$ in $L^p(0,T;C(\bar{\Omega}))$, for all $1<p<+\infty$. This proves Proposition 58. □

3.3.2 Solving the coupled heat equation in $C^1([0,T];C^1(\bar{\Omega}))$

In this subsection, we want to prove that the initial nonlinear boundary value problem for the coupled heat equation (3.1):

$$\begin{cases} \partial_t y-\operatorname{div}(\alpha\nabla y)=S(y) & \text{in } Q,\\ \alpha\frac{\partial y}{\partial n}+hy=hy_b & \text{on } \Sigma,\\ y(\cdot,0)=y_0 & \text{in } \Omega, \end{cases} \tag{3.101}$$

admits at least one local solution i.e. a solution on a time interval $[0,t_f]$, with $0<t_f\leq T$.

(H7) We assume our initial condition y_0 to belong to $C_b^4(\Omega)$. We assume $\alpha\in C_b^3(\Omega)$, $\alpha>0$ on $\bar{\Omega}$, $h\in C^1(\partial\Omega)$, $h>0$ on $\partial\Omega$ and that $y_b\in C^2(\bar{\Sigma})$. In particular, it is assumed that the functions α and h do not depend on the time variable t. We assume that our absorption coefficient $\mu_a(.,.)$ appearing in formula (3.95) for the heat source term $S(.)$ belongs to the Banach space $C_b^2(\Omega\times\mathbb{R})$. Finally, we assume that $\varphi_a\in C_c^2(\mathbb{R}^3)$.

In this subsection we will need the following Sobolev spaces [25, 32], for $5<p<+\infty$:

$$W_p^{2-2/p}(\Omega):=\{u\in L^p(\Omega);\frac{\partial u}{\partial x_i}\in L^p(\Omega),\ \text{and}\ \int\int_{\Omega\times\Omega}\frac{|\ \frac{\partial u}{\partial x_i}(x)-\frac{\partial u}{\partial x_i}(y)\ |^p}{|\ x-y\ |^{1+p}}dxdy<+\infty,$$
$$i=1,\cdots,3\},$$

$$W_p^{1-3/p}(\partial\Omega):=\{u\in L^p(\partial\Omega);\int\int_{\partial\Omega\times\partial\Omega}\frac{|\ u(x)-u(y)\ |^p}{|\ x-y\ |^{p-1}}dS(x)dS(y)<+\infty\},$$

$$W_p^{2,1}(Q) := \{u \in L^p(Q);\ \| u \|_{W_p^{2,1}(Q)} = \|u\|_{p,Q} + \sum_{i=1}^{3} \left\| \frac{\partial u}{\partial x_i} \right\|_{p,Q} + \left\| \frac{\partial u}{\partial t} \right\|_{p,Q} + \sum_{i=1}^{3}\sum_{j=1}^{3} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{p,Q} < +\infty\},$$

$$W_p^{1-1/p,1/2-1/2p}(\Sigma) := \{u \in L^p(\Sigma) \text{ such that } \inf\{\| v \|_{W_p^{2,1}(Q)};\ v_{|\Sigma} = u\} < +\infty\}.$$

Firstly, in view to reduce us from problem (3.101) to a problem with homogeneous boundary condition and zero initial condition, we introduce the auxilary linear initial boundary value problem for the heat equation:

$$\begin{cases} \partial_t \omega - \operatorname{div}(\alpha \nabla \omega) = 0 & \text{in } Q, \\ \alpha \frac{\partial \omega}{\partial n} + h\ \omega = h\ y_b & \text{on } \Sigma, \\ \omega(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases} \tag{3.102}$$

Proposition 62. *Assuming $p > 5$ and that the compatibility conditions*

$$\alpha \frac{\partial y_0}{\partial n} + h y_0 = h\ y_b(., 0), \tag{3.103}$$

and

$$\alpha \frac{\partial}{\partial n}(\operatorname{div}(\alpha \nabla y_0)) + h \operatorname{div}(\alpha \nabla y_0) = h\ \dot{y}_b(., 0), \tag{3.104}$$

are satisfied in the sense of traces in $W_p^{1-3/p}(\partial\Omega)$, problem (3.102) possesses one and only one solution ω in $C^1([0,T]; C^1(\bar{\Omega}))$.

Proof. By the compatibility condition (3.103), it follows from [32] that problem (3.103) possesses one and only one solution in $W_p^{2,1}(Q)$ (see (3.121)). Thus, we consider the derivated problem with respect to time:

$$\begin{cases} \partial_t v - \operatorname{div}(\alpha \nabla v) = 0 & \text{in } Q, \\ \alpha \frac{\partial v}{\partial n} + h v = h \dot{y}_b & \text{on } \Sigma, \\ v(\cdot, 0) = \operatorname{div}(\alpha \nabla y_0) & \text{in } \Omega. \end{cases} \tag{3.105}$$

From our hypotheses (H7) follows that $\dot{y}_b \in W_p^{1-1/p,1/2-1/2p}(\Sigma)$ and $\operatorname{div}(\alpha \nabla y_0) \in W_p^{2-2/p}(\Omega)$ for any $p \in]5, +\infty[$. By the compatibility condition (3.104), it still follows from [32] that the inhomogeneous initial boundary value problem (3.105) possesses a unique solution v in $W_p^{2,1}(Q)$ for any $p > 5$ (the same). By [32, Lemma II.3.3 p.80], v and its first order derivatives with respect to the "spatial variables" x_1, x_2, x_3 are Hölder continuous functions in $\bar{Q}$. Thus, a fortiori, $v \in C([0,T]; C^1(\bar{\Omega}))$. Let us now set:

$$\omega(x,t) = y_0(x) + \int_0^t v(x,s)\ ds,\ \ \forall t \in [0,T],\ \ \forall x \in \Omega. \tag{3.106}$$

One verifies easily that $\omega \in C^1([0,T]; C^1(\bar{\Omega}))$ and is solution of equation $(3.102)_{(i)}$. That ω verifies boundary condition $(3.102)_{(ii)}$ follows by using the compatibility condition (3.103). Thus ω defined by formula (3.106) is the solution of the linear initial boundary value problem (3.102). □

Now, setting $\breve{y} := y - \omega$, we are reduced to prove that the initial nonlinear boundary value problem:

$$\begin{cases} \partial_t \breve{y} - \operatorname{div}(\alpha \nabla \breve{y}) = S(\breve{y} + \omega) & \text{in } Q, \\ \alpha \frac{\partial \breve{y}}{\partial n} + h\breve{y} = 0 & \text{on } \Sigma, \\ \breve{y}(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \tag{3.107}$$

admits at least one local solution. To prove this, given any $p > 5$, let us introduce the following "fixed point problem": to $z \in \bar{B}(0, R) \subset C^1([0,T]; C^1(\bar{\Omega}))$ such that $z(0) = 0$, we associate $\breve{y}_z \in W_p^{2,1}(Q)$, solution of the linear initial boundary value problem [32]:

$$\begin{cases} \partial_t \breve{y}_z - \operatorname{div}(\alpha \nabla \breve{y}_z) = S(z + \omega) & \text{in } Q, \\ \alpha \frac{\partial \breve{y}_z}{\partial n} + h\breve{y}_z = 0 & \text{on } \Sigma, \\ \breve{y}_z(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \tag{3.108}$$

p being greater than 5, $W_p^{2,1}(Q) \hookrightarrow C([0,T]; C^1(\bar{\Omega}))$ [32]. As $\frac{d}{dt}(S(z+\omega)) \in L^p(0,T; C(\bar{\Omega}))$, and thus a fortiori to $L^p(Q)$, we may consider the derivated problem with respect to time:

$$\begin{cases} \partial_t v - \operatorname{div}(\alpha \nabla v) = \frac{d}{dt}(S(z+\omega)) & \text{in } Q, \\ \alpha \frac{\partial v}{\partial n} + hv = 0 & \text{on } \Sigma, \\ v(\cdot, 0) = \mu_a(\cdot, y_0(\cdot)) \mid (\mathbf{E}_0 * \varphi_a)(\cdot) \mid^2 & \text{on } \Omega. \end{cases} \tag{3.109}$$

To guarantee the existence and uniqueness of a solution in $W_p^{2,1}(Q)$ to the derivated problem (3.109), we suppose that its initial condition $\mu_a(\cdot, y_0(\cdot)) \mid (\mathbf{E}_0 * \varphi_a)(\cdot) \mid^2$ is admissible [32] i.e. verifies on Γ :

$$\alpha \frac{\partial}{\partial n} \left(\mu_a(\cdot, y_0(\cdot)) \mid (\mathbf{E}_0 * \varphi_a)(\cdot) \mid^2 \right) + h\, \mu_a(\cdot, y_0(\cdot)) \mid (\mathbf{E}_0 * \varphi_a)(\cdot) \mid^2 = 0, \tag{3.110}$$

in the sense of traces in the space $W_p^{1-3/p}(\Gamma)$. By an argument similar to (3.106), using the fact that $z(0) = 0$, one deduces easily that the solution v of the derivated problem (3.109) is $v = \frac{d\breve{y}_z}{dt}$. To alleviate the notations, we will denote $\frac{d\breve{y}_z}{dt}$ rather $\dot{\breve{y}}_z$. Thus $\dot{\breve{y}}_z$ is solution of:

$$\begin{cases} \partial_t \dot{\breve{y}}_z - \operatorname{div}(\alpha \nabla \dot{\breve{y}}_z) = \frac{d}{dt}(S(z+\omega)) & \text{in } Q, \\ \alpha \frac{\partial \dot{\breve{y}}_z}{\partial n} + h\dot{\breve{y}}_z = 0 & \text{on } \Sigma, \\ \dot{\breve{y}}_z(\cdot, 0) = \mu_a(\cdot, y_0(\cdot)) \mid (\mathbf{E}_0 * \varphi_a)(\cdot) \mid^2 & \text{on } \Omega. \end{cases} \tag{3.111}$$

The operator which sends $z \in C^1([0,T]; C^1(\bar{\Omega}))$ such that $z(0) = 0$ onto $\breve{y}_z$, does not leave the closed convex subset $K(0;R) := \bar{B}(0;R) \cap \{z \in \bar{B}(0;R); z(0) = 0\}$ of $C^1([0,T]; C^1(\bar{\Omega}))$ stable, so that we can not apply to it the Schauder's fixed point theorem. In order to define an operator from the nonvoid closed convex subset $K(0;R)$ of $C^1([0,T]; C^1(\bar{\Omega}))$ into itself, we will need firstly to restrict $\breve{y}_z$ to a fixed subinterval $[0, t_f]$ of $[0,T]$ of sufficiently small length and then to extend its restriction appropriately to the whole interval $[0,T]$ in order to obtain an element of $C^1([0,T]; C^1(\bar{\Omega}))$ corresponding to z of norm less than or equal to R. This extension will be made using the following lemma:

Lemma 63. *Let X be a Banach space and $C^1([0,T]; X)$ the Banach space of C^1-functions on the interval $[0,T]$ with values in X endowed with its usual norm (in the application we have in mind, X will be $C^1(\bar{\Omega})$). Let $t_f \in]0, T/2]$ and $\theta(t_f; \cdot) \in C^\infty([0,T]; \mathbb{R})$ such*

that $0 \leq \theta(t_f;\cdot) \leq 1$, $\theta(t_f;\cdot)|_{[0,t_f]} = 1$ *and* $\theta(t_f;t) = 0$ *if* $t \geq 1.5\ t_f$. *For* $f \in C^1([0,t_f];X)$ *such that* $f(0) = 0$, *we define its extension* $\tilde{f}(t_f;\cdot)$ *to the interval* $[0,T]$ *by*

$$\tilde{f}'(t_f;t) = \begin{cases} f'(t), & \text{if} \quad 0 \leq t \leq t_f \\ 2f'(t_f)\theta(t_f;t) - f'(2t_f - t)\theta(t_f;t), & \text{if} \quad t_f \leq t \leq T, \end{cases} \tag{3.112}$$

and $\tilde{f}(t_f;t) = \int_0^t \tilde{f}'(t_f;\xi)\ d\xi$, *for all* $t \in [0,T]$.

Then:

1. $\left\|\tilde{f}'(t_f;\cdot)\right\|_{C([0,T];X)} \leq 3\left\|f'\right\|_{C([0,t_f];X)}$;
2. $\left\|\tilde{f}(t_f;\cdot)\right\|_{C([0,T];X)} \leq 3T\left\|f'\right\|_{C([0,t_f];X)}$;
3. $\left\|\tilde{f}(t_f;\cdot)\right\|_{C^1([0,T];X)} \leq 3(T+1)\left\|f'\right\|_{C([0,t_f];X)} \leq 3(T+1)\left\|f\right\|_{C^1([0,t_f];X)}$.

The proof is elementary and may be left to the reader. In formula (3.112), the product $f'(2t_f - t)\theta(t_f;t)$ must be understood as being equal to 0 in case t would be greater than $2t_f$. Using the third assertion of the lemma, it follows that the extension operator $f \mapsto \tilde{f}(t_f;\cdot)$ is a linear continuous operator from the Banach space $C^1([0,t_f];X)$ into the Banach space $C^1([0,T];X)$.

If we apply the preceding lemma to the restriction to the interval $[0,t_f]$ of the function

$$f : [0,T] \to C^1(\bar{\Omega})) : t \mapsto \breve{y}_z(t) - \dot{\breve{y}}_z(0)t$$

nul at $t = 0$ and belonging to the Banach space $C^1([0,T];C^1(\bar{\Omega}))$, we obtain by linearity of the extension operator ($\breve{y}_z(0) = 0$) the function:

$$\tilde{f}(t_f;.) : [0,T] \to C^1(\bar{\Omega})) : t \mapsto \widetilde{(\breve{y}_z|_{[0,t_f]})}(t_f;t) - \dot{\breve{y}}_z(0)\int_0^t \theta(t_f;\xi)\ d\xi.$$

We have that $\tilde{f}(t_f;.)|_{[0,t_f]} = f|_{[0,t_f]}$. Let us define the function

$$\breve{f}_0(t_f;\cdot) : [0,T] \to C^1(\bar{\Omega})) : t \mapsto \dot{\breve{y}}(0)\int_0^t \theta(t_f;\xi)\ d\xi\ . \tag{3.113}$$

$\dot{\breve{y}}_z(0)$ does not depend on z as $\dot{\breve{y}}_z(0) = \mu_a(\cdot,y_0(\cdot))\ |\ (\mathbf{E}_0 * \varphi_a)(\cdot)\ |^2$ on Ω, so that we have denoted it simply $\dot{\breve{y}}(0)$ in formula (3.113). But the function $\breve{f}_0(t_f;\cdot)$ depends on t_f: it is the reason of our notation. However as $\dot{\breve{f}}_0(t_f;0) = \mu_a(\cdot,y_0(\cdot))\ |\ (\mathbf{E}_0 * \varphi_a)(\cdot)\ |^2$, $\dot{\breve{f}}_0(t_f;0)$ does not depend on t_f also.

The following lemma shows that if we take the radius $R > 0$ of our closed ball $\bar{B}(0;R)$ in the Banach space $C^1([0,T];C^1(\bar{\Omega}))$ sufficiently large, then the function $\breve{f}_0(t_f;\cdot)$ will be contained in its interior and also in the interior of $K(0;R) := \bar{B}(0;R) \cap \{z \in \bar{B}(0;R); z(0) = 0\}$ for any $t_f \in]0, \frac{T}{2}]$.

Lemma 64. *Whatever is* $t_f \in]0, \frac{T}{2}]$ *in Lemma 63, the norm of* $\breve{f}_0(t_f;\cdot)$ *in* $C^1([0,T];C^1(\bar{\Omega}))$ *satisfies always the following inequality:*

$$\| \breve{f}_0(t_f;\cdot) \|_{C^1([0,T];C^1(\bar{\Omega}))} \leq \| \mu_a(\cdot, y_0(\cdot)) \mid (\mathbf{E}_0 * \varphi_a)(\cdot) \mid^2 \|_{C^1(\bar{\Omega}))} (1+T). \tag{3.114}$$

Moreover:

$$\begin{aligned} &\| \widetilde{(\breve{y}_z|_{[0,t_f]})} - \breve{f}_0(t_f;\cdot) \|_{C^1([0,T];C^1(\bar{\Omega}))} \\ &\leq 3(T+1) \| (\breve{y}_z|_{[0,t_f]})' - \mu_a(\cdot, y_0(\cdot)) \mid (\mathbf{E}_0 * \varphi_a)(\cdot) \mid^2 \|_{C^0([0,t_f];C^1(\bar{\Omega}))} . \end{aligned} \tag{3.115}$$

The proof of (3.114) follows immediately from the definition by formula (3.113) of the function $\breve{f}_0(t_f;\cdot)$. The proof of (3.115) follows from the definition of $\tilde{f}$ and the third point of the previous extension lemma, Lemma 63.

Let us set

$$UB(y_0, \mathbf{E}_0) := \| \mu_a(\cdot, y_0(\cdot)) \mid (\mathbf{E}_0 * \varphi_a)(\cdot) \mid^2 \|_{C^1(\bar{\Omega}))} . \tag{3.116}$$

Let us observe that $\| \breve{f}_0(t_f;\cdot) \|_{C^1([0,T];C^1(\bar{\Omega}))}$ satisfies the inequality:

$$\| \breve{f}_0(t_f;\cdot) \|_{C^1([0,T];C^1(\bar{\Omega}))} \leq (1+T) UB(y_0, \mathbf{E}_0),$$

whatever is $t_f \in]0, T/2]$.

We want to prove that for $R > 3(T+1)UB(y_0, \mathbf{E}_0)$, there exists $t_f \in]0, T/2]$ such that for all $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$ satisfying $z(0) = 0$: $\widetilde{(\breve{y}_z|_{[0,t_f]})} \in \bar{B}(0;R)$. To prove that, we will need the following bounds on $\breve{y}_z$ and their time derivatives:

Lemma 65. *Let us fix some* $p \in]5, +\infty[$. *For every* $R > 0$ *fixed, there exists some constant* $C(R) > 0$ *depending only on* R *such that for all* $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$ *satisfying* $z(0) = 0$*:*

$$\|\breve{y}_z\|_{W_p^{2,1}(Q)} \leq C(R,p),$$

and

$$\left\|\dot{\breve{y}}_z\right\|_{W_p^{2,1}(Q)} \leq C(R,p).$$

Proof. We know from the results in [32, Ch.IV-Sec.9, Ch.VII-Sec.10] about general boundary value problems for parabolic equations applied to the homogeneous Robin boundary value problem (3.108) (resp.(3.111) with the compatibility condition (3.110)), that

$$\|\breve{y}_z\|_{W_p^{2,1}(Q)} \leq C(p) \|S(z+\omega)\|_{L^p(Q)} , \tag{3.117}$$

and:

$$\left\|\dot{\breve{y}}_z\right\|_{W_p^{2,1}(Q)} \leq C(p) \left(\left\| \frac{d}{dt} S(z+\omega) \right\|_{L^p(Q)} + \left\| \mu_a(\cdot, y_0(\cdot)) \mid (\mathbf{E}_0 * \varphi_a)(\cdot) \mid^2 \right\|_{W_p^{2-2/p}(\Omega)} \right). \tag{3.118}$$

From Corollary 49, Corollary 50 and using our Proposition 45, it follows for any $p \in]1, +\infty[$, the following inequalities:

$$\| \mathbf{E}_z * \varphi_a \|_{L^p(0,T;C^1(\bar{\Omega})^3)} \leq C(R,p)$$

and

$$\| \dot{\mathbf{E}}_z * \varphi_a \|_{L^p(0,T;C^1(\bar{\Omega})^3)} \leq C(R,p).$$

Using now our hypothesis (H6) that the absorption coefficient $\mu_a(.,.) \in C_b^1(\bar{\Omega} \times \mathbb{R})$, it follows from these inequalities that for any $p \in]1,+\infty[$:

$$\| S(z) \|_{L^p(0,T;C^1(\bar{\Omega}))} \leq C(R,p)$$

and

$$\| \dot{S}(z) \|_{L^p(0,T;C^1(\bar{\Omega}))} \leq C(R,p).$$

A fortiori for any $p \in]1,+\infty[$:

$$\| S(z) \|_{L^p(Q)} \leq C(R,p)$$

and

$$\| \dot{S}(z) \|_{L^p(Q)} \leq C(R,p).$$

Now, for z in the closed ball $\bar{B}(0;R)$ of radius R with centrum at the origin of the Banach space $C^1([0,T];C^1(\bar{\Omega}))$, $z+\omega \in \bar{B}(0;R+ \| \omega \|_{C^1([0,T];C^1(\bar{\Omega}))})$ so that a fortiori

$$\|S(z+\omega)\|_{L^p(Q)} \leq C(R,p)$$

and

$$\left\| \frac{d}{dt} S(z+\omega) \right\|_{L^p(Q)} \leq C(R,p).$$

Using now the inequalities (3.117) and (3.118), the result follows. □

Theorem 66. *Let us suppose that we have choosen*

$$R > 3(T+1)UB(y_0, \mathbf{E}_0), \tag{3.119}$$

where $UB(y_0,\mathbf{E}_0)$ has been defined in formula (3.116). Then, there exists $t_f \in]0,T/2]$ such that $\forall z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$ satisfying $z(0)=0$:

$$\widetilde{(\breve{y}_z|_{[0,t_f]})} \in \bar{B}(0;R).$$

Proof. $\forall z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$ satisfying $z(0)=0$:

$$\|\breve{y}_z\|_{W_p^{2,1}(Q)} \leq C(R,p),$$

and

$$\left\| \dot{\breve{y}}_z \right\|_{W_p^{2,1}(Q)} \leq C(R,p).$$

Let us consider $\rho > 0$ such that $\rho \leq \frac{R-3(T+1)UB(y_0,\mathbf{E}_0)}{3(T+1)}$. $\bar{B}(\breve{f}_0(t_f;\cdot);\rho) \subset \bar{B}(0;\frac{R}{3(T+1)})$ for every $t_f \in]0,T/2]$. As a consequence of [32, Lemma 3.3 p.80, second inequality], we have that $W_p^{2,1}(Q) \hookrightarrow C^\alpha([0,T];C^1(\bar{\Omega}))$ for $\alpha = \frac{1}{2}(1-\frac{5}{p})$. Thus:

$$\|\breve{y}_z\|_{C^\alpha([0,T];C^1(\bar{\Omega}))} \leq C(R,p),$$

and

$$\left\| \dot{\breve{y}}_z \right\|_{C^\alpha([0,T];C^1(\bar{\Omega}))} \leq C(R,p).$$

As $\breve{y}_z(0) = 0$, from the first inequality results that choosing $t_f > 0$ sufficiently small, we will have $\forall t \in [0, t_f]$:

$$\|\breve{y}_z(.,t)\|_{C^1(\bar{\Omega}))} \leq \frac{\rho}{4},$$

for all $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$ satisfying $z(0) = 0$. Similarly, as $\dot{\breve{y}}_z(0) = \dot{\breve{f}}_0(t_f;0) = \mu_a(\cdot, y_0(\cdot)) \mid (\mathbf{E}_0 * \varphi_a)(\cdot) \mid^2$ (in particular is independent of t_f and z), from the second inequality results that choosing $t_f > 0$ smaller if necessary, we will have $\forall t \in [0, t_f]$:

$$\left\|\dot{\breve{y}}_z(.,t) - \dot{\breve{f}}_0(t_f;0)\right\|_{C^1(\bar{\Omega}))} \leq \frac{\rho}{4},$$

for all $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$ satisfying $z(0) = 0$.

Now, $\breve{f}_0(t_f;t) = \mu_a(\cdot, y_0(\cdot)) \mid (\mathbf{E}_0 * \varphi_a)(\cdot) \mid^2 \int_0^t \theta(t_f;\xi)\, d\xi$ so that

$$\left\|\breve{f}_0(t_f;t)\right\|_{C^1(\bar{\Omega}))} \leq \left\|\mu_a(\cdot, y_0(\cdot)) \mid (\mathbf{E}_0 * \varphi_a)(\cdot) \mid^2\right\|_{C^1(\bar{\Omega}))} t_f$$

for all $t \in [0, t_f]$, so that reducing still t_f if necessary, we will have for all $t \in [0, t_f]$: $\left\|\breve{f}_0(t_f;t)\right\|_{C^1(\bar{\Omega}))} \leq \frac{\rho}{4}$. Thus:

$$\left\|\breve{y}_z(.,t) - \breve{f}_0(t_f;t)\right\|_{C^1(\bar{\Omega}))} \leq \|\breve{y}_z(.,t)\|_{C^1(\bar{\Omega}))} + \left\|\breve{f}_0(t_f;t)\right\|_{C^1(\bar{\Omega}))}$$

so that for all $t \in [0, t_f]$:

$$\left\|\breve{y}_z(.,t) - \breve{f}_0(t_f;t)\right\|_{C^1(\bar{\Omega}))} \leq \frac{\rho}{2}.$$

On the other hand $\dot{\breve{f}}_0(.,t) = \dot{\breve{f}}_0(.,0)$, for all $t \in [0, t_f]$ so that

$$\left\|\dot{\breve{y}}_z(.,t) - \dot{\breve{f}}_0(t_f;t)\right\|_{C^1(\bar{\Omega}))} \leq \left\|\dot{\breve{y}}_z(.,t) - \dot{\breve{f}}_0(t_f;0)\right\|_{C^1(\bar{\Omega}))} \leq \frac{\rho}{4}.$$

BY Summing these two last inequalities, we obtain:

$$\left\|\breve{y}_z|_{[0,t_f]} - \breve{f}_0(t_f;\cdot)|_{[0,t_f]}\right\|_{C^1([0,t_f];C^1(\bar{\Omega}))} \leq \frac{3\rho}{4} \leq \rho.$$

By lemma 63:

$$\left\|\widetilde{\breve{y}_z|_{[0,t_f]}} - \widetilde{\breve{f}_0(t_f;\cdot)|_{[0,t_f]}}\right\|_{C^1([0,T];C^1(\bar{\Omega}))} \leq 3(T+1)\rho.$$

and

$$\begin{aligned}\left\|\widetilde{(\breve{f}_0(t_f;\cdot)|_{[0,t_f]})}\right\|_{C^1([0,T];C^1(\bar{\Omega}))} &\leq 3(T+1)\left\|\breve{f}_0(t_f;\cdot)'|_{[0,t_f]}\right\|_{C([0,t_f];C^1(\bar{\Omega}))} \\ &\leq 3(T+1)UB(y_0,\mathbf{E}_0).\end{aligned}$$

Thus

$$\left\|\widetilde{\breve{y}_z|_{[0,t_f]}}\right\|_{C^1([0,T];C^1(\bar{\Omega}))} \leq 3(T+1)(\rho + UB(y_0,\mathbf{E}_0)).$$

As $3(T+1)(\rho + UB(y_0,\mathbf{E}_0)) \leq R$, it follows that $\widetilde{\breve{y}_z|_{[0,t_f]}} \in \bar{B}(0;R)$. □

Lemma 67. *For $p > 5$, the embedding from $W_p^{2,1}(Q)$ into $C([0,T];C^1(\bar{\Omega}))$ is a linear compact mapping.*

Proof. By the inequality [32, p.343, top], we have for all $p > 5$ that:

$$|y|^{(2-5/p)} \leq c \, \|y\|_{p,Q}^{(2)} , \ \forall y \in W_p^{2,1}(Q). \tag{3.120}$$

We have written this inequality following the notations of [32]. Let us explain these notations. Firstly, $\|y\|_{p,Q}^{(2)} = \sum_{j=0}^{2} \langle\langle y \rangle\rangle_{p,Q}^{(j)}$, where $\langle\langle y \rangle\rangle_{p,Q}^{(j)} = \sum_{2r+s=j} \|D_t^r D_x^s y\|_{p,Q}$ [32, p.5]. Consequently:

$$\|y\|_{p,Q}^{(2)} = \|y\|_{p,Q} + \sum_{i=1}^{3} \left\| \frac{\partial y}{\partial x_i} \right\|_{p,Q} + \left\| \frac{\partial y}{\partial t} \right\|_{p,Q} + \sum_{i=1}^{3} \sum_{j=1}^{3} \left\| \frac{\partial^2 y}{\partial x_i \partial x_j} \right\|_{p,Q} = \|y\|_{W_p^{2,1}(Q)} . \tag{3.121}$$

On the other hand [32, p.7], $|y|^{(2-5/p)}$ means as $1 < 2 - \frac{5}{p} < 2$ for $p > 5$:

$$|y|^{(2-5/p)} = \|y\|_{\infty,Q} + \sum_{i=1}^{3} \left\| \frac{\partial y}{\partial x_i} \right\|_{\infty,Q} + \sup_{(x,t),(x,t') \in \bar{Q}} \frac{|y(x,t) - y(x,t')|}{|t-t'|^{1-\frac{5}{2p}}}$$
$$+ \sum_{i=1}^{3} \left\{ \sup_{(x,t),(x,t') \in \bar{Q}} \frac{|\frac{\partial y}{\partial x_i}(x,t) - \frac{\partial y}{\partial x_i}(x,t')|}{|x-x'|^{1-\frac{5}{p}}} + \sup_{(x,t),(x,t') \in \bar{Q}} \frac{|\frac{\partial y}{\partial x_i}(x,t) - \frac{\partial y}{\partial x_i}(x,t')|}{|t-t'|^{\frac{1}{2}(1-\frac{5}{p})}} \right\} .$$

Let us consider now a bounded set E in $W_p^{2,1}(Q)$. There exists thus a real number $R > 0$, such that for all $y \in E : \|y\|_{W_p^{2,1}(Q)} \leq R$. As a consequence of inequality (3.120), the families of continuous functions on $\bar{Q}$: $\left\{ y \in C(\bar{Q}); y \in E \right\}$, $\left\{ \frac{\partial y}{\partial x_i} \in C(\bar{Q}); y \in E \right\}$ $(i = 1,2,3)$ are equicontinous families in $C(\bar{Q})$ endowed with the supremum norm. Also for every fixed $(x,t) \in \bar{Q}$, the sets $\{y(x,t); y \in E\}$, $\left\{ \frac{\partial y}{\partial x_i}(x,t); y \in E \right\}$ $(i = 1,2,3)$ are relatively compact subsets of $\mathbb{R}$. Thus by Ascoli's theorem [20, (7.5.7)], the sets $\left\{ y \in C(\bar{Q}); y \in E \right\}$, $\left\{ \frac{\partial y}{\partial x_i} \in C(\bar{Q}); y \in E \right\}$ $(i = 1,2,3)$ are relatively compact subsets of $C(\bar{Q})$. But the mapping

$$C([0,T];C^1(\bar{\Omega})) \to C(\bar{Q})^4 : y \mapsto (y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \frac{\partial y}{\partial x_3})$$

is an isometry onto the closed subspace

$$\{(y, y_1, y_2, y_3) \in C(\bar{Q})^4; y_1 = \frac{\partial y}{\partial x_1}, y_2 = \frac{\partial y}{\partial x_2}, y_3 = \frac{\partial y}{\partial x_3}\}$$

of $C(\bar{Q})^4$. The set $\left\{ (y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \frac{\partial y}{\partial x_3}) \in C(\bar{Q})^4; y \in E \right\}$ is relatively compact in that closed subspace of $C(\bar{Q})^4$ being the intersection of the cartesian product

$$\left\{ y \in C(\bar{Q}); y \in E \right\} \times \prod_{i=1}^{3} \left\{ \frac{\partial y}{\partial x_i} \in C(\bar{Q}); y \in E \right\}$$

relatively compact subset of $C(\bar{Q})^4$ with that closed subspace of $C(\bar{Q})^4$. By the previous isometry $\left\{ y \in C([0,T];C^1(\bar{\Omega})); y \in E \right\}$ is a relatively compact subset $C([0,T];C^1(\bar{\Omega}))$.

□

Proposition 68. *Let us denote by* $K(0;R) := \{z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}));\, z(0) = 0\}$. *Supposing that* $R > 0$ *satisfies condition (3.119), then the range of the mapping from the closed convex set* $K(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$ *into* $C^1([0,T];C^1(\bar{\Omega}))$ *which sends* $z \in K(0;R)$ *onto* $(\breve{y}_z|_{[0,t_f]})\sim$ *is a relatively compact subset in* $C^1([0,T];C^1(\bar{\Omega}))$.

Proof. By Lemma 65, we know that

$$\|\breve{y}_z\|_{W_p^{2,1}(Q)} \leq C(R,p),$$

and

$$\left\|\dot{\breve{y}}_z\right\|_{W_p^{2,1}(Q)} \leq C(R,p).$$

Thus, a fortiori, the image of $K(0;R)$ by the mapping $z \mapsto (\breve{y}_z|_{[0,t_f]}), \dot{\breve{y}}_z|_{[0,t_f]})$ in $W_p^{2,1}(Q_{t_f}) \times W_p^{2,1}(Q_{t_f})$ is bounded, where we have set $Q_{t_f} := \Omega\times]0,t_f[$. As $p > 5$, by Lemma 67 applied to Q_{t_f}, the embedding from $W_p^{2,1}(Q_{t_f})$ into $C([0,t_f];C^1(\bar{\Omega}))$ is a linear compact mapping. It follows that the image of $K(0;R)$ by the mapping $z \mapsto (\breve{y}_z|_{[0,t_f]}), \dot{\breve{y}}_z|_{[0,t_f]})$ in $C([0,t_f];C^1(\bar{\Omega})) \times C([0,t_f];C^1(\bar{\Omega}))$ is a relatively compact subset. Being contained in the closed subspace

$$E := \left\{(y,v) \in C([0,t_f];C^1(\bar{\Omega})) \times C([0,t_f];C^1(\bar{\Omega}));\, v = \dot{y}\right\}$$

of $C([0,t_f];C^1(\bar{\Omega})) \times C([0,t_f];C^1(\bar{\Omega}))$, it is also a relatively compact subset of E (in the definition of E, $v = \dot{y}$ must be understood in the sense of distributions on $]0,t_f[$ with values in $C^1(\bar{\Omega})$). Now, E is isomorphic to $C^1([0,t_f];C^1(\bar{\Omega}))$. Thus the image of $K(0;R)$ by the mapping $z \mapsto \breve{y}_z|_{[0,t_f]}$ is a relatively compact subset of the Banach space $C^1([0,t_f];C^1(\bar{\Omega}))$. By Lemma 63, the mapping from $C^1([0,t_f];C^1(\bar{\Omega}))$ into $C^1([0,T];C^1(\bar{\Omega}))$ which sends f onto its extension $\tilde{f}$ is linear and continuous. Consequently, the image of $K(0;R)$ by the mapping $z \mapsto (\breve{y}_z|_{[0,t_f]})\sim$ is also a relatively compact subset of $C^1([0,T];C^1(\bar{\Omega}))$. □

Proposition 69. *Supposing that* $R > 0$ *satisfies condition (3.119), then the the mapping from the closed convex set* $K(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$ *into* $C^1([0,T];C^1(\bar{\Omega}))$ *which sends* $z \in K(0;R)$ *onto* $(\breve{y}_z|_{[0,t_f]})\sim$ *is a continuous mapping.*

Proof. Let us consider a sequence $(z_n)_{n\in\mathbb{N}}$ in $K(0;R)$ converging to $z \in K(0;R)$ for the norm of $C^1([0,T];C^1(\bar{\Omega}))$. It follows from Propositions 57 and 58 that the sequence $(S(z_n+\omega))_{n\in\mathbb{N}}$ converges strongly to $S(z+\omega)$ in $L^p(0,T;C(\bar{\Omega}))$ and that the sequence of the time derivatives $(\frac{d}{dt}S(z_n+\omega))_{n\in\mathbb{N}}$ converges weakly to $\frac{d}{dt}S(z+\omega)$ in $L^p(0,T;C(\bar{\Omega}))$. By inequality (3.118) the solution of Problem (3.111) being an affine continuous mapping of the right-hand side, the sequence $(\dot{\breve{y}}_{z_n})_{n\in\mathbb{N}}$ is weakly convergent to $\dot{\breve{y}}_z$ in $W_p^{2,1}(Q)$ but due to the compact embedding from $W_p^{2,1}(Q)$ into $C([0,T];C^1(\bar{\Omega}))$ by Lemma 67, strongly convergent in $C([0,T];C^1(\bar{\Omega}))$. Also by inequality (3.117), the solution of Problem (3.108) being a continuous mapping of the right-hand side, the sequence $(\breve{y}_{z_n})_{n\in\mathbb{N}}$ is strongly convergent to $\breve{y}_z$ in $C([0,T];C^1(\bar{\Omega}))$. In conclusion, the sequence $(\breve{y}_{z_n})_{n\in\mathbb{N}}$ is strongly convergent to $\breve{y}_z$ in $C^1([0,T];C^1(\bar{\Omega}))$. A fortiori, the sequence $(\breve{y}_{z_n}|_{[0,t_f]})_{n\in\mathbb{N}}$ is strongly convergent to $\breve{y}_z|_{[0,t_f]}$ in $C^1([0,t_f];C^1(\bar{\Omega}))$. From Lemma 63 now follows, that the sequence $(\breve{y}_{z_n}|_{[0,t_f]})\sim_{n\in\mathbb{N}}$ strongly converges to $(\breve{y}_z|_{[0,t_f]})\sim$ in $C^1([0,T];C^1(\bar{\Omega}))$. So, the result follows. □

We come now to our existence theorem of a local solution to our nonlinear initial boundary value problem (3.101):

Theorem 70. *Let us suppose that $R > 0$ has been choosen such that*

$$R > 3(T+1)UB(y_0, \mathbf{E}_0) \tag{3.122}$$

where $UB(y_0, \mathbf{E}_0)$ has been defined in formula (3.116). Then, there exists $t_f \in]0, T/2]$ such that the mapping from the closed convex set $K(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$, which sends $z \in K(0;R)$ to $(\breve{y}_z|_{[0,t_f]}) \in C^1([0,T];C^1(\bar{\Omega}))$, has a fixed point $\zeta \in K(0;R)$. Consequently $\zeta|_{[0,t_f]} + \omega|_{[0,t_f]} \in C^1([0,t_f];C^1(\bar{\Omega}))$ is a weak solution of our nonlinear initial boundary value problem (3.101).

Proof. By Theorem 66, and the hypothesis (3.122) on R, there exists $t_f \in]0, \frac{T}{2}]$ such that for every $z \in K(0;R)$, $(\breve{y}_z|_{[0,t_f]}) \in K(0;R)$. By Proposition 69, the mapping from $K(0;R)$ into $K(0;R)$ which sends $z \in K(0;R)$ onto $(\breve{y}_z|_{[0,t_f]})$ is a continuous mapping. By Proposition 68, its range is a relatively compact subset in the Banach space $C^1([0,T];C^1(\bar{\Omega}))$. Thus by the Schauder fixed point theorem [23, p.171], this mapping has a fixed point ζ whose restriction to the time interval $[0,t_f]$ is a weak solution of the nonlinear initial boundary value problem (3.108) by the "causality principle". Adding to $\zeta|_{[0,t_f]}$, $\omega|_{[0,t_f]}$ weak solution on the time interval $[0,t_f]$ of the auxilary non-homogeneous linear initial boundary value problem (3.102), we obtain a weak solution of our nonlinear non-homogeneous initial boundary value problem (3.101) on the time interval $[0,t_f]$ $(t_f > 0)$. □

3.3.3 Case of a right-hand side $\mathbf{G} \in C^1([0,T];\mathcal{H})$

To prove that Theorem 70 is also true when the right-hand side $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2)$ in the Maxwell system (3.50) belongs to $C^1([0,T];\mathcal{H})$, we have basically to prove that the bounds (3.68) and (3.69) remain valid in the case $\mathbf{G} \in C^1([0,T];\mathcal{H})$.

Proposition 71. *For every fixed $(\mathbf{E}_0, \mathbf{H}_0) \in Y$, and every fixed $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2) \in C^1([0,T];\mathcal{H})$, the Y-valued solutions $(\mathbf{E}_z, \mathbf{H}_z) : [0,T] \to Y : t \mapsto (\mathbf{E}_z, \mathbf{H}_z)(t)$ of the initial value problem (3.50) given by formula (3.51) for z running over $\bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$ have uniformly bounded norms in the Banach space $C([0,T];Y)$ by a constant $C(R)$ depending only on the radius R of the ball $\bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$.*

Proof. It suffices to prove that the norm of the mappings $w_z : [0,T] \to Y : t \mapsto w_z(t) := \int_0^t U_z(t,r)\mathbf{G}(r)dr$ in $C([0,T];Y)$ are uniformly bounded by a constant $C(R)$ for $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$. By the proof of Proposition 41, we know that $w_z(t) = Q_z(t)^{-1}\mathbf{G}(t) - v_z(t)$, where

$$v_z(t) := U_z(t,0)Q_z(0)^{-1}\mathbf{G}(0) + \int_0^t U_z(t,r)[Q_z(r)^{-1}(g_z(r) - G(r)) - \mathcal{M}_z(r)Q_z(r)^{-1}\mathbf{G}(r)]dr, \tag{3.123}$$

with $g_z(r) := \mathbf{G}'(r) - \dot{Q}_z(r)Q_z(r)^{-1}\mathbf{G}(r)$.

1. We have shown in the proof of Proposition 48, that $\|Q_z(t)^{-1}\|_{\mathcal{L}(\mathcal{H},Y)} \leq C(R)$ independent of z and t for $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$ and $t \in [0,T]$. $\mathbf{G}(.)$ belonging a fortiori to $C([0,T];\mathcal{H})$ is bounded. Thus $\| Q_z(t)^{-1}\mathbf{G}(t) \|_Y \leq C(R)$.

2. Taking $t = 0$ in the preceding point, we get: $\| Q_z(0)^{-1}\mathbf{G}(0) \|_Y \leq C(R)$. Proposition 48, implies in particular that the families of operators $(U_z(t,0)|_Y)_{0\leq t\leq T}$ are uniformly bounded in $\mathcal{L}(Y)$ by a constant $C(R)$ for $z \in \bar{B}(0,R)$, so that $\| U_z(t,0)Q_z(0)^{-1}\mathbf{G}(0) \|_Y \leq C(R)$ for all $t \in [0,T]$.

3. Let us now consider the third term $-\int_0^t U_z(t,r)\mathcal{M}_z(r)Q_z(r)^{-1}\mathbf{G}(r)dr$ in the right-hand side of (3.123). We want to bound its norm in the Banach space Y by a constant depending only on R. By Proposition 48: $\| U_z(t,r)|_Y \|_{\mathcal{L}(Y)} \leq C(R)$. By the first point $\| Q_z(r)^{-1}\mathbf{G}(r) \|_Y \leq C(R)$. Looking to formula (3.44) in the proof of Proposition 36, we deduce that $\| \mathcal{M}_z(r) \|_{\mathcal{L}(Y)} \leq C(R)$. Thus

$$\| -\int_0^t U_z(t,r)\mathcal{M}_z(r)Q_z(r)^{-1}\mathbf{G}(r)dr \|_Y \leq C(R).$$

4. Finally, let us consider the second term $\int_0^t U_z(t,r)Q_z(r)^{-1}(g_z(r) - \mathbf{G}(r))dr$ in the righthand side of (3.123). We want also to bound its norm in the Banach space Y. In view of the reasonings already made in the two preceding points, it suffices still to prove that the norm of $g_z(r)$ in the Banach space $\mathcal{H}$ is bounded by a constant depending only on R. $g_z(r) = \mathbf{G}'(r) - \dot{Q}_z(r)Q_z(r)^{-1}\mathbf{G}(r)$. Recalling, that $\mathbf{G}$ belongs to $C^1([0,T];\mathcal{H})$, we have only to prove that $\| \dot{Q}_z(r) \|_{\mathcal{L}(Y;\mathcal{H})}$ is bounded by a constant depending only on R. From the explicit expression of $\dot{Q}_z(r)$:

$$\dot{Q}_z(r)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix} -\frac{\partial\epsilon}{\partial z}(.,z(.,r))\frac{\partial z}{\partial t}(.,r)\,\mathrm{curl}\,\psi \\ 0\end{pmatrix}, \text{for all } (\varphi,\psi) \in Y,$$

the definitions of Y and $\mathcal{H}$ and $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$ it is clear.

From $w_z(t) = Q_z(t)^{-1}\mathbf{G}(t) - v_z(t)$, formula (3.123), and the four preceding points, it is now clear that $\| w_z \|_{C([0,T];Y)}$ is uniformly bounded by a constant $C(R)$ depending only on the radius R of the ball $\bar{B}(0;R)$ for $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$. □

Thus bound (3.68) is still valid. Now, we want to prove that bound (3.69) remains also valid for a right-hand side $\mathbf{G} \in C^1([0,T];\mathcal{H})$:

Proposition 72. *For every fixed $(\mathbf{E}_0,\mathbf{H}_0) \in Y$, and every fixed $\mathbf{G} = (\mathbf{G}_1,\mathbf{G}_2) \in C^1([0,T];\mathcal{H})$, the time derivatives of the Y-valued solutions $(\mathbf{E}_z,\mathbf{H}_z) : [0,T] \to Y : t \mapsto (\mathbf{E}_z,\mathbf{H}_z)(t)$ of the initial value problem (3.50) given by formula (3.51), for z running over the closed ball $\bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$, belong to $C([0,T];\mathcal{H})$, and have their norms in this space uniformly bounded by a constant $C(R)$ depending only on the radius of the ball $\bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$.*

Proof. $w_z : [0,T] \to Y : t \mapsto w_z(t) := \int_0^t U_z(t,r)\mathbf{G}(r)dr$ is solution of:

$$\frac{dw_z}{dt}(t) = (\mathcal{A}_z(t) + \mathcal{M}_z(t))w_z(t) + \mathbf{G}(t). \tag{3.124}$$

From Proposition 71, we know that $\| w_z \|_{C([0,T];Y)}$ is uniformly bounded by a constant $C(R)$ depending only on the radius R of the ball $\bar{B}(0;R)$ for $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$. From (4.12),(3.20),(3.24),(3.25), it is clear that $\| \mathcal{A}_z(t)+\mathcal{M}_z(t) \|_{\mathcal{L}(Y;\mathcal{H})}$ is uniformly bounded by a constant $C(R)$ depending only on the radius R of the ball $\bar{B}(0;R)$ for $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$. Thus from formula (3.124), follows now that $\| \frac{dw_z}{dt} \|_{C([0,T];\mathcal{H})}$ is uniformly bounded by a constant $C(R)$ depending only on the radius R of the ball $\bar{B}(0;R)$ for $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$. On the other hand

$$\| U_z(t,0)\,(\mathbf{E}_0,\mathbf{H}_0) \|_Y \leq \| U_z(t,0)|_Y \|_{\mathcal{L}(Y)} \| (\mathbf{E}_0,\mathbf{H}_0) \|_Y \leq C(R) \| (\mathbf{E}_0,\mathbf{H}_0) \|_Y$$

by Proposition 48. Thus

$$\begin{aligned} \| (\mathcal{A}_z(t)+\mathcal{M}_z(t))U_z(t,0)\,(\mathbf{E}_0,\mathbf{H}_0) \|_{\mathcal{H}} \leq & \| \mathcal{A}_z(t)+\mathcal{M}_z(t) \|_{\mathcal{L}(Y;\mathcal{H})} \| U_z(t,0)\,(\mathbf{E}_0,\mathbf{H}_0) \|_Y \\ & \leq C(R) \| (\mathbf{E}_0,\mathbf{H}_0) \|_Y . \end{aligned} \tag{3.125}$$

In conclusion

$$\begin{aligned} \| \frac{d}{dt}(\mathbf{E}_z,\mathbf{H}_z) \|_{C([0,T];\mathcal{H})} = & \| \frac{d}{dt}(U_z(.,0)\,(\mathbf{E}_0,\mathbf{H}_0)) + \frac{dw_z}{dt} \|_{C([0,T];\mathcal{H})} \\ \leq & \| \frac{d}{dt}(U_z(.,0)\,(\mathbf{E}_0,\mathbf{H}_0)) \|_{C([0,T];\mathcal{H})} + \| \frac{dw_z}{dt} \|_{C([0,T];\mathcal{H})} \end{aligned} \tag{3.126}$$

is uniformly bounded by a constant $C(R)$ depending only on the radius R of the ball $\bar{B}(0;R)$ for $z \in \bar{B}(0;R) \subset C^1([0,T];C^1(\bar{\Omega}))$. □

Thus, we know now, that the basic estimates (3.68) and (3.69) remain valid for right-hand sides $\mathbf{G} \in C^1([0,T];\mathcal{H})$ in the Maxwell system (3.50). Also, all the reasonings of subsections 3.1 and 3.2 which follow the estimates (3.68) and (3.69) remain valid. Consequently, Theorem 70 on the existence of a local weak solution to our coupled nonlinear initial boundary value problem (3.101) between the heat equation and the Maxwell system (3.50) with $z = y$, is also valid when the right-hand side $G = (\mathbf{G}_1, \mathbf{G}_2)$ in the Maxwell system (3.50) belongs to $C^1([0,T];\mathcal{H})$.

Chapter 4

Optimal electromagnetic external source field in the Heat-Maxwell coupled system

4.1 Introduction

In this chapter, we want to study a related optimal control problem to the Heat-Maxwell coupled system studied in chapter 3. To simplify matters, we suppose here that the permittivity ϵ depends only on the space and not on the temperature. Consequently, the electric field $\mathbf{E}$ does not depend on the temperature y.

We suppose that the electric field $\mathbf{E}_{ext}$ of the exterior electromagnetic field $(\mathbf{E}_{ext}, \mathbf{H}_{ext})$ hitting Ω is in the complementary of Ω, $\mathbb{R}^3 \setminus \overline{\Omega}$, of the form

$$\mathbf{E}_{ext}(x,t) = \sum_{j=1}^{N} f_j(t)\mathbf{e}_{ext,j}(x,t),\ N \geq 1, \tag{4.1}$$

where

$$\mathbf{e}_{ext,j} \in \mathbf{C}^{1,1}([0,T]; \mathbf{H}(\mathrm{curl}\,, \mathbb{R}^3 \setminus \overline{\Omega})) \tag{4.2}$$

for $j = 1, \ldots, N$ are such that

$$(\mathrm{curl}\,\mathbf{e}_{ext,j}) \cdot n = 0 \text{ on } \Gamma. \tag{4.3}$$

Ω is supposed to be an open bounded subset of $\mathbb{R}^3$ with Lipschitz boundary. f_j for $j = 1, \ldots, N$ are given real-valued functions depending of the time variable t only. Our optimal control problem is the following:

$$\begin{aligned} \min J(y,v) :=& \frac{1}{2}\int_0^T \int_\Omega |\ \nabla y(x,t)\ |^2\ dx\ dt + \sum_{j=1}^{M} \frac{\lambda_{j,Q}}{2} \int_{T_{j,1}}^{T_{j,2}} \int_{\Omega_j} |\ y(x,t) - y_{j,Q}(x,t)\ |^2\ dx\ dt \\ &+ \frac{\lambda_\Omega}{2} \int_\Omega |\ y(x,T) - y_d(x)\ |^2\ dx + \frac{\lambda}{2} \sum_{k=1}^{N} \int_0^T |v_k(t)|^2 dt, \end{aligned} \tag{4.4}$$

with $\Omega_j \subset \Omega$, $T_{j,1} \leq T_{j,2}$, $[T_{j,1}, T_{j,2}] \subset [0,T]$, $\lambda_{j,Q} \geq 0$ $(j = 1, \ldots, M)$, $\lambda > 0$, $\lambda_\Omega \geq 0$, subject the following Heat-Maxwell system:

$$\begin{cases} \partial_t y - \operatorname{div}(\alpha \nabla y) = S(y) & \text{in } Q := \Omega\times]0,T[, \\ -\alpha \frac{\partial y}{\partial n} = h(y - y_b) & \text{on } \Sigma := \Gamma\times]0,T[, \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases} \tag{4.5}$$

where the heat source is defined by

$$S(y)(x,t) := \mu_a(x, y(x,t)) \mid (\mathbf{E} * \varphi_a)(x,t) \mid^2, \quad \text{for all } (x,t) \in Q, \tag{4.6}$$

We assume that the absorption coefficient $\mu_a \in C_b^1(\bar{\Omega} \times \mathbb{R})$ is monotone decreasing with respect to y, this condition will be necessary to prove the well posedness of (4.5) in the sense of [53, Theorem 5.5 p.268]. The initial condition $y_0 \in C(\bar{\Omega})$ and $y_b \in C(\overline{\Sigma})$. φ_a being a given regularization function in $C_c^1(\mathbb{R}^3)$ and $\mathbf{E}$ the electric field in Ω solution of the following Maxwell system:

$$\begin{cases} \partial_t(\epsilon \mathbf{E}) - \operatorname{curl} \mathbf{H} + \sigma \mathbf{E} = 0 & \text{in } Q. \\ \partial_t \mathbf{H} + \hat{\mu} \operatorname{curl} \mathbf{E} = 0 & \text{in } Q, \\ \mathbf{E} \times n = \mathbf{E}_{ext} \times n & \text{on } \Sigma, \\ \mathbf{H} \cdot n = 0 & \text{on } \Sigma, \\ \mathbf{E}(\cdot,0) = \mathbf{E}_0,\ \mathbf{H}(\cdot,0) = \mathbf{H}_0 & \text{in } \Omega, \end{cases} \tag{4.7}$$

where $\alpha > 0$ belongs to $C(\bar{\Omega})$, $h > 0$ belongs to $C(\Gamma)$, $\epsilon \geq \epsilon_0 > 0$ belongs to $L^\infty(\Omega)$, $\hat{\epsilon} := \frac{1}{\epsilon} \in L^\infty(\Omega)$, $\sigma \geq 0$ belongs to $L^\infty(\Omega)$, $\mu \geq \mu_0 > 0$ belongs $L^\infty(\Omega)$ and $\hat{\mu} := \frac{1}{\mu} \in L^\infty(\Omega)$ are only functions of x with $\mathbf{E}_{ext}$ given by (4.1), such that

$$\begin{cases} -f_j'' + f_j = v_j \quad in \quad]0,T[, \\ f_j(0) = f_{j,0}\,, \\ f_j'(0) = f_{j,1}\,, \end{cases} \tag{4.8}$$

where $v = (v_j)_{j=1}^N \in \mathcal{U}$ are the controls belonging to the control space $\mathcal{U} := [L^2(0,T)]^N$. The set of admissible controls is the closed convex subset $\mathcal{U}_{ad} := \prod_{j=1}^N U_{ad,j} \subset \mathcal{U}$, where for $j = 1, \ldots, N$:

$$U_{ad,j} = \{v_j \in L^2(0,T); v_{j,a} \leq v_j(t) \leq v_{j,b} \text{ a.e. } t \in [0,T]\}. \tag{4.9}$$

The numbers $f_{j,0} \in \mathbb{R}$ and $f_{j,1} \in \mathbb{R}$ will remain fixed. $y_{j,Q}$ in the cost functional (4.4) is a given template temperature distribution in $L^2(Q)$ and $y_d \in L^2(\Omega)$.

Remark 14. We have considered a more general cost functional than previously in chapter 2 by replacing the term

$$\frac{\lambda_Q}{2} \int_0^T \int_\Omega \mid y(x,t) - y_Q(x,t) \mid^2 dx\, dt$$

by the more flexible expression

$$\sum_{j=1}^M \frac{\lambda_{j,Q}}{2} \int_{T_{j,1}}^{T_{j,2}} \int_{\Omega_j} \mid y(x,t) - y_{j,Q}(x,t) \mid^2 dx\, dt,$$

the idea being that it is not clear how to choose adequately the function y_Q. We could choose for example $y_{j,Q}$ equals to a fixed temperature a little greater than the temperature

of fusion of the powder e.g. 1600 ^0C [54] as we want the powder to have fused everywhere during a certain subinterval of the time of treatment. It seems also natural to require that $\bigcup_{j=1}^{N} \Omega_j = \Omega$ and the different pieces Ω_j to overlap near their boundaries to glue perfectly together .

In [58] an optimal control problem for microwave heating was studied, where the electric and magnetic fields are time harmonic, with fixed frequency.

4.2 Well posednes of the heat-Maxwell system

Similarly, as we have made in chapter 3, we reduce our Maxwell problem (4.7) to an intial homogeneous boundary value problem by extending $\mathbf{e}_{ext,j}$ to Ω in a vector field

$$\mathbf{e}_j \in \mathbf{C}^{1,1}([0,T];\mathbf{H}(\mathrm{curl}\,,\Omega))$$

such that

$$\mathbf{e}_j \times n = \mathbf{e}_{ext,j} \times n \text{ and } (\mathrm{curl}\,\mathbf{e}_j)\cdot n = 0 \text{ on } \Gamma.$$

Then, we introduce the new vector field $\mathcal{E}$ on Ω by

$$\mathcal{E}(x,t) = \mathbf{E}(x,t) - \sum_{j=1}^{N} f_j(t)\mathbf{e}_j(x,t), \text{ for all } (x,t) \in \Omega \times [0,T].$$

The new couple of vector fields $(\mathcal{E},\mathbf{H})$ is solution of the following intial homogeneous boundary value problem for the Maxwell's equations:

$$\begin{cases} \partial_t \mathcal{E} - \hat{\epsilon}\ \mathrm{curl}\,\mathbf{H} + \hat{\epsilon\sigma}\ \mathcal{E} = -\sum_{j=1}^{N}(f_j'\mathbf{e}_j + f_j\mathbf{e}_j' + \hat{\epsilon\sigma} f_j\mathbf{e}_j) & \text{in } Q, \\ \partial_t \mathbf{H} + \hat{\mu}\ \mathrm{curl}\,\mathcal{E} = -\hat{\mu}\sum_{j=1}^{N} f_j\ \mathrm{curl}\,\mathbf{e}_j & \text{in } Q, \\ \mathcal{E}\times n = 0 & \text{on } \Sigma, \\ \mathbf{H}\cdot n = 0 & \text{on } \Sigma, \\ \mathcal{E}(0) = \mathbf{E}_0 - \sum_{j=1}^{N} f_{j,0}\mathbf{e}_j(0),\ \mathbf{H}(0) = \mathbf{H}_0 & \text{in } \Omega. \end{cases} \tag{4.10}$$

Supposing that

$$\mathbf{E}_0 \times n = \sum_{j=1}^{N} f_{j,0}\ \mathbf{e}_{ext,j}(0) \times n \quad \text{on } \Gamma, \tag{4.11}$$

we have that $\mathcal{E}(0)\times n = 0$ on Γ and thus that

$$\mathcal{E}_0 := \mathbf{E}_0 - \sum_{j=1}^{N} f_j(0)\mathbf{e}_j(0) \in \mathbf{H}_0(\mathrm{curl}\,,\Omega).$$

We also suppose that $\mathbf{H}_0 \in \mathbf{J}^1_n(\Omega,\mu)$. Consequently, the initial condition $(\mathcal{E}_0,\mathbf{H}_0)$ to our initial boundary value problem (4.10) belongs to the domain

$$D(\mathcal{A}) = \mathbf{H}_0(\mathrm{curl}\,,\Omega) \times \mathbf{J}^1_n(\Omega,\mu) \tag{4.12}$$

of the infinitesimal generator $A = \mathcal{A} + \mathcal{M}$ in the Hilbert space

$$\mathcal{H} = \mathbf{L}^2(\Omega) \times \mathbf{J}_n(\Omega, \mu).$$

Let us recall from chapter 3, that

$$\mathcal{A}\phi := (\hat{\epsilon}\,\mathrm{curl}\,\psi, -\hat{\mu}\,\mathrm{curl}\,\varphi), \text{ for all } \phi = (\varphi, \psi) \in D(\mathcal{A}) \tag{4.13}$$

and that

$$\mathcal{M}\phi = (-\hat{\epsilon}\sigma\varphi, 0) \text{ for all } \phi = (\varphi, \psi) \in \mathcal{H}. \tag{4.14}$$

Lemma 73. *There exists a constant $C(|f_{j,0}|, |f_{j,1}|, \max(|v_{j,a}|, |v_{j,b}|)) > 0$, such that for every $v_j \in U_{ad,j}$, the solution $f_j \in C^1([0,T])$ of the Cauchy problem (4.8), satisfies the following bound:*

$$\|f_j\|_{\infty,[0,T]} + \left\|\frac{df_j}{dt}\right\|_{\infty,[0,T]} \leq C(|f_{j,0}|, |f_{j,1}|, \max(|v_{j,a}|, |v_{j,b}|)). \tag{4.15}$$

Proof. Applying the method of variation of arbitrary constants to solve nonhomogeneous second-order linear differential equations as explained in [45, par. 23, pp.92-93], f can be easily computed:

$$f_j(t) = f_{j,0}\cosh(t) + f_{j,1}\sinh(t) + \int_0^t \sinh(y-t)v_j(y)dy. \tag{4.16}$$

From that formula follows immediately that

$$\|f_j\|_{\infty,[0,T]} \leq C(|f_{j,0}| + |f_{j,1}| + \max(|v_{j,a}|, |v_{j,b}|)). \tag{4.17}$$

Differentiating with respect to t formula (4.16), we obtain

$$\frac{df_j}{dt}(t) = f_{j,0}\sinh(t) + f_{j,1}\cosh(t) - \int_0^t \cosh(y-t)v_j(y)dy. \tag{4.18}$$

From that formula for $\frac{df_j}{dt}$ follows immediately also that

$$\left\|\frac{df_j}{dt}\right\|_{\infty,[0,T]} \leq C(|f_{j,0}|, |f_{j,1}|, \max(|v_{j,a}|, |v_{j,b}|)). \tag{4.19}$$

Summing inequalities (4.17) and (4.18), we obtain (4.15). □

Corollary 74. *For $v_j \in U_{ad,j}$, f_j and $\frac{df_j}{dt}$ are lipschitzian functions on $[0,T]$.*

Proof. • Let $t_1, t_2 \in [0,T]$ we have

$$f_j(t_2) - f_j(t_1) = \int_{t_1}^{t_2} \frac{df_j}{dt}\,dt,$$

and by (4.19), $\left\|\frac{df_j}{dt}\right\|_{\infty,[0,T]}$ is bounded by $C(|f_{j,0}|, |f_{j,1}|, \max(|v_{j,a}|, |v_{j,b}|))$. Thus

$$|f_j(t_2) - f_j(t_1)| \leq C(|f_{j,0}|, |f_{j,1}|, \max(|v_{j,a}|, |v_{j,b}|))\ |t_2 - t_1|\,.$$

- We have

$$-f_j'' + f_j = v_j. \tag{4.20}$$

Let $t_1, t_2 \in [0,T]$. Integrating both sides from t_1 to t_2 in (4.20), we obtain

$$\frac{df_j}{dt}(t_1) - \frac{df_j}{dt}(t_2) + \int_{t_1}^{t_2} f_j(t)\ dt = \int_{t_1}^{t_2} v_j(t)\ dt.$$

By the previous result, f_j is bounded. Also v_j, as $v_j \in U_{ad,j}$. Thus, by the previous equation:

$$\left|\frac{df_j}{dt}(t_2) - \frac{df_j}{dt}(t_1)\right| \leq C(|f_{j,0}|, |f_{j,1}|, \max(|v_{j,a}|, |v_{j,b}|))\ \ |t_2 - t_1|\,,$$

proving that $\frac{df_j}{dt}$ is also lipschitzian.

□

From the previous corollary, it follows immediately:

Corollary 75. $\mathbf{G} : t \in [0,T] \mapsto \mathbf{G}(t) \in \mathcal{H}$ *where*

$$\mathbf{G}(t) := (-\sum_{j=1}^{N}(f_j'(t)\mathbf{e}_j(t) + f_j(t)\mathbf{e}_j'(t) + \hat{\epsilon}\sigma f_j(t)\mathbf{e}_j(t)), -\hat{\mu}\sum_{j=1}^{N} f_j(t)\ curl\,\mathbf{e}_j(t)), \tag{4.21}$$

is a Lipschitz continuous function.

We will use the following functional space:

$$W^{1,1}(0,T;\mathcal{H}) := \{u \in L^1(0,T;\mathcal{H}) \text{ such that } \frac{du}{dt} \in L^1(0,T;\mathcal{H})\}.$$

We want to prove that the intial homogeneous boundary value problem for the Maxwell's equations (4.10) possesses a unique strong solution on the time interval $[0,T]$ by using theorem [44, Corollary 2.11 p.109]:

Theorem 76. *(Abstract Cauchy Problem) Let X be a reflexive Banach space and let A be the infinitesimal generator of a C_0 semigroup $(T_t)_{t\geq 0}$ on X. If $f : [0,T] \to X$ is Lipschitz continuous on $[0,T]$, then for every $x \in D(A)$ the initial value problem*

$$\begin{cases} \frac{dw}{dt}(t) = Aw(t) + f(t), \ t > 0, \\ w(0) = x, \end{cases} \tag{4.22}$$

has a unique strong solution w on $[0,T]$ i.e. $w \in W^{1,1}(0,T;X)$, $w(0) = x$, $w(t) \in D(A)$ for a.e. $t \in [0,T]$, and $w'(t) = Aw(t) + f(t)$ for a.e. $t \in [0,T]$. This solution is given by

$$w(t) = T_t x + \int_0^t T_{t-s} f(s) ds,\ t \in [0,T]. \tag{4.23}$$

Let us denote by $(T_t)_{t\geq 0}$ the semigroup generated by the operator $A = \mathcal{A} + \mathcal{M}$ in the Hilbert space $\mathcal{H} = \mathbf{L}^2(\Omega) \times \mathbf{J}_n(\Omega,\mu)$ with domain $D(A)$. Applying theorem 76 to the abstract Cauchy problem defined by the infinitesimal generator A in the Hilbert space $\mathcal{H}$, the righthand side $\mathbf{G}$ and the initial condition $(\mathcal{E}_0, \mathbf{H}_0)$, we obtain:

Theorem 77. *The initial boundary value problem for the Maxwell's equations (4.7) possesses one and only one strong solution* $(\mathbf{E},\mathbf{H})$ *for any given control* $v=(v_j)_{j=1}^N \in \mathcal{U}_{ad}$, *and any given initial condition* $(\mathbf{E}_0,\mathbf{H}_0)\in \mathbf{H}(curl,\Omega)\times \mathbf{J}_n^1(\Omega,\mu)$ *verifying (4.11),* $\mathbf{E}_{ext}$ *being given by (4.1) and (4.8) i.e. possesses one and only one function* $(\mathbf{E},\mathbf{H})\in W^{1,1}(0,T;\mathcal{H})$ *verifying* $(\mathbf{E},\mathbf{H})(0)=(\mathbf{E}_0,\mathbf{H}_0)$ *and equations (4.7) for a.e.* $t\in[0,T]$. *This solution is given by*

$$(\mathbf{E},\mathbf{H})(t)=(\mathcal{E}(t)+\sum_{j=1}^{N} f_j(t)\mathbf{e}_j(t),\mathbf{H}(t)) \text{ for all } t\in[0,T], \tag{4.24}$$

where $(\mathcal{E}(t),\mathbf{H}(t))$ *is given by the equation:*

$$(\mathcal{E}(t),\mathbf{H}(t))=T_t\left(\mathcal{E}_0,\mathbf{H}_0\right)+\int_0^t T_{t-s}\mathbf{G}(s)ds, \tag{4.25}$$

with $\mathbf{G}$ *given by formula (4.21) and*

$$\mathcal{E}_0\in \mathbf{H}_0(curl,\Omega). \tag{4.26}$$

Now, that we know that the initial boundary value problem for the Maxwell's equations (4.7) , possesses a unique strong solution for any given initial condition $(\mathbf{E}_0,\mathbf{H}_0)\in\mathbf{H}(\mathrm{curl},\Omega)\times\mathbf{J}_n^1(\Omega,\mu)$ and any given $v\in U_{ad}$, we will prove that the semilinear parabolic initial boundary value problem for the heat equation (4.5) with electromagnetic heat source (4.6), possesses a unique weak solution $y\in W(0,T)\cap C(\bar{Q})$.

Remark 15. In chapter 3, we have supposed that $\varphi_a\in C_c^1(\mathbb{R}^3)$, but in the present chapter, as we have supposed the electric permittivity ϵ independent of the temperature y, we will need only for our following reasonings that $\varphi_a\in L^2(\mathbb{R}^3)$. As we know by the above semi-group theory, that $\mathbf{E}\in C([0,T];\mathbf{L}^2(\Omega))$, it follows from proposition 51 in chapter 3 that $\mathbf{E}*\varphi_a\in\mathbf{C}(\bar{Q})=C([0,T];\mathbf{C}(\bar{\Omega}))$. This will suffice us here. Examples of functions $\varphi_a\in L^2(\mathbb{R}^3)$, could be e.g. $\varphi_a=\mathbf{1}_{B(0;r)}$ or $\varphi_a(\cdot)=e^{-(\frac{\cdot}{r})^2}$ [55, p.1522,(2)] for a small $r>0$.

To prove that, we are going to use [53, Theorem 5.5 p.268] about existence and uniqueness of the weak solution of general semilinear parabolic initial-boundary value problems of the form [53, (5.1)p.265]:

$$\begin{cases} \partial_t y-\operatorname{div}(\alpha\nabla y)+d(x,t,y)=0 & \text{in } Q,\\ \alpha\frac{\partial y}{\partial\nu}+b(x,t,y)=g & \text{on } \Sigma,\\ y(\cdot,0)=y_0 & \text{in } \Omega. \end{cases} \tag{4.27}$$

To apply theorem [53, Theorem 5.5 p.268] we recall that d and b must satisfy the following assumptions:

- $d:Q\times\mathbb{R}\to\mathbb{R}$ is measurable with respect to $(x,t)\in Q$ for any fixed $y\in\mathbb{R}$.
- $b:\Sigma\times\mathbb{R}\to\mathbb{R}$ is measurable with respect to $(x,t)\in\Sigma$ for any fixed $y\in\mathbb{R}$.
- d and b are monotone increasing with respect to y for almost every $(x,t)\in Q$ and $(x,t)\in\Sigma$, respectively [53, Assumption 5.1 p.266].

- d and b satifies the boundedness condition

$$| d(x,t,0) | \leq K \text{ and } b(x,t,0) \leq K' \text{ for a.e. } (x,t) \in Q \text{ respectively } \in \Sigma, \quad (4.28)$$

K and K' are positive constants [53, Assumption 5.2 p.266].

- For every $(x,t) \in Q$ locally Lipschitz continuous with respect to y, that is, for any $M > 0$ there is some $L(M) > 0$ such that

$$| d(x,t,y_1) - d(x,t,y_2) | \leq L(M) | y_1 - y_2 | \text{ for all } y_i \in \mathbb{R}, \quad (4.29)$$

with $| y_i | \leq M$, $i = 1, 2$. The same is assumed to hold for b on Σ [53, Assumption 5.2 p.266].

We have now to check the above hypotheses on the semilinear parabolic partial differential equation (4.5). Firstly, let us check the hypotheses on the nonlinear term

$$d(x,t,y) = -S(x,t,y) = -\mu_a(x,y) \ |(\mathbf{E} * \varphi_a)(x,t)|^2.$$

- So that

$$d(x,t,0) = -\mu_a(x,0) \ |(\mathbf{E} * \varphi_a)(x,t)|^2,$$

where $\mathbf{E} * \varphi_a \in \mathbf{C}(\bar{Q})$ and $\mu_a \in C_b^1(\bar{\Omega} \times \mathbb{R})$ and thus is a fortiori bounded. Consequently, $|d(x,t,0)|$ is uniformly bounded on Q.

- Moreover

$$|d(x,t,y_1) - d(x,t,y_2| = |\mu_a(x,y_1) - \mu_a(x,y_2)| \ |(\mathbf{E} * \varphi_a)(x,t)|^2.$$

But $\mu_a \in C_b^1(\bar{\Omega} \times \mathbb{R})$ which implies that

$$|\mu_a(x,y_1) - \mu_a(x,y_2)| \leq C|y_1 - y_2|$$

so that

$$|d(x,t,y_1) - d(x,t,y_2| \leq C|y_1 - y_2|.$$

Due the hypothesis on the absorption coefficient μ_a, d is monotone increasing with respect to y.

Secondly, let us check the hypothesis on the boundary condition. In the present case:

$$b(x,t,y) = h(x)(y - y_b(x,t)).$$

- We have

$$b(x,t,0) = -h(x)y_b(x,t).$$

Thus

$$|b(x,t,0)| \leq K := \sup_{x \in \Gamma} h(x) \sup_{(x,t) \in \Sigma} |y_b(x,t)|.$$

Let us recall that $h > 0$ and belongs to $C^1(\partial\Omega)$.

- Also, we have that

$$|b(x,t,y_1)-b(x,t,y_2)| = h(x)|y_1-y_2| \leq (\sup_{x\in\Gamma} h(x))|y_1-y_2|,$$

for all $y_1,\ y_2 \in \mathbb{R}$.

As $h > 0$ belongs to $C(\Gamma)$, $b(x,t,y)$ is monotone increasing with respect to y. Also, as the coefficient $\alpha > 0$ belongs to $C(\bar{\Omega})$ implies that the partial differential operator $-\text{div}\,(\alpha\,\nabla\cdot)$ verifies the hypotheses [53, pp.37-38] i.e. satisfy the condition of uniform ellipticity: the hypotheses made on α implies in particular [53, (2.20)p.37] (also the conormal derivative [53, p.37] is in the present case $\alpha\frac{\partial}{\partial n}$ and α is continuous and strictly positive on $\partial\Gamma$).

We are thus allowed to apply [53, Theorem 5.5 p.268], so that, consequently:

Theorem 78. *The semi-linear parabolic initial boundary value problem (4.5), (4.6) has a unique weak solution $y \in W(0,T)\cap C(\bar{Q})$ for any $\mathbf{E} \in C([0,T];\mathbf{L}^2(\Omega))$, any $y_b \in C(\overline{\Sigma})$ and any initial condition $y_0 \in C(\bar{\Omega})$. Moreover for any $r > 5/2$ and any $s > 4$, there exists a constant $C(r,s)$ such that*

$$\begin{aligned}\|y\|_{W(0,T)} + \|y\|_{C(\bar{Q})} \leq & C(r,s)(\left\|\mu_a(\cdot,0)|\mathbf{E}*\varphi_a|^2(\cdot,\cdot)\right\|_{L^r(Q)} \\ & + \|h(\cdot)y_b(\cdot,\cdot)\|_{L^s(\Sigma)} + \|y_0\|_{C(\bar{\Omega})}).\end{aligned} \tag{4.30}$$

Here $W(0,T)$ is defined by

$$W(0,T) := \{u \in L^2(0,T;H^1(\Omega)) \text{ such that } \frac{du}{dt} \in L^2(0,T;(H^1(\Omega))^*)\}.$$

Let us point out that the temperature y depends on the electric field $\mathbf{E}$ (4.5)(4.6), wich depends on $\mathbf{E}_{ext}$ (4.7), thus on $f = (f_j)_{j=1}^N \in L^2(0,T)^N$ which itself depends on the control $v = (v_j)_{j=1}^N \in \mathcal{U}_{ad}$ (4.8).

4.3 Existence of an optimal control

Now, we want to prove the existence of an optimal control $v = (v_j)_{j=1}^N \in \mathcal{U}_{ad}$. We indicate this dependence of the temperature y with respect to the control v, by writing y_v. Let us introduce the reduced cost functional:

$$\hat{J} : \mathcal{U}_{ad} \to \mathbb{R} : v \mapsto J(y_v, v).$$

Let us consider a minimizing sequence $(v^{(k)})_{k\geq 0}$ as $k \to +\infty$ in $\mathcal{U}_{ad}$, such that

$$\hat{J}(v^{(k)}) \to \inf_{v\in U_{ad}} \hat{J}(v).$$

From the definition of the cost functional (4.4), and of the reduced cost functional, follows immediately that the sequence $(v^{(k)})_{k\geq 0}$ is bounded in $L^2([0,T])^N$. By (4.21), the right-hand side corresponding to $v^{(k)}$ in the auxilary intial homogeneous boundary value problem for the Maxwell's equations (4.10) is:

$$\mathbf{G}^{(k)}(t) = \sum_{j=1}^{N}(-f_j'^{(k)}(t)\mathbf{e}_j(t) - f_j^{(k)}(t)\mathbf{e}_j'(t) - \hat{\epsilon}\sigma f_j^{(k)}(t)\mathbf{e}_j(t), -\hat{\mu}f_j^{(k)}(t)\text{curl}\,\mathbf{e}_j(t)), \tag{4.31}$$

where $f_j^{(k)}$ is the solution of the Cauchy problem (4.8) with right-hand side $v_j^{(k)}$. From (4.15) and the boundedness of $(v^{(k)})_{k\geq 0}$ in $L^2([0,T])^N$ follows that $(f^{(k)})_{k\geq 0}$ and $(f'^{(k)})_{k\geq 0}$ are bounded in $C([0,T])^N$. A fortiori the sequence $(f^{(k)})_{k\geq 0}$ is bounded in $H^1([0,T])^N$. Also, as the sequence $(v^{(k)})_{k\geq 0}$ is bounded in $L^2([0,T])^N$, it follows by (4.8) that the sequence $(f'^{(k)})_{k\geq 0}$ is also bounded in $H^1([0,T])^N$. Modulo, extracting subsequences, we may suppose that the sequence $(v^{(k)})_{k\geq 0}$ is weakly convergent in $L^2([0,T])^N$ and that the sequences $(f^{(k)})_{k\geq 0}$ and $(f'^{(k)})_{k\geq 0}$ are weakly convergent in $H^1([0,T])^N$. By the compact embedding of $H^1([0,T])^N$ into $C([0,T])^N$, follows that $(f^{(k)})_{k\geq 0}$ and $(f'^{(k)})_{k\geq 0}$ are strongly convergent in $C([0,T])^N$. Consequently, $(f^{(k)})_{k\geq 0}$ is strongly convergent to some function f in $C^1([0,T])^N$ corresponding by (4.8) to the weak limit v of the sequence $(v_k)_{k\geq 0}$ in $L^2([0,T])^N$. From (4.31), follows that

$$\mathbf{G}_1^{(k)} = \sum_{j=1}^{N}(-f_j'^{(k)}(t)\mathbf{e}_j(t) - f_j^{(k)}(t)\mathbf{e}_j'(t) - \hat{\epsilon}\sigma f_j^{(k)}(t)\mathbf{e}_j(t))$$

converges to

$$\sum_{j=1}^{N}(-f_j'(t)\mathbf{e}_j(t) - f_j(t)\mathbf{e}_j'(t) - \hat{\epsilon}\sigma f_j(t)\mathbf{e}_j(t)) \text{ in } C([0,T];\mathbf{L}^2(\Omega)) \text{ as } k\to+\infty,$$

and

$$\mathbf{G}_2^{(k)} = \sum_{j=1}^{N}(-\hat{\mu}f_j^{(k)}\text{curl}\,\mathbf{e}_j)$$

to

$$\sum_{j=1}^{N}(-\hat{\mu}f_j\text{curl}\,\mathbf{e}_j) \text{ in } C([0,T];\mathbf{J}_n(\Omega,\mu)) \text{ as } k\to+\infty.$$

We have thus that

$$\mathbf{G}^{(k)} \to \mathbf{G} \text{ in } C([0,T];\mathcal{H}) \text{ as } k\to+\infty. \tag{4.32}$$

Let us consider $(\mathbf{E}_{v^{(k)}}, \mathbf{H}_{v^{(k)}})$ solution of (4.7) with

$$\mathbf{E}_{ext}^{(k)}(x,t) = \sum_{j=1}^{N} f_j^{(k)}(t)\mathbf{e}_{ext,j}(x,t), \quad (x,t)\in\Sigma.$$

We recall that

$$(\mathbf{E}_{v^{(k)}}(t), \mathbf{H}_{v^{(k)}}(t)) = (\mathcal{E}_{v^{(k)}}(t) + \sum_{j=1}^{N} f_j^{(k)}(t)\mathbf{e}_j(\cdot,t), \mathbf{H}^{(k)}(t)), \tag{4.33}$$

with

$$(\mathcal{E}_{v^{(k)}}(t), \mathbf{H}_{v^{(k)}}(t)) = T_t\,(\mathcal{E}_0, \mathbf{H}_0) + \int_0^t T_{t-s}\mathbf{G}^{(k)}(s)ds, \tag{4.34}$$

for every $t\in[0,T]$. Passing to the limit as $k\to+\infty$ in (4.33) and (4.34), it is clear by (4.25), that

$$(\mathbf{E}_{v^{(k)}}, \mathbf{H}_{v^{(k)}}) \to (\mathbf{E},\mathbf{H}) \text{ in } C([0,T];\mathcal{H}) \text{ as } k\to+\infty.$$

Let us now denote by $y_{v^{(k)}}$, the weak solution in $W(0,T)\cap C(\bar{Q})$ of

$$\begin{cases} \partial_t y_{v^{(k)}} - \operatorname{div}(\alpha\,\nabla y_{v^{(k)}}) = S(y_{v^{(k)}}) & \text{in } Q,\\ -\alpha\frac{\partial y_{v^{(k)}}}{\partial n} = h(y_{v^{(k)}} - y_b) & \text{on } \Sigma,\\ y_{v^{(k)}}(\cdot,0) = y_0 & \text{in } \Omega, \end{cases} \tag{4.35}$$

with

$$S(y_{v^{(k)}})(x,t) \coloneqq \mu_a(x, y_{v^{(k)}}(x,t)) \mid (\mathbf{E}_{v^{(k)}} * \varphi_a)(x,t) \mid^2, \quad \text{for all } (x,t)\in Q. \tag{4.36}$$

We know by Theorem 78, that this weak solution exists and is unique and satisfies

$$\begin{aligned} \|y_{v^{(k)}}\|_{W(0,T)} + \|y_{v^{(k)}}\|_{C(\bar{Q})} \leq & C(r,s)\big(\big\|\mu_a(\cdot,0)|\mathbf{E}_{v^{(k)}} * \varphi_a|^2(\cdot,\cdot)\big\|_{L^r(Q)} \\ & + \|h(\cdot)y_b(\cdot,\cdot)\|_{L^s(\Sigma)} + \|y_0\|_{C(\bar{\Omega})}\big), \end{aligned} \tag{4.37}$$

for any $r > 5/2$ and any $s > 4$.

The sequence $(\mathbf{E}_{v^{(k)}})_{k\geq 0}$ is bounded in $C([0,T];\mathbf{L}^2(\Omega))$, so that $(\mathbf{E}_{v^{(k)}} * \varphi_a)_{k\geq 0}$ is bounded in $C([0,T];\mathbf{C}(\bar{\Omega})) = \mathbf{C}(\bar{Q})$. Therefore this latest result with estimate (4.37) show us, that $(y_{v^{(k)}})_{k\geq 0}$ is bounded in $W(0,T)$. Modulo extracting a new subsequence, we may prove that $(y_{v^{(k)}})_{k\geq 0}$ converges weakly in $W(0,T)$. By the compact embedding from $H^1(\Omega)$ into $L^p(\Omega)$ for $1 < p < 6$ and the Lions-Aubin compactness lemma [47, p.106], $(y_{v^{(k)}})_{k\geq 0}$ is strongly convergent in $L^2(Q)$.

Proposition 79. *The sequence $(y_{v^{(k)}})_{k\geq 0}$ converges weakly to y_v in $W(0,T)$.*

Proof. Let us call y the weak limit in $W(0,T)$ of $(y_{v^{(k)}})_{k\geq 0}$. $y_{v^{(k)}}$ is the weak solution of the parabolic initial boundary value problem:

$$\begin{cases} \frac{\partial y_{v^{(k)}}}{\partial t} - \operatorname{div}(\alpha\nabla y_{v^{(k)}}) - \mu_a(\cdot, y_{v^{(k)}})|\mathbf{E}_{v^{(k)}} * \varphi_a)|^2 = 0 & \text{in} \quad Q,\\ \alpha\frac{\partial y_{v^{(k)}}}{\partial n} + h y_{v^{(k)}} = h y_b & \text{on} \quad \Gamma,\\ y_{v^{(k)}}(\cdot,0) = y_0 & \text{in} \quad \Omega, \end{cases}$$

i.e. verifies [53, p.140]:
for every $\phi \in W_2^{1,1}(Q) := L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ such that $\phi(\cdot,T) = 0$

$$\begin{aligned} & -\iint_Q y_{v^{(k)}}(x,t)\tfrac{\partial\phi}{\partial t}(x,t)dx\;dt + \iint_Q \alpha(x)\nabla y_{v^{(k)}}(x,t)\cdot\nabla\phi(x,t)dx\;dt \\ & + \iint_\Sigma h(x) y_{v^{(k)}}(x,t)\phi(x,t)dS(x)\;dt = \iint_\Sigma h(x)y_b(x,t)\phi(x,t)dS(x)\;dt \\ & + \int_\Omega y_0(x)\phi(x,0)dx + \iint_Q \mu_a(x, y_{v^{(k)}}(x,t))|(\mathbf{E}_{v^{(k)}} * \varphi_a)(x,t)|^2\phi(x,t)dx\;dt. \end{aligned} \tag{4.38}$$

We want to pass to the limit in (4.38). We have seen previously, that $(y_{v^{(k)}})_{k\geq 0}$ converges a fortiori strongly to y in $L^2(Q)$. We know also that $\frac{\partial\phi}{\partial t} \in L^2(Q)$. Thus, the first term in the left-hand side of (4.38)

$$-\iint_Q y_{v^{(k)}}(x,t)\frac{\partial\phi}{\partial t}(x,t)dx\;dt \to -\iint_Q y(x,t)\frac{\partial\phi}{\partial t}(x,t)dx\;dt \text{ as } k\to+\infty.$$

As $(y_{v^{(k)}})_{k\geq 0}$ converges weakly to y in $W(0,T)$, a fortiori $(y_{v^{(k)}})_{k\geq 0}$ converges weakly to y in $L^2(0,T;H^1(\Omega))$ implying that $\nabla y_{v^{(k)}}$ converges weakly to ∇y in $L^2(Q)$. $\nabla\phi$ belongs also to $L^2(Q)$. Thus the second term in the left-hand side of (4.38)

$$\iint_Q \nabla y_{v^{(k)}}(x,t)\cdot\nabla\phi(x,t)dx\;dt \to \iint_Q \nabla y(x,t)\cdot\nabla\phi(x,t)dx\;dt \text{ as } k\to+\infty.$$

The weak convergence of $y_{v^{(k)}}$ to y in $L^2(0,T;H^1(\Omega))$, implies that the traces $y_{v^{(k)}}|_\Sigma$ converge to $y|_\Sigma$ in $L^2(\Sigma)$, so that the third term in the left-hand side of (4.38)

$$\iint_\Sigma h(x) y_{v^{(k)}}(x,t)\phi(x,t)dS(x)\ dt \to \iint_\Sigma h(x) y(x,t)\phi(x,t)dS(x)\ dt \text{ as } k \to +\infty.$$

In the right-hand side of (4.38),

$$\mathbf{E}_{v^{(k)}} * \varphi_a \to \mathbf{E}_v * \varphi_a \text{ in } C\left(\bar{Q}\right)^3 \text{ as } k \to +\infty,$$

and

$$\mu_a(x, y_{v^{(k)}}(x,t)) \to \mu_a(x, y(x,t)) \text{ as } k \to +\infty,$$

so that by the Lebesgue bounded convergence theorem:

$$\iint_Q \mu_a(x, y_{v^{(k)}}(x,t))|(\mathbf{E}_{v^{(k)}} * \varphi_a)(x,t)|^2\phi(x,t)dx\ dt \to$$
$$\iint_Q \mu_a(x, y(x,t))|(\mathbf{E}_v * \varphi_a)(x,t)|^2\phi(x,t)dx\ dt \text{ as } k \to +\infty.$$

So, we are allowed to pass to the limit in (4.38), and we obtain:

$$\begin{aligned} -\iint_Q y(x,t)\tfrac{\partial\phi}{\partial t}(x,t)dx\ dt + \iint_Q \nabla y(x,t)\cdot\nabla\phi(x,t)dx\ dt \\ +\iint_\Sigma h(x)y(x,t)\phi(x,t)dS(x)\ dt = \iint_\Sigma h(x)y_b(x,t)\phi(x,t)dS(x)\ dt \\ +\int_\Omega y_0(x)\phi(x,0)dx + \iint_Q \mu_a(x,y(x,t))|(\mathbf{E}_v * \varphi_a)(x,t)|^2\phi(x,t)dx\ dt, \end{aligned}$$

which means that y is the weak solution in $W(0,T)$ of

$$\begin{cases} \frac{\partial y}{\partial t} - \operatorname{div}(\alpha\nabla y) - \mu_a(\cdot,y)|(\mathbf{E}_v * \varphi_a)|^2 = 0 & \text{in} \quad Q, \\ \alpha\frac{\partial y}{\partial n} + hy = hy_b & \text{on} \quad \Sigma, \\ y(\cdot,0) = y_0(\cdot) & \text{on} \quad \Omega. \end{cases}$$

Therefore $y = y_v$. □

Corollary 80. $\hat{J}(v) = \inf_{w\in\mathcal{U}_{ad}} \hat{J}(w)$.

Proof. We know that $(y_{v^{(k)}})_{k\geq 0}$ converges weakly to y_v in $W(0,T)$. This implies by the Lions-Aubin compactness lemma [47, p.106],

$$y_{v^{(k)}} - y_Q \to y_v - y_Q \text{ in } L^2(Q) \text{ as } k \to +\infty.$$

This implies the strong convergence of

$$(y_{v^{(k)}} - y_Q)\,.\mathbf{1}_{[T_{j,1},T_{j,2}]\times\Omega_j} \to (y_v - y_Q)\,.\mathbf{1}_{[T_{j,1},T_{j,2}]\times\Omega_j} \text{ in } L^2([T_{j,1},T_{j,2}]\times\Omega_j)$$

as $k \to +\infty$. Thus

$$\|y_v - y_Q\|^2_{L^2([T_{j,1},T_{j,2}]\times\Omega_j)} = \lim_{k\to+\infty} \|y_{v^{(k)}} - y_Q\|^2_{L^2([T_{j,1},T_{j,2}]\times\Omega_j)}\,. \tag{4.39}$$

$y_v \in W(0,T) \mapsto \nabla y_v \in L^2(0,T;\mathbf{L}^2(\Omega))$ being a linear continuous mapping, it follows that $(\nabla y_{v^{(k)}})_{k\geq 0}$ converges weakly to ∇y_v in $L^2(0,T;\mathbf{L}^2(\Omega))$. The function $\|\cdot\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}$ is convex and continuous [5, p.71] and thus weakly lower semi-continuous [53, p.47]. Thus

$$\|\nabla y_v\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} \leq \liminf_{k\to+\infty} \|\nabla y_{v^{(k)}}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}\,. \tag{4.40}$$

By the linear continuous injection from $W(0,T)$ into $C([0,T];L^2(\Omega))$ follows that $(y_{v^{(k)}})_{k\geq 0}$ converges also weakly to y_v in $C([0,T];L^2(\Omega))$. Considering the linear continuous mapping

$$y \in C([0,T];L^2(\Omega)) \mapsto y(\cdot,T) \in L^2(\Omega),$$

it follows that $y_{v^{(k)}}(\cdot,T)$ converges also weakly to $y_v(\cdot,T)$ in $L^2(\Omega)$. Thus $y_{v^{(k)}}(\cdot,T) - y_d$ converges also weakly to $y_v(\cdot,T) - y_d$ in $L^2(\Omega)$. Moreover, the function $\|\cdot\|^2_{L^2(\Omega)}$ is convex and continuous [5, p.71] and thus weakly lower semi-continuous [53, p.47], so that

$$\|y_v(\cdot,T) - y_d\|^2_{L^2(\Omega)} \leq \liminf_{k\to+\infty} \|y_{v^{(k)}}(\cdot,T) - y_d\|^2_{L^2(\Omega)}\,. \tag{4.41}$$

Concerning the last term in the definition of the cost functional (4.4) , $\left(v^{(k)}\right)_{k\geq 0}$ converges weakly to $v \in L^2([0,T])^N$ and the function $\|\cdot\|^2_{L^2([0,T])^N}$ is convex and continuous [5, p.71] and thus weakly lower semi-continuous [53, p.47]. Thus, we have also:

$$\|v\|^2_{L^2([0,T])^N} \leq \liminf_{k\to+\infty} \left\|v^{(k)}\right\|^2_{L^2([0,T])^N}\,. \tag{4.42}$$

In conclusion

$$\begin{aligned}
&\hat{J}(y_v) \leq \liminf_{k\to+\infty} \frac{1}{2}\int_0^T \int_\Omega \mid \nabla y_{v^{(k)}}(x,t) \mid^2 dx\, dt + \sum_{j=1}^{M} \frac{\lambda_{j,Q}}{2} \liminf_{k\to+\infty} \int_{T_{j,1}}^{T_{j,2}} \int_{\Omega_j} \mid y_{v^{(k)}}(x,t) \\
&- y_{j,Q}(x,t) \mid^2 dx\, dt + \frac{\lambda_\Omega}{2} \liminf_{k\to+\infty} \|y_{v^{(k)}}(\cdot,T) - y_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2} \liminf_{k\to+\infty} \left\|v^{(k)}\right\|^2_{L^2([0,T])^N} \\
&\leq \liminf_{k\to+\infty} [\frac{1}{2}\int_0^T \int_\Omega \mid \nabla y_{v^{(k)}}(x,t) \mid^2 dx\, dt + \sum_{j=1}^{M} \frac{\lambda_{j,Q}}{2} \int_{T_{j,1}}^{T_{j,2}} \int_{\Omega_j} \mid y_{v^{(k)}}(x,t) \\
&- y_{j,Q}(x,t) \mid^2 dx\, dt + \frac{\lambda_\Omega}{2} \|y_{v^{(k)}}(\cdot,T) - y_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2} \left\|v^{(k)}\right\|^2_{L^2([0,T])^N}] \\
&= \liminf_{k\to+\infty} \hat{J}(y_{v^{(k)}}) = \inf_{w\in U_{ad}} \hat{J}(w).
\end{aligned} \tag{4.43}$$

We have thus proved, the existence of at least one optimal control. □

4.4 Differentiability of the control to space mapping

Our purpose now is to prove that the reduced cost functional $\hat{J}$ is Fréchet differentiable. The proof reduces essentially to prove that the control-to-state mapping is Fréchet differentiable.

Theorem 81. *The mapping*

$$\mathbf{E} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \mapsto y \in W(0,T) \cap C(\bar{Q}),$$

y, weak solution of the semilinear initial parabolic boundary value problem (4.5)- (4.6), is Fréchet differentiable at any point $\mathbf{E} \in L^\infty(0,T;\mathbf{L}^2(\Omega))$. Its Fréchet derivative at point $\mathbf{E}$ is the linear continuous mapping

$$\delta\mathbf{E} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \mapsto \delta y \in W(0,T) \cap C(\bar{Q}),$$

δy, solution of the linear initial parabolic boundary value problem

$$\left\{\begin{array}{ll} \frac{\partial \delta y}{\partial t}(x,t) - \mathrm{div}(\alpha \nabla \delta y)(x,t) - \frac{\partial \mu_a}{\partial y}(x,y(x,t))|\mathbf{E}*\varphi_a|^2_{\mathbb{R}^3}(x,t)\delta y(x,t) & \\ \qquad = 2\mu_a(x,y(x,t))\left(\mathbf{E}*\varphi_a\right)(x,t)\cdot\left(\delta\mathbf{E}*\varphi_a\right)(x,t), & \text{in} \quad Q, \\ \alpha\frac{\partial \delta y}{\partial n}(x,t) + h\delta y(x,t) = 0, & \text{on} \quad \Sigma, \\ \delta y(\cdot,0) = 0, & \text{in} \quad \Omega. \end{array}\right. \tag{4.44}$$

The proof of Theorem 81 will follow from Lemma 82, and Proposition 85 presented below.

Lemma 82. *The mapping*

$$\mathbf{E} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \mapsto |\mathbf{E}*\varphi_a|^2_{\mathbb{R}^3} \in \mathbf{L}^\infty(Q)$$

is continuously Fréchet differentiable and its Fréchet derivative at the point $\mathbf{E} \in L^\infty(0,T;\mathbf{L}^2(\Omega))$ is the linear continuous mapping

$$\delta\mathbf{E} \mapsto 2\left(\mathbf{E}*\varphi_a\right)\cdot\left(\delta\mathbf{E}*\varphi_a\right).$$

Proof. • The mapping

$$\mathbf{E} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \mapsto \mathbf{E}*\varphi_a(x,t) = \int_\Omega \mathbf{E}(y,t)\varphi_a(x-y)dy \in \mathbf{L}^\infty(Q)$$

is linear, and continuous due to the estimate:

$$\begin{aligned} |(\mathbf{E}*\varphi_a)(x,t)| &\leq \int_\Omega |\mathbf{E}(y,t)\varphi_a(x-y)|dy \\ &\leq \left(\int_\Omega |\mathbf{E}(y,t)|^2 dy\right)^{1/2}\left(\int_\Omega |\varphi_a(x-y)|^2 dy\right)^{1/2} \\ &\leq \|\mathbf{E}\|_{L^\infty(0,T;L^2(\Omega)^3)}\|\varphi_a\|_{L^2(\mathbb{R})^3}, \end{aligned}$$

which implies

$$\|\mathbf{E}*\varphi_a\|_{\mathbf{L}^\infty(Q)} \leq \|\varphi_a\|_{L^2(\mathbb{R})}\|\mathbf{E}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}.$$

Its Fréchet derivative at any point $\mathbf{E} \in L^\infty(0,T;\mathbf{L}^2(\Omega))$ is the linear continuous mapping

$$\delta\mathbf{E} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \mapsto \delta\mathbf{E}*\varphi_a \in \mathbf{L}^\infty(Q).$$

- The mapping
$$\mathbf{H} \in \mathbf{L}^\infty(Q) \mapsto (\mathbf{H}, \mathbf{H}) \in \mathbf{L}^\infty(Q) \times \mathbf{L}^\infty(Q)$$
is linear and continuous. Therefore its Fréchet derivative at any point $\mathbf{H} \in \mathbf{L}^\infty(Q)$ is itself.

- The mapping
$$(\mathbf{F}, \mathbf{G}) \in \mathbf{L}^\infty(Q) \times \mathbf{L}^\infty(Q) \mapsto \mathbf{F} \cdot \mathbf{G} \in L^\infty(Q)$$
is bilinear and continuous. Thus it is continuously Fréchet differentiable and its Fréchet derivative at the point $(\mathbf{F}, \mathbf{G})$ is the linear continuous mapping
$$(\delta\mathbf{F}, \delta\mathbf{G}) \in \mathbf{L}^\infty(Q) \times \mathbf{L}^\infty(Q) \mapsto \mathbf{F} \cdot \delta\mathbf{G} + \delta\mathbf{F} \cdot \mathbf{G} \in L^\infty(Q).$$

The result follows by the chain rule for the Fréchet derivative of a composite mapping. □

We consider now the mapping which sends $f \in L^\infty_+(Q)$ onto the solution of the semilinear parabolic initial boundary value problem

$$\begin{cases} \partial_t y - \operatorname{div}(\alpha \nabla y) - f\mu_a(.,y) = 0 & \text{in } Q := \Omega\times]0,T[, \\ \alpha\frac{\partial y}{\partial n} + hy = hy_b & \text{on } \Sigma := \Gamma\times]0,T[, \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases} \tag{4.45}$$

(in view of (4.5)-(4.6), the particular case is when $f = |\mathbf{E} * \varphi_a|^2_{\mathbb{R}^3}$). $L^\infty_+(Q)$ is defined by

$$L^\infty_+(Q) := \{u \in L^\infty(Q) \text{ such that } u \geq 0 \text{ a.e. in } Q\}.$$

Firstly, we must prove that problem (4.45) possesses one and only one weak solution. This time, $d(x,t,y) = -f(x,t)\mu_a(x,y)$. As $f \geq 0$, and $\mu_a(x,y)$ is decreasing as y is increasing, $d(x,t,y)$ is increasing as y is increasing. Thus $d(x,t,y)$ is monotically increasing with y. Also

$$|d(x,t,0)| = |f(x,t)|\mu_a(x,0)$$

is bounded as f and μ_a are bounded. Moreover as

$$|d(x,t,y_1) - d(x,t,y_2)| = |f(x,t)||\mu_a(x,y_1) - \mu_a(x,y_2)| \leq \| f \|_\infty \left\| \frac{\partial \mu_a}{\partial y} \right\|_\infty |y_1 - y_2|,$$

d is uniformly lipschitzian in y. We are thus allowed to apply [53, Theorem 5.5 p.268], so that, consequently:

Proposition 83. *The semi-linear parabolic initial boundary value problem (4.45) has a unique weak solution $y \in W(0,T) \cap C(\bar{Q})$ for any $f \in L^\infty_+(Q)$, any $y_b \in C(\bar{\Sigma})$ and any initial condition $y_0 \in C(\bar{\Omega})$. Moreover there exists a constant $C > 0$ such that*

$$\begin{aligned} \|y\|_{W(0,T)} + \|y\|_{C(\bar{Q})} \leq C(&\|f\mu_a(\cdot,0)\|_{L^\infty(Q)} \\ &+ \|h(\cdot)y_b(\cdot,\cdot)\|_{L^\infty(\Sigma)} + \|y_0\|_{C(\bar{\Omega})}). \end{aligned} \tag{4.46}$$

In fact, this proposition remains true for any $f \in L^\infty(Q)$. Let us multiply both sides of equation $(4.45)_{(i)}$: $\partial_t y - \operatorname{div}(\alpha \nabla y) - f\mu_a(.,y) = 0$ by $e^{\lambda t}$. We obtain:

$$\partial_t(e^{\lambda t}y) - \operatorname{div}(\alpha \nabla(e^{\lambda t}y)) - e^{\lambda t}(f\mu_a(.,e^{-\lambda t}(e^{\lambda t}y)) + \lambda y) = 0. \tag{4.47}$$

Considering as new unknown $y := e^{\lambda t}y$, this semilinear parabolic equation [53, (5.1)p.265] is of the form $(4.27)_{(i)}$ with

$$d(x,t,y) = -e^{\lambda t}(f(x,t)\mu_a(x,e^{-\lambda t}y) + \lambda e^{-\lambda t}y).$$

Now

$$\frac{\partial d}{\partial y}(x,t,y) = -e^{\lambda t}(f(x,t)e^{-\lambda t}\frac{\partial \mu_a}{\partial y}(x,e^{-\lambda t}y) + \lambda e^{-\lambda t}).$$

Let us consider $\lambda \leq 0$, then

$$\frac{\partial d}{\partial y}(x,t,y) = |\lambda| - f(x,t)\frac{\partial \mu_a}{\partial y}(x,e^{-\lambda t}y) \geq |\lambda| - \| f \|_{\infty,Q} \| \frac{\partial \mu_a}{\partial y} \|_{\infty,\bar{\Omega}\times\mathbb{R}}$$

which will be positive if we choose λ "sufficiently negative ". In that case d is monotonically increasing with y, so that [53, Assumption 5.1, p.266] is verified. We have

$$|d(x,t,0)| = e^{\lambda t}|f(x,t)|\mu_a(x,0) \leq \|f\mu_a(\cdot,0)\|_{L^\infty(Q)} \leq \| f \|_{\infty,Q} \| \mu_a(\cdot,0) \|_{\infty,\bar{\Omega}} .$$

Moreover

$$|d(x,t,y_1) - d(x,t,y_2)| \leq (\| f \|_{\infty,Q} \| \frac{\partial \mu_a}{\partial y} \|_{\infty,\bar{\Omega}\times\mathbb{R}} + |\lambda|) \, |y_1 - y_2|$$

so that the mapping $y \mapsto d(x,t,y)$ is uniformly lipschitzian in y. Thus [53, Assumption 5.2, p.266] is also verified. Consequently:

Proposition 84. *The previous proposition remains true for $f \in L^\infty(Q)$.*

Proof. For $|\lambda| > \| f \|_{\infty,Q} \| \frac{\partial \mu_a}{\partial y} \|_{\infty,\bar{\Omega}\times\mathbb{R}}$, we have by [53, inequality (5.6) p.268]:

$$\begin{aligned} \left\|e^{\lambda t}y\right\|_{W(0,T)} + \left\|e^{\lambda t}y\right\|_{C(\bar{Q})} \leq C(r,s)(\left\|e^{\lambda t}f\mu_a(.,0)\right\|_{L^\infty(Q)} \\ + \left\|h(\cdot)e^{\lambda t}y_b(\cdot,\cdot)\right\|_{L^\infty(\Sigma)} + \|y_0\|_{C(\bar{\Omega})}). \end{aligned} \tag{4.48}$$

λ being fixed, we have

$$\begin{aligned} \left\|e^{\lambda t}y\right\|_{W(0,T)} &\approx \left\|e^{\lambda t}y\right\|_{L^2(0,T;H^1(\Omega))} + \left\|\lambda e^{\lambda t}y + e^{\lambda t}\tfrac{dy}{dt}\right\|_{L^2(0,T;H^1(\Omega)^*)} \\ &\approx \|y\|_{L^2(0,T;H^1(\Omega))} + \left\|\tfrac{dy}{dt}\right\|_{L^2(0,T;H^1(\Omega)^*)} \approx \|y\|_{W(0,T)} , \end{aligned}$$

and thus inequality (4.48) implies inequality (4.46). □

Let us now increment f in equation $(4.45)_{(i)}$ by $\delta f \in L^\infty(Q)$, $f + \delta f$ still belongs to $L^\infty(Q)$, so that by the previous proposition, the initial boundary value problem (4.45) with f replaced by $f+\delta f$ still possesses one and only one solution $y+\delta y$ in $W(0,T)\cap C(\bar{Q})$. The difference $\delta y = (y + \delta y) - y$ is solution of the semilinear parabolic initial boundary value problem

$$\begin{cases} \partial_t \delta y - \operatorname{div}(\alpha \nabla \delta y) - f.(\mu_a(.,y+\delta y) - \mu_a(.,y)) - \mu_a(.,y+\delta y)\delta f = 0 & \text{in } Q, \\ \alpha \frac{\partial \delta y}{\partial n} + h\delta y = 0 & \text{on } \Sigma, \\ \delta y(\cdot,0) = 0 & \text{in } \Omega. \end{cases} \tag{4.49}$$

(4.49) may be rewritten:

$$\begin{cases} \partial_t \delta y - \operatorname{div}(\alpha \nabla \delta y) - f.(\mu_a(.,y+\delta y) - \mu_a(.,y)) = \mu_a(.,y+\delta y)\delta f & \text{in } Q, \\ \alpha \frac{\partial \delta y}{\partial n} + h\delta y = 0 & \text{on } \Sigma, \\ \delta y(\cdot,0) = 0 & \text{in } \Omega. \end{cases} \tag{4.50}$$

Applying[53, inequality (5.6) p.268], we have:

$$\| \delta y \| \lesssim \| \delta f \| . \tag{4.51}$$

Let us introduce dy the weak solution of the linear parabolic initial boundary value problem:

$$\begin{cases} \partial_t dy - \operatorname{div}(\alpha \nabla dy) - f\ \frac{\partial \mu_a}{\partial y}(.,y) dy = \mu_a(.,y)\delta f & \text{in } Q, \\ \alpha \frac{\partial dy}{\partial n} + h\ dy = 0 & \text{on } \Sigma, \\ dy(\cdot,0) = 0 & \text{in } \Omega. \end{cases} \tag{4.52}$$

Subtracting (4.52) from (4.49), we obtain:

$$\begin{cases} \partial_t(\delta y - dy) - \operatorname{div}(\alpha \nabla(\delta y - dy)) - f\ \frac{\partial \mu_a}{\partial y}(.,y)(\delta y - dy) = h & \text{in } Q, \\ \alpha \frac{\partial(\delta y - dy)}{\partial n} + h(\delta y - dy) = 0 & \text{on } \Sigma, \\ (\delta y - dy)(\cdot,0) = 0 & \text{in } \Omega. \end{cases} \tag{4.53}$$

where $h := f.(\mu_a(.,y+\delta y) - \mu_a(.,y) - \frac{\partial \mu_a}{\partial y}(.,y)(\delta y)) + (\mu_a(.,y+\delta y) - \mu_a(.,y))\delta f$.

$$\begin{aligned} h &= f\ (\mu_a(.,y+\delta y) - \mu_a(.,y) - \tfrac{\partial \mu_a}{\partial y}(.,y)\delta y) + (\mu_a(.,y+\delta y) - \mu_a(.,y))\delta f \\ &= f \int_y^{y+\delta y} (y+\delta y - v) \tfrac{\partial^2 \mu_a}{\partial v^2}(.,v) dv + \int_y^{y+\delta y} \tfrac{\partial \mu_a}{\partial v}(.,v) dv\ \delta f \\ &= f \int_0^{\delta y} (\delta y - w) \tfrac{\partial^2 \mu_a}{\partial v^2}(.,y+w) dw + \int_0^{\delta y} \tfrac{\partial \mu_a}{\partial v}(.,y+w) dw\ \delta f\ . \end{aligned}$$

Assuming that the absorption coefficient $\mu_a \in C_b^2(\Omega \times \mathbb{R})$ like in [3, hypothesis (H7)], it follows that $\| h \|_\infty \lesssim (\| f \|_\infty \| \delta y \|_\infty^2 + \| \delta y \|_\infty \| \delta f \|_\infty)$. By inequality (4.51), now follows: $\| h \|_\infty \lesssim \| \delta f \|_\infty^2$, so that by the linear parabolic initial boundary value problem (4.53) (the coefficient of the 0-order term of equation $(4.53)_{(i)}$: $-f\ \frac{\partial \mu_a}{\partial y}(.,y)$ is nonnegative):

$$\| \delta y - dy \|_{W(0,T)\cap C(\bar{Q})} \lesssim \| \delta f \|_\infty^2 . \tag{4.54}$$

We have thus proved:

Proposition 85. *The mapping which sends $f \in L^\infty(Q)$ to $y \in W(0,T) \cap C(\bar{Q})$ the unique weak solution of the semilinear parabolic initial boundary value problem (4.45) is Fréchet differentiable and its Fréchet derivative at point $y \in W(0,T) \cap C(\bar{Q})$ is given by the linear continuous mapping from $L^\infty(Q)$ to $W(0,T) \cap C(\bar{Q})$ which sends $\delta f \in L^\infty(Q)$ to $\delta y \in W(0,T) \cap C(\bar{Q})$ solution of the linear parabolic initial boundary value problem (4.52).*

Proof. (of Theorem 81) The proof follows from Lemma 82, and Proposition 85 by the chain's rule formula for the derivation of a composite function. □

We now consider the mapping

$$v = (v_j)_{j=1}^N \in L^2(0,T)^N \mapsto \mathbf{E} \in L^2(0,T;\mathbf{L}^2(\Omega)).$$

Proposition 86. *We suppose (4.11). Then, the mapping which sends* $v = (v_j)_{j=1}^N \in L^2(0,T)^N$ *onto* $(\mathbf{E},\mathbf{H}) \in W^{1,1}(0,T;\mathcal{H})$ *the strong solution of (4.7) with*

$$\mathbf{E}_{ext} = \sum_{j=1}^{N} f_j \mathbf{e}_{ext,j} \times n,$$

the f_j *solutions of (4.8)* $(j = 1,\dots,N)$*, is affine and continuous from* $L^2(0,T)^N$ *into* $C([0,T];\mathcal{H})$*. Its Fréchet derivative at any point* $v = (v_j)_{j=1}^N \in L^2(0,T)^N$ *is the linear part of this mapping i.e. the linear continuous mapping from* $L^2(0,T)^N$ *into* $C([0,T];\mathcal{H})$ *which sends* $\delta v = (\delta v_j)_{j=1}^N$ *onto* $(\delta\mathbf{E},\delta\mathbf{H}) \in W^{1,1}(0,T;\mathcal{H}) \hookrightarrow C([0,T];\mathcal{H})$ *the strong solution of*

$$\begin{cases} \partial_t(\epsilon\delta\mathbf{E}) - curl\,\delta\mathbf{H} + \sigma\delta\mathbf{E} = 0 & in\ Q, \\ \partial_t\delta\mathbf{H} + \hat{\mu}\ curl\,\delta\mathbf{E} = 0 & in\ Q, \\ \delta\mathbf{E}\times n = \displaystyle\sum_{j=1}^{N} \delta f_j \mathbf{e}_{ext,j}\times n & on\ \Sigma := \Gamma\times]0,T[, \\ \delta\mathbf{H}\cdot n = 0 & on\ \Sigma, \\ \delta\mathbf{E}(\cdot,0) = 0,\ \delta\mathbf{H}(\cdot,0) = 0 & in\ \Omega. \end{cases} \tag{4.55}$$

with the δf_j $(j = 1,\dots,N)$ *solution of the two-point homogeneous boundary value problems:*

$$\begin{cases} -\delta f_j{}'' + \delta f_j = \delta v_j \quad in \quad]0,T[, \\ \delta f_j(0) = 0\,, \\ \delta f_j{}'(0) = 0\,. \end{cases} \tag{4.56}$$

Proof. By formulas (4.8) and (4.21), the mapping which sends $v = (v_j)_{j=1}^N$ onto $\mathbf{G}$ from $L^2([0,T])^N$ into $C([0,T];\mathcal{H})$ is affine; it is also continuous by (4.32). By formula (4.25), the mapping from $C([0,T];\mathcal{H})$ into $C([0,T];\mathcal{H})$ which sends $\mathbf{G}$ onto $(\mathcal{E},\mathbf{H})$ is also affine and continuous. By formula (4.24), the mapping from $C([0,T];\mathcal{H})$ into $C([0,T];\mathcal{H})$ which sends $\mathbf{G}$ onto $(\mathbf{E},\mathbf{H})$ is also affine and continuous. So, the composite function which sends $v = (v_j)_{j=1}^N$ onto $(\mathbf{E},\mathbf{H})$ from $L^2([0,T])^N$ into $C([0,T];\mathcal{H})$ is affine and continuous. □

Corollary 87. *The control-to-state mapping*

$$\mathcal{S} : L^2(0,T)^N \to W(0,T)\cap C(\bar{Q}) \tag{4.57}$$

which sends $v = (v_j)_{j=1}^N \in L^2(0,T)^N$ *onto* y *weak solution of the semilinear initial parabolic boundary value problem (4.5)-(4.6), is Fréchet differentiable at any point* $v = (v_j)_{j=1}^N \in L^2(0,T)^N$*. Its Fréchet derivative at the point* $v = (v_j)_{j=1}^N \in L^2(0,T)^N$ *is the linear continuous mapping from the space* $L^2(0,T)^N$ *into* $W(0,T)\cap C(\bar{Q})$ *which sends* $\delta v = (\delta v_j)_{j=1}^N \in L^2(0,T)^N$ *onto* $\delta y \in W(0,T)\cap C(\bar{Q})$ *solution of the linear initial boundary value problem (4.44) with* $(\delta\mathbf{E},\delta\mathbf{H}) \in W^{1,1}(0,T;\mathcal{H})$ *strong solution of (4.55)-(4.56).*

Proof. The proof follows from Proposition 86 and Theorem 81 by the chain's rule formula for the derivation of a composite function. □

Let us recall the type of cost functional that we consider in this paper (4.4):

$$\begin{aligned} J(y,v) := \tfrac{1}{2}\int_0^T \int_\Omega |\ \nabla y(x,t)\ |^2\ dx\ dt + \sum_{j=1}^{M} \frac{\lambda_{j,Q}}{2} \int_{T_{j,1}}^{T_{j,2}} \int_{\Omega_j} |\ y(x,t) - y_{j,Q}(x,t)\ |^2\ dx\ dt \\ + \tfrac{\lambda_\Omega}{2}\int_\Omega |\ y(x,T) - y_d(x)\ |^2\ dx + \tfrac{\lambda}{2}\sum_{k=1}^{N}\int_0^T |v_k(t)|^2 dt, \end{aligned} \tag{4.58}$$

with $\Omega_j \subset \Omega$, $T_{j,1} \leq T_{j,2}$, $[T_{j,1}, T_{j,2}] \subset [0,T]$, $\lambda_{j,Q} \geq 0$ $(j = 1, \ldots, M)$.

We now consider its corresponding reduce cost functional:

$$\hat{J}(v) = J(y_v, v) = J(\mathcal{S}(v), v), \tag{4.59}$$

where $\mathcal{S}$ denotes the control-to-state mapping (4.57). Let us compute its Fréchet derivative at the given point $v \in L^2(0,T)^N$:

$$D\hat{J}(v)\delta v = J_y(y,v)D\mathcal{S}(v)\delta v + J_v(y,v)\delta v. \tag{4.60}$$

J being given by formula (4.58), we have:

$$\begin{aligned} D\hat{J}(v)\delta v = &\int_0^T \int_\Omega \nabla y(x,t) \cdot \nabla (D\mathcal{S}(v)\delta v)(x,t)\ dxdt \\ &+ \sum_{j=1}^M \lambda_{j,Q} \int_{T_{j,1}}^{T_{j,2}} \int_{\Omega_j} (y(x,t) - y_{j,Q}(x,t))(D\mathcal{S}(v)\delta v)(x,t)\ dxdt \\ &+ \lambda_\Omega \int_\Omega (y(x,T) - y_d(x))(D\mathcal{S}(v)\delta v)(x,T)\ dx + \lambda \sum_{k=1}^N \int_0^T v_k(t)\delta v_k(t) dt. \end{aligned} \tag{4.61}$$

4.5 First order necessary optimality condition

Let us now introduce the adjoint problem associated to (4.5) :

$$\begin{cases} \frac{\partial p}{\partial t} + \operatorname{div}(\alpha \nabla p) + \frac{\partial \mu_a}{\partial y}(x,y)|E * \varphi_a|^2 p = \Delta y - \sum_{j=1}^M \lambda_{j,Q}(y - y_{j,Q}) 1_{\Omega_j \times]T_{j,1}, T_{j,2}[} & \text{in} \quad Q, \\ \alpha \frac{\partial p}{\partial n} + hp = \frac{\partial y}{\partial n} & \text{on} \quad \Sigma, \\ p(\cdot, T) = \lambda_\Omega (y(.,T) - y_d) & \text{on} \quad \Omega, \end{cases} \tag{4.62}$$

By a weak solution $p \in W(0,T)$ of the backward parabolic boundary value problem, (4.62), we mean that for every $\varphi \in L^2(0,T; H^1(\Omega))$:

$$\begin{aligned} &\int_0^T \int_\Omega \frac{\partial p}{\partial t}(x,t)\varphi(x,t) dxdt - \int_0^T \int_\Omega \alpha(x) \nabla p(x,t) \nabla \varphi(x,t) dxdt \\ &\quad + \int_0^T \int_\Omega \frac{\partial \mu_a}{\partial y}(x, y(x,t))|\mathbf{E} * \varphi_a|^2(x,t) p(x,t) \varphi(x,t) dxdt \\ &= -\int_0^T \int_\Omega \nabla y(x,t) \nabla \varphi(x,t) dxdt - \sum_{j=1}^M \lambda_{j,Q} \int_{T_{j,1}}^{T_{j,2}} \int_{\Omega_j} (y - y_{j,Q})(x,t) \varphi(x,t) dxdt \\ &\quad + \int_0^T \int_\Gamma h(x) p(x,t) \varphi(x,t) dS(x) dt, \end{aligned} \tag{4.63}$$

and

$$p(x,T) = \lambda_\Omega (y(x,T) - y_d(x)), \text{ for a.e. } x \in \Omega. \tag{4.64}$$

Proposition 88.

$$\begin{aligned} D\hat{J}(v)\delta v = &2 \int_0^T \int_\Omega \mu_a(x, y(x,t)) p(x,t) (E * \varphi_a)(x,t) \cdot (\delta E * \varphi_a)(x,t)\ dxdt \\ &+ \lambda \sum_{k=1}^N \int_0^T v_k(t) \delta v_k(t) dt, \end{aligned} \tag{4.65}$$

δE being deduced from δv by solving the linear problem (4.55)-(4.56) independent of v.

Proof. By formulas (4.61), (4.63), and $\mathcal{S}'(v)\delta v = dy$ (4.52) with

$$f : (x,t) \mapsto |\mathbf{E} * \varphi_a|^2(x,t),$$

we get:

$$\begin{aligned} D\hat{J}(v)\delta v = &\int_0^T \int_\Gamma h(x)p(x,t)dy(x,t)\ dS(x)dt - \int_0^T \int_\Omega \frac{\partial p}{\partial t}(x,t)dy(x,t)\ dxdt \\ &+ \int_0^T \int_\Omega \alpha(x)\nabla p(x,t)\nabla dy(x,t)dxdt - \int_0^T \int_\Omega \frac{\partial \mu_a}{\partial y}(x,y(x,t))|\mathbf{E} * \varphi_a|^2(x,t)p(x,t)dy(x,t)dxdt \\ &+ \lambda_\Omega \int_\Omega (y(x,T) - y_d(x))dy(x,T)dx + \lambda \sum_{k=1}^N \int_0^T v_k(t)\delta v_k(t)dt. \end{aligned} \tag{4.66}$$

Let us note that the formel integral $-\int_0^T \int_\Omega \frac{\partial p}{\partial t}(x,t)dy(x,t)\ dxdt$ must be understood in the sense $-\int_0^T < \frac{\partial p}{\partial t}(\cdot,t), dy(\cdot,t) >_{(H^1(\Omega))^*,H^1(\Omega)}\ dt$. By the formula of integration by parts in $W(0,T)$ [53, p.148], $dy(\cdot,0) = 0$ and (4.64), we have:

$$\begin{aligned} -\int_0^T < \frac{\partial p}{\partial t}(\cdot,t), dy(\cdot,t) >_{H^1(\Omega)^*,H^1(\Omega)}\ dt = &\int_0^T < \frac{\partial dy}{\partial t}(\cdot,t), p(\cdot,t) >_{H^1(\Omega)^*,H^1(\Omega)}\ dt \\ &- \lambda_\Omega \int_\Omega (y(x,T) - y_d(x))dy(x,T)dx. \end{aligned} \tag{4.67}$$

As dy is the weak solution of the linear parabolic initial boundary value problem (4.52), we have in particular that:

$$\begin{aligned} \int_0^T < \tfrac{\partial dy}{\partial t}(\cdot,t), p(\cdot,t) >_{H^1(\Omega)^*,H^1(\Omega)}\ dt = &-\int_0^T \int_\Omega \alpha(x)\nabla p(x,t)\nabla dy(x,t)dxdt \\ &+ \int_0^T \int_\Omega \tfrac{\partial \mu_a}{\partial y}(x,y(x,t))|\mathbf{E} * \varphi_a|^2(x,t)p(x,t)dy(x,t)dxdt \\ + \int_0^T \int_\Omega \mu_a(x,y(x,t))\delta f(x,t)p(x,t)dxdt - &\int_0^T \int_\Gamma h(x)dy(x,t)p(x,t)dS(x)dt. \end{aligned} \tag{4.68}$$

From (4.66), (4.67), (4.68) and $\delta f(x,t) = 2\ (\mathbf{E} * \varphi_a) \cdot (\delta \mathbf{E} * \varphi_a)$, we get □

Corollary 89. *(first order necessary condition) If $\bar{v} \in \mathcal{U}_{ad}$ is an optimal control, then*

$$\begin{aligned} 2\int_0^T \int_\Omega \mu_a(x,\bar{y}(x,t))\bar{p}(x,t)(\bar{E} * \varphi_a)(x,t) \cdot (\delta E * \varphi_a)(x,t)\ dxdt \\ + \lambda \sum_{k=1}^N \int_0^T \bar{v}_k(t)\delta v_k(t)dt \geq 0, \end{aligned} \tag{4.69}$$

for all $\delta v = v - \bar{v}$, $v \in \mathcal{U}_{ad}$, δE being deduced from δv by solving (4.55)-(4.56).

4.6 Discussion and outlook

This chapter deals with an optimal control problem related to the coupled heat-Maxwell system studied in chapter 3. Here, we have considered the permittivity independent of the temperature to simplify matters. We have also considered in this chapter a more general type of cost functional, more flexible for concrete applications, the choice of y_Q being in practice not obvious. The perspectives we want to consider for this work are the following:

- Our immediate priority will be to improve the variational inequality (4.69). In fact the variational inequality we derived depend on $\delta\mathbf{E}$ which means that for every δv we have to solve (4.55)-(4.56). We would like to find a variational inequality independant of $\delta\mathbf{E}$.
- Furthermore, we would like to consider the same optimal control problem presented in this chapter but with the heat-Maxwell system studied in chapter 3 i.e with a temperature dependent permittivity.

Appendix A

Mathematical study of the nonlinear thermal model in the selective laser melting process

We want to prove existence of a solution to problem (2.2). For simplicity, we consider the following initial boundary value problem.

$$\begin{cases} c(y)\,\partial_t y - div(\kappa(y)\,\nabla y) = 0 & \text{in} \quad Q, \\ -\kappa(y)\frac{\partial y}{\partial \nu} = h(y)\,(y - y_{ex}) + \varepsilon(y)\,\sigma_{SB}\,(y^4 - y_{ex}^4) - \alpha(y) I(.,.) & \text{on} \quad \Sigma_1 \\ \frac{\partial y}{\partial \nu} = 0 & \text{on} \quad \Sigma_2, \\ y(x,0) = y_0(x) & \text{in} \quad \Omega. \end{cases} \tag{A.1}$$

We make the following assumptions:
(H1) c et κ are real function belonging to $C^1(\mathbb{R})$ and satisfy :

$$0 < c_1 \leq c(s) \leq c_2 < \infty \quad \text{for all } s \in \mathbb{R},$$

$$0 < \kappa_1 \leq \kappa(s) \leq \kappa_2 < \infty \quad \text{for all } s \in \mathbb{R}.$$

We have neglected here the density $\rho(y)$.
(H2) h, α et ϵ are real positive function, continuous and increasing. Moreover,

$$0 < \alpha(s) < 1 \quad \text{for all } s \in \mathbb{R}$$

(H3) $I \in L^\infty(\Sigma_1), \quad I(x,t) \geq 0$ for a.e. $(x,t) \in \Sigma_1$

(H4) $y_0 \in L^\infty(\Omega)$ and $y_0 \geq y_{ex} > 0$ a.e. in Ω

Let us set $\beta(s) = h(y) \cdot (y - y_{ex}) + \varepsilon(y) \cdot \sigma_{SB} \cdot (y^4 - y_{ex}^4)$ if $s \geq y_{ex}$ and $\beta(s) = 0$ if $s \leq y_{ex}$. It results from this latest hypothesis that β is a real continuous, positive and increasing function. Note that $\beta(y_{ex}) = 0$. We suppose:

(H5) β is a Lipschitz function and bounded. This last assumption is justified by the fact that the temperature in $\bar{\Omega}$ cannot exceed a certain constant depending on the data.

We want to examinate the following problem:

$$\begin{cases} c(y)\,\partial_t y - div(\kappa(y)\,\nabla y) = 0 & \text{in} \quad Q, \\ -\kappa(y)\frac{\partial y}{\partial \nu} = \beta(y) - \alpha(y) I(.,.) & \text{on} \quad \Sigma_1, \\ \frac{\partial y}{\partial \nu} = 0 & \text{on} \quad \Sigma_2, \\ y(x,0) = y_0(x) & \text{in} \quad \Omega, \end{cases} \tag{A.2}$$

We will use the first two changes of variables made in the thesis of Jean-Marc Talbot at pages 15-16 [50]. We then obtain the following equations:

$$\begin{cases} \frac{\partial}{\partial t}(b(v)) - \Delta v = 0 & \text{in} \quad Q, \\ -\frac{\partial v}{\partial \nu} = (\beta \circ g)(v) - (\alpha \circ g)(v) I(.,.) & \text{on} \quad \Sigma_1, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on} \quad \Sigma_2, \\ v(x,0) = G(y_0(x)) & \text{in} \quad \Omega. \end{cases} \tag{A.3}$$

Here, $G(s) = \int_{y_{ex}}^{s} \kappa(\nu) d\nu$ is the Kirchoff transformation. It follows from (H1) that G is a continuously differentiable function, with strictly positive derivative and therefore G is invertible. We will denote by g its inverse which is itself continuously differentiable and strictly increasing from $\mathbb{R}$ in $\mathbb{R}$. In the initial boundary value problem (A.3): $v = G(y)$ and therefore $y = g(v)$. $b(s) := \int_0^s \rho(\xi)\ d\xi$, where $\rho : \mathbb{R} \to \mathbb{R}$ is a continuous function defined by

$$\rho(s) = \frac{(c \circ g)(s)}{(\kappa \circ g)(s)}, \ \forall s \in \mathbb{R},$$

which satisfies

$$0 < \frac{c_1}{\kappa_2} \leq \rho(s) \leq \frac{c_2}{\kappa_1} < +\infty \quad \forall s \in \mathbb{R}$$

Therefore, $b(.)$ Is a strictly increasing continuous function of $\mathbb{R}$ in $\mathbb{R}$, and therefore invertible. Denote by d its inverse i.e. $d = b^{-1}$. According to the hypotheses below we have that $\rho \in C^1(\mathbb{R})$ et $b \in C^2(\mathbb{R})$. We now introduce the new variable $\omega = b(v)$, so Δv becomes:

$$\Delta v = d'(\omega)\Delta\omega + d''(\omega)|\nabla\omega|^2,$$

and $\frac{\partial v}{\partial \nu}$ becomes:

$$\frac{\partial v}{\partial \nu} = d'(\omega)\frac{\partial \omega}{\partial \nu}$$

Then (A.3) becomes:

$$\begin{cases} \frac{\partial \omega}{\partial t} - d'(\omega)\Delta\omega - d''(\omega)|\nabla\omega|^2 = 0 & \text{in} \quad Q, \\ -d'(\omega)\frac{\partial \omega}{\partial \nu} = (\beta \circ g \circ d)(\omega) - (\alpha \circ g \circ d)(\omega) I(.,.) & \text{on} \quad \Sigma_1, \\ \frac{\partial \omega}{\partial \nu} = 0 & \text{on} \quad \Sigma_2, \\ \omega(x,0) = (b \circ G)(y_0(x)) & \text{in} \quad \Omega. \end{cases} \tag{A.4}$$

in addition, since d is the inverse function of b:

$$d'(\omega) = \frac{1}{b'(b^{-1}(\omega))} \tag{A.5}$$

and

$$d''(\omega) = -\frac{b''(b^{-1}(\omega))}{[b'(b^{-1}(\omega))]^3}. \tag{A.6}$$

From (A.5) and (A.6) we have that $d \in C^2(\mathbb{R})$. Now we can still transform the equation (A.4) $_{(i)}$ observing that:

$$d'(\omega)\Delta\omega + d''(w)|\nabla\omega|^2 = \operatorname{div}(d'(\omega)\nabla\omega). \tag{A.7}$$

Therefore the initial boundary value problem (A.4), can be written as:

$$\begin{cases} \frac{\partial\omega}{\partial t} - \operatorname{div}(d'(\omega)\nabla\omega) = 0 & \text{in} \quad Q, \\ -d'(\omega)\frac{\partial\omega}{\partial\nu} = (\beta\circ g\circ d)(\omega) - (\alpha\circ g\circ d)(\omega)I(.,.) & \text{on} \quad \Sigma_1, \\ \frac{\partial\omega}{\partial\nu} = 0 & \text{on} \quad \Sigma_2, \\ \omega(x,0) = (b\circ G)(y_0(x)) & \text{in} \quad \Omega. \end{cases} \tag{A.8}$$

To prove that this problem possesses a unique weak solution, we are going to adapt to our problem (A.8), the proof given in the book of F. Chipot ([13], pp.207-214), by replacing in his proof of the existence of a weak solution, the space $L^2(0,T;L^2(\Omega))$ by the space $L^2(0,T;H^\varepsilon(\Omega))$, $\frac{1}{2} < \varepsilon < 1$. Thus given $y \in L^2(0,T;H^\varepsilon(\Omega))$, we consider

$$\omega \in W(0,T) := \{u \in L^2(0,T;H^1(\Omega)); \frac{du}{dt} \in L^2(0,T;H^1(\Omega)^*)\}, \tag{A.9}$$

the weak solution of the nonhomogeneous linear initial boundary value problem:

$$\begin{cases} \frac{\partial\omega}{\partial t} - \sum_{i=1}^{3}\frac{\partial}{\partial x_i}(d'(y)\frac{\partial\omega}{\partial x_i}) = 0 \text{ in } Q, \\ -d'(y)\frac{\partial\omega}{\partial\nu} = (\beta\circ g\circ d)(y) - (\alpha\circ g\circ d)(y)I(.,.) \text{ on } \Sigma_1, \\ \frac{\partial\omega}{\partial\nu} = 0 \text{ on } \Sigma_2, \\ \omega(x,0) = (b\circ G)(y_0(x)) \text{ for } x \in \Omega, \end{cases} \tag{A.10}$$

and we have to prove that the mapping $y \mapsto \omega$ operates in the closed convex set

$$B = \{y \in L^2(0,T;H^\varepsilon(\Omega)); \|y\|_{L^2(0,T;H^\varepsilon(\Omega))} \leq C\}$$

for C a well choosen strictly positive constant and possesses a fixed point in B by using Schauder's fixed point theorem. Let us note that:

1°) the embedding from $W(0,T) := \{y \in L^2(0,T;H^1(\Omega)); \frac{dy}{dt} \in L^2(0,T;H^1(\Omega)^*)\}$ into $L^2(0,T;H^\varepsilon(\Omega))$ is a linear continous compact mapping by the Compacity Lemma ([34], pp.57-60);

2°) The trace mapping $y \mapsto y_{|\Sigma}$ is a linear and continuous mapping from $L^2(0,T;H^\varepsilon(\Omega))$ into $L^2(0,T;H^{\varepsilon-1/2}(\partial\Omega)$ [35], and thus a fortiori from $L^2(0,T;H^\varepsilon(\Omega))$ into $L^2(\Sigma) = L^2(0,T;L^2(\partial\Omega))$.

These latest two ingredients will be used in the proof.

In [50], J.-M. Talbot makes the hypothesis that $\beta : \mathbb{R} \to \mathbb{R}$ is a lipschitzian function; we will suppose moreover that this function is bounded, setting to which we can reduce us by considering $\inf(\sup(-n,\beta),n)$ for some large $n \in \mathbb{N}$. This hypothesis can be supported by thermal considerations showing that the temperature in $\bar{\Omega}$ can not exeeced a certain bound depending on the data. Now, let us introduce the new unknown:

$$\omega_\lambda(x,t) := e^{-\lambda t}\omega(x,t).$$

The equations (A.10), for the new unknown $\omega_\lambda(.,.)$ become:

$$\begin{cases} \frac{\partial \omega_\lambda}{\partial t} + \lambda\omega_\lambda(x,t) - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(d'(e^{\lambda t}y_\lambda)\frac{\partial \omega_\lambda}{\partial x_i}) = 0 \text{ in } Q, \\ -d'(e^{\lambda t}y_\lambda)\frac{\partial \omega_\lambda}{\partial \nu} = e^{-\lambda t}(\beta \circ g \circ d)(e^{\lambda t}y_\lambda) - e^{-\lambda t}(\alpha \circ g \circ d)(e^{\lambda t}y_\lambda)I(.,.) \text{ on } \Sigma_1, \\ \frac{\partial \omega_\lambda}{\partial \nu} = 0 \text{ on } \Sigma_2, \\ \omega_\lambda(x,0) = (b \circ G)(y_0(x)) \text{ for } x \in \Omega, \end{cases} \tag{A.11}$$

here we have introduced $y_\lambda(x,t) := e^{-\lambda t}y(x,t)$, for some $\lambda > 0$. Now let us write, what is a weak solution of our new fixed point problem (A.11). Let us consider a "test function" $\varphi \in H^1(\Omega)$. Multiplying, both sides of equation (A.11) by $\varphi \in H^1(\Omega)$, and integrating over Ω:

$$\begin{aligned} &\int_\Omega \frac{\partial \omega_\lambda}{\partial t}(x,t)\varphi(x)dx + \lambda\int_\Omega \omega_\lambda(x,t)\varphi(x)dx + \int_\Omega d'((e^{\lambda t}y_\lambda)(x,t))\nabla\omega_\lambda(x,t)\nabla\varphi(x)dx \\ &= -\int_{\Gamma_1} e^{-\lambda t}(\beta \circ g \circ d)((e^{\lambda t}y_\lambda)(x,t))\varphi(x)dS(x) \\ &+ \int_{\Gamma_1} e^{-\lambda t}(\alpha \circ g \circ d)((e^{\lambda t}y_\lambda)(x,t))I(x,t)\varphi(x)dS(x). \end{aligned} \tag{A.12}$$

Definition 90. By a weak solution of the nonhomogeneous linear initial boundary value problem (A.11), we mean a function $\omega_\lambda \in W(0,T)$, such that the variational equation (A.12) is satisfied $\forall \varphi \in H^1(\Omega)$, and such that $\omega_\lambda(.,0) = (b \circ G)(y_0(.))$. This last condition has sense since $W(0,T) \hookrightarrow C([0,T];L^2(\Omega))$.

Let us suppose that $I \in L^2(\Sigma_1)$. Then the r.h.s. in the variational equation (A.12) defines an element of $L^2(0,T;(H^1(\Omega))^*)$. Also the initial condition $(b \circ G)(y_0(.)) \in L^2(\Omega)$ because $b \circ G : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function and we have supposed that $y_0 \in L^\infty(\Omega)$, so that $(b \circ G)(y_0(.)) \in L^\infty(\Omega) \subset L^2(\Omega)$. Thus by Theorem 1 and Theorem 2 of [19], pp.512-513, the variational equation (A.12) possesses one and only one weak solution ω_λ in $W(0,T)$.

Let us consider as test functions: $\varphi := \omega_\lambda(.,t)$. By the variational equation (A.12):

$$\begin{aligned} &\tfrac{1}{2}\int_\Omega \frac{\partial \omega_\lambda^2}{\partial t}(x,t)dx + \lambda\int_\Omega \omega_\lambda(x,t)^2dx + \int_\Omega d'((e^{\lambda t}y_\lambda)(x,t))\left|\nabla\omega_\lambda(x,t)\right|^2 dx \\ &= \int_{\Gamma_1} e^{-\lambda t}(\alpha \circ g \circ d)((e^{\lambda t}y_\lambda)(x,t))I(x,t)\omega_\lambda(x,t)dS(x) \\ &- \int_{\Gamma_1} e^{-\lambda t}(\beta \circ g \circ d)((e^{\lambda t}y_\lambda)(x,t))\omega_\lambda(x,t)dS(x). \end{aligned} \tag{A.13}$$

Thus:

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\left\|\omega_\lambda(.,t)\right\|^2_{L^2(\Omega)} + \lambda\left\|\omega_\lambda(.,t)\right\|^2_{L^2(\Omega)} + \int_\Omega d'((e^{\lambda t}y_\lambda)(x,t))\left|\nabla\omega_\lambda(x,t)\right|^2 dx \\ &= \int_{\Gamma_1} e^{-\lambda t}(\alpha \circ g \circ d)((e^{\lambda t}y_\lambda)(x,t))I(x,t)\omega_\lambda(x,t)dS(x) \\ &- \int_{\Gamma_1} e^{-\lambda t}(\beta \circ g \circ d)((e^{\lambda t}y_\lambda)(x,t))\omega_\lambda(x,t)dS(x). \end{aligned} \tag{A.14}$$

Integrating both sides from 0 to T, we have:

$$\begin{aligned}
&\tfrac{1}{2}\left\|\omega_\lambda(.,T)\right\|^2_{L^2(\Omega)} - \tfrac{1}{2}\left\|\omega_\lambda(.,0)\right\|^2_{L^2(\Omega)} + \lambda\int_0^T\int_\Omega \omega_\lambda(x,t)^2 dx\ dt \\
&+\int_0^T\int_\Omega d'((e^{\lambda t}y_\lambda)(x,t))\left|\nabla\omega_\lambda(x,t)\right|^2 dx\ dt \\
&=\int_0^T\int_{\Gamma_1} e^{-\lambda t}(\alpha\circ g\circ d)((e^{\lambda t}y_\lambda)(x,t))I(x,t)\omega_\lambda(x,t)dS(x)dt \\
&-\int_0^T\int_{\Gamma_1} e^{-\lambda t}(\beta\circ g\circ d)((e^{\lambda t}y_\lambda)(x,t))\omega_\lambda(x,t)dS(x)dt.
\end{aligned} \tag{A.15}$$

By (B.1) and (1.11) page 15 [13], $b' \geq \frac{k_2}{c_1}$, so that for a constant $c(\lambda) > 0$, we have for all $\varepsilon > 0$:

$$\begin{aligned}
&\tfrac{1}{2}\left\|\omega_\lambda(.,T)\right\|^2_{L^2(\Omega)} + c(\lambda)\left\|\omega_\lambda\right\|^2_{L^2(0,T;H^1(\Omega))} \leq \left\|I(.,.)\right\|_{L^2(\Sigma_1)}\left\|\omega_\lambda\right\|_{L^2(\Sigma_1)} \\
&+\left\|\beta\right\|_{L^\infty(\mathbb{R})}\left|\Sigma_1\right|^{1/2}\left\|\omega_\lambda\right\|_{L^2(\Sigma_1)} + \tfrac{1}{2}\left\|b\circ G\circ y_0\right\|^2_{L^2(\Omega)} \leq \tfrac{\varepsilon^2}{2}\left\|\omega_\lambda\right\|^2_{L^2(\Sigma_1)} + \tfrac{1}{2\varepsilon^2}\left\|I(.,.)\right\|^2_{L^2(\Sigma_1)} \\
&+\left\|\beta\right\|_{L^\infty(\mathbb{R})}\left|\Sigma_1\right|^{1/2}\tfrac{\varepsilon^2}{2}\left\|\omega_\lambda\right\|^2_{L^2(\Sigma_1)} + \left\|\beta\right\|_{L^\infty(\mathbb{R})}\left|\Sigma_1\right|^{1/2}\tfrac{1}{2\varepsilon^2} + \tfrac{1}{2}\left\|b\circ G\circ y_0\right\|^2_{L^2(\Omega)} \\
&\leq \tfrac{\varepsilon^2}{2}(1+\left\|\beta\right\|_{L^\infty(\mathbb{R})}\left|\Sigma_1\right|^{1/2})\gamma^2\left\|\omega_\lambda\right\|^2_{L^2(0,T;H^1(\Omega))} + \tfrac{1}{2\varepsilon^2}\left\|I(.,.)\right\|^2_{L^2(\Sigma_1)} + \tfrac{1}{2}\left\|b\circ G\circ y_0\right\|^2_{L^2(\Omega)},
\end{aligned} \tag{A.16}$$

where γ denotes the norm of the trace operator from $L^2(0,T;H^1(\Omega))$ into $L^2(0,T;L^2(\Sigma_1))$. By choosing $\varepsilon > 0$ strictly smaller than $\sqrt{\frac{2c(\lambda)}{(1+\left\|\beta\right\|_{L^\infty(\mathbb{R})}\left|\Sigma_1\right|^{1/2})\gamma^2}}$, it follows from the previous inequality that there exists a constant $\tilde{c}(\lambda) > 0$ such that

$$\left\|\omega_\lambda\right\|_{L^2(0,T;H^1(\Omega))} \leq \tilde{c}(\lambda)(\left\|I(.,.)\right\|_{L^2(\Sigma_1)} + \left\|b\circ G\circ y_0\right\|_{L^2(\Omega)}), \tag{A.17}$$

and thus $\left\|\omega_\lambda\right\|_{L^2(0,T;H^1(\Omega))}$ is bounded independently of y. Now, multiplying both sides of the variational equation (A.12), by an arbitrary function $\xi(.) \in L^2(0,T)$, we obtain:

$$\begin{aligned}
&\int_0^T\int_\Omega \tfrac{\partial\omega_\lambda}{\partial t}(x,t)\varphi(x)\xi(t)dxdt + \lambda\int_0^T\int_\Omega \omega_\lambda(x,t)\varphi(x)\xi(t)dxdt \\
&+\int_0^T\int_\Omega d'((e^{\lambda t}y_\lambda)(x,t))\nabla\omega_\lambda(x,t)\nabla\varphi(x)\xi(t)dxdt \\
&=-\int_0^T\int_{\Gamma_1} e^{-\lambda t}(\beta\circ g\circ d)((e^{\lambda t}y_\lambda)(x,t))\varphi(x)\xi(t)dS(x)dt \\
&+\int_0^T\int_{\Gamma_1} e^{-\lambda t}(\alpha\circ g\circ d)((e^{\lambda t}y_\lambda)(x,t))I(x,t)\varphi(x)\xi(t)dS(x)dt.
\end{aligned} \tag{A.18}$$

Using the density of $H^1(\Omega) \times L^2(0,T)$ into $L^2(0,T;H^1(\Omega))$, it follows that:

$$\begin{aligned}
&\int_0^T \int_\Omega \tfrac{\partial \omega_\lambda}{\partial t}(x,t)\psi(x,t)dxdt + \lambda \int_0^T \int_\Omega \omega_\lambda(x,t)\psi(x,t)dxdt \\
&+ \int_0^T \int_\Omega d'((e^{\lambda t}y_\lambda)(x,t))\nabla\omega_\lambda(x,t)\nabla\psi(x,t)dxdt \\
&= -\int_{\Sigma_1} e^{-\lambda t}(\beta \circ g \circ d)((e^{\lambda t}y_\lambda)(x,t))\psi(x,t)dS(x)dt \\
&+ \int_{\Sigma_1} e^{-\lambda t}(\alpha \circ g \circ d)((e^{\lambda t}y_\lambda)(x,t))I(x,t)\psi(x,t)dS(x)dt, \ \forall \psi \in L^2(0,T;H^1(\Omega)).
\end{aligned} \tag{A.19}$$

Thus:

$$\begin{aligned}
&\left\| \tfrac{\partial \omega_\lambda}{\partial t} \right\|_{L^2(0,T;H^1(\Omega)^*)} \leq \lambda \left\| \omega_\lambda \right\|_{L^2(0,T;H^1(\Omega))} + \tfrac{k_2}{c_1} \left\| \omega_\lambda \right\|_{L^2(0,T;H^1(\Omega))} \\
&+ \left\| \beta \right\|_{L^\infty(\mathbb{R})} |\Sigma_1|^{1/2} \gamma + \left\| I(.,.) \right\|_{L^2(\Sigma_1)} \gamma.
\end{aligned} \tag{A.20}$$

By inequalities (A.17) and (A.20), we have thus proved the following proposition:

Proposition 91. *For every $y \in L^2(0,T;H^\varepsilon(\Omega))$, the norm in the space $W(0,T)$ of the weak solution ω_λ of the nonhomogeneous linear initial boundary value problem (A.11), in the sense of the variational equality (A.12), is bounded by a constant independent of y.*

So let us consider now

$$B = \{y_\lambda \in L^2(0,T;H^\varepsilon(\Omega)); \left\| y_\lambda \right\|_{L^2(0,T;H^\varepsilon(\Omega))} \leq C\},$$

where $C > 0$ is the constant mentioned in the above proposition. B is a closed convex subset of $L^2(0,T;H^\varepsilon(\Omega))$. Moreover, the mapping which sends $y_\lambda \in L^2(0,T;H^\varepsilon(\Omega))$ onto $\omega_\lambda \in W(0,T)$ weak solution of the nonhomogeneous linear initial boundary value problem (A.11), in the sense of the variational inequality (A.12). Let us call this mapping Φ. By the compacity of the embedding from $W(0,T)$ into $L^2(0,T;H^\varepsilon(\Omega))$ ([34], pp.57-60), the range of Φ is relatively compact in B. Thus, to apply Schauder's fixed theorem, we must prove the continuity of the mapping Φ.

We need to show that Φ is continuous on B. So let us consider a sequence $y_{n,\lambda} \in B$ such that

$$y_{n,\lambda} \to y_\lambda \text{ in } B \text{ as } n \to +\infty \tag{A.21}$$

we want to proof that

$$\omega_{n,\lambda} = \phi(y_{n,\lambda}) \to \phi(y_\lambda) \text{ in } B \text{ as } n \to +\infty$$

where $\omega_{n,\lambda} = \phi(y_{n,\lambda})$ denotes the weak solution to (A.11) corresponding to $y_\lambda = y_{n,\lambda}$. (A.21) implies that

$$y_{n,\lambda} \to y_\lambda \text{ in } L^2(Q) \text{ as } n \to +\infty$$

and by the reciproque of Lebesgue's Dominated Convergence theorem there exist a subsequence still labeled by n such that

$$y_{n,\lambda} \to y_\lambda \text{ a.e in } \Omega \times]0,T[\text{ as } n \to +\infty. \tag{A.22}$$

By proposition 91, $\omega_{n,\lambda}$ is bounded in $W(0,T)$ and by the compacity of the embedding from $W(0,T)$ into $L^2(0,T;H^\epsilon(\Omega))$ we can extract from n a subsequence, still labeled by n, such that

$$\begin{aligned}
&\omega_{n,\lambda} \to \omega_\infty && \text{in } L^2(0,T;H^\epsilon(\Omega)),\\
&\omega_{n,\lambda} \rightharpoonup \omega_\infty && \text{in } L^2(0,T;H^1(\Omega)),\\
&\tfrac{\partial \omega_{n,\lambda}}{\partial t} \rightharpoonup \tfrac{\partial \omega_\infty}{\partial t} && \text{in } L^2(0,T;(H^1(\Omega))^*),\\
&\tfrac{\partial \omega_{n,\lambda}}{\partial x_i} \rightharpoonup \tfrac{\partial \omega_\infty}{\partial x_i} && \text{in } L^2(0,T;L^2(\Omega)),
\end{aligned} \tag{A.23}$$

as $n \to +\infty$, where $\omega_\infty \in L^2(0,T;H^1(\Omega)) \subset L^2(0,T;H^\epsilon(\Omega))$. From (A.12) we deduce that for all $\varphi \in H^1(\Omega)$, for all $\xi \in D(0,T)$

$$\begin{aligned}
&\int_0^T -(\omega_{n,\lambda}(t),\varphi)\xi'(t)\,dt + \lambda\int_0^T\int_\Omega \omega_{n,\lambda}(x,t)\varphi(x)\xi(t)\,dxdt\\
&+\int_0^T\int_\Omega d'((e^{\lambda t}y_{n,\lambda})(x,t))\nabla\omega_{n,\lambda}(x,t)\cdot\nabla\varphi(x)\,\xi(t)\,dxdt\\
&=-\int_0^T\int_{\Gamma_1} e^{-\lambda t}(\beta\circ g\circ d)((e^{\lambda t}y_{n,\lambda})(x,t))\varphi(x)\xi(t)dS(x)dt\\
&+\int_0^T\int_{\Gamma_1} e^{-\lambda t}(\alpha\circ g\circ d)((e^{\lambda t}y_{n,\lambda})(x,t))I(x,t)\varphi(x)\xi(t)dS(x)dt.
\end{aligned} \tag{A.24}$$

where

$$(\omega_{n,\lambda}(t),\varphi) = \int_\Omega \omega_{n,\lambda}(x,t)\varphi(x)dx.$$

Recalling the continuity of d', $\beta\circ g\circ d$ and $\alpha\circ g\circ d$, by (A.22) we have

$$d'((e^{\lambda t}y_{n,\lambda})(x,t))\nabla\varphi(x)\,\xi(t) \to d'((e^{\lambda t}y_\lambda)(x,t))\nabla\varphi(x)\,\xi(t),$$

$$e^{-\lambda t}(\beta\circ g\circ d)((e^{\lambda t}y_{n,\lambda})(x,t))\varphi(x)\xi(t) \to e^{-\lambda t}(\beta\circ g\circ d)((e^{\lambda t}y_\lambda)(x,t))\varphi(x)\xi(t),$$

$$e^{-\lambda t}(\alpha\circ g\circ d)((e^{\lambda t}y_{n,\lambda})(x,t))I(x,t)\varphi(x)\xi(t) \to e^{-\lambda t}(\alpha\circ g\circ d)((e^{\lambda t}y_\lambda)(x,t))I(x,t)\varphi(x)\xi(t),$$

as $n \to +\infty$, we have also

$$|\, d'((e^{\lambda t}y_{n,\lambda})(x,t))\nabla\varphi(x)\,\xi(t)\,| \le \tfrac{\kappa_2}{c_1}\,|\,\nabla\varphi(x)\,\xi(t)\,|,\ (x,t)\in Q,$$

$$|\, e^{-\lambda t}(\beta\circ g\circ d)((e^{\lambda t}y_{n,\lambda})(x,t))\varphi(x)\xi(t)\,| \le \|\,\beta\,\|_{\infty,\mathbb{R}}|\,\varphi(x)\xi(t)\,|,\ (x,t)\in\Sigma_1,$$

$$|\, e^{-\lambda t}(\alpha\circ g\circ d)((e^{\lambda t}y_{n,\lambda})(x,t))I(x,t)\varphi(x)\xi(t)\,| \le |\,I(x,t)\varphi(x)\xi(t)\,|,\ (x,t)\in\Sigma_1,$$

By the Lebesgue dominated convergence theorem we have

$$d'((e^{\lambda t}y_{n,\lambda})(x,t))\nabla\varphi(x)\,\xi(t) \to d'((e^{\lambda t}y_\lambda)(x,t))\nabla\varphi(x)\,\xi(t) \text{ in } L^2(Q),$$

$$e^{-\lambda t}(\beta\circ g\circ d)((e^{\lambda t}y_{n,\lambda})(x,t))\varphi(x)\xi(t) \to e^{-\lambda t}(\beta\circ g\circ d)((e^{\lambda t}y_\lambda)(x,t))\varphi(x)\xi(t) \text{ in } L^2(\Sigma_1),$$

$$e^{-\lambda t}(\alpha\circ g\circ d)((e^{\lambda t}y_{n,\lambda})(x,t))I(x,t)\varphi(x)\xi(t) \to e^{-\lambda t}(\alpha\circ g\circ d)((e^{\lambda t}y_\lambda)(x,t))I(x,t)\varphi(x)\xi(t),$$

in $L^2(\Sigma_1)$. Passing to the limit in (A.24) leads to

$$\begin{aligned}\tfrac{d}{dt}(\omega_\infty,\varphi)+\lambda\int_\Omega \omega_\infty(x,t)\varphi(x)dx+\int_\Omega d'((e^{\lambda t}y_\lambda)(x,t))\nabla\omega_\infty(x,t)\nabla\varphi(x)dx\\ =-\int_{\Gamma_1} e^{-\lambda t}(\beta\circ g\circ d)((e^{\lambda t}y_\lambda)(x,t))\varphi(x)dS(x)\\ +\int_{\Gamma_1} e^{-\lambda t}(\alpha\circ g\circ d)((e^{\lambda t}y_\lambda)(x,t))I(x,t)\varphi(x)dS(x)\quad \text{in } D'(0,T).\end{aligned} \tag{A.25}$$

As $\omega_\infty\in W(0,T)$ we have

$$\omega_\infty\in L^2(0,T;H^1(\Omega)),\qquad \frac{d\,\omega_\infty}{dt}\in L^2(0,T;(H^1(\Omega))^*). \tag{A.26}$$

Now, by the theorem 11.6 in [13] page 191 we can write for a.e. $t\in(0,T)$, for all $\varphi\in H^1(\Omega)$

$$(\omega_{n,\lambda}(t),\varphi)-(\omega_\lambda(0),\varphi)=\int_0^t\frac{d}{dt}(\omega_{n,\lambda},\varphi)dt=\int_0^t\langle y_{(n,\lambda)_t},\varphi\rangle dt \tag{A.27}$$

By (A.23), and $W(0,T)\hookrightarrow C(0,T;L^2(\Omega))$ we have

$$\begin{aligned}|\ \langle\omega'_{n,\lambda},\varphi\rangle_{(H^1(\Omega))^*,H^1(\Omega)}\ |\ &\le\|\ \omega'_{n,\lambda}(t)\ \|_{(H^1(\Omega))^*}\|\ \varphi\ \|_{H^1(\Omega)}\\ &\le\|\ \omega'_{n,\lambda}(t)\ \|_{L^2(\Omega)}\|\ \varphi\ \|_{H^1(\Omega)}\\ &\le\|\ \omega'_{n,\lambda}\ \|_{C((0,T;L^2(\Omega))}\|\ \varphi\ \|_{H^1(\Omega)}\\ &\le\|\ \omega'_{n,\lambda}\ \|_{L^2(0,T;H^1(\Omega))}\|\ \varphi\ \|_{H^1(\Omega)}\\ &\le C\ \|\ \varphi\ \|_{H^1(\Omega)}\ .\end{aligned}$$

By Lebesgue dominated convergence we pass to the limit in (A.27) we get

$$(\omega_\infty(t),\varphi)-(\omega_\lambda(0),\varphi)=\int_0^t\langle\omega_{\infty_t},\varphi\rangle dt=(\omega_\infty(t),\varphi)-(\omega_\infty(0),\varphi)$$

a.e. $t\in$, for all $\varphi\in H^1(\Omega)$. Thus

$$\omega_\lambda(x,0)=\omega_\infty(x,0)\text{ for }x\in\Omega. \tag{A.28}$$

By (A.25), (A.26),(A.28) one sees that ω_∞ the limit of $\omega_{n,\lambda}=\phi(y_{n,\lambda})$ in $L^2(0,T;H^\epsilon(\Omega))$ is a weak solution of the nonhomogenous linear initial boundary value problem (A.11), in the sense of the definition (90). Thus $\omega_\infty=\omega_\lambda=\phi(y_\lambda)$ since $\omega_{n,\lambda}$ has only $\phi(y_\lambda)$ as possible limit in $L^2(0,T;H^\epsilon(\Omega))$.
Since any subsequence of $\omega_{n,\lambda}$ have the same limit

$$\omega_{n,\lambda}=\phi(y_{n,\lambda})\to\omega_\lambda=\phi(y_\lambda)\text{ in }L^2(0,T;H^\epsilon(\Omega))\text{ as }n\to+\infty.$$

This complete the proof of the continuity of ϕ and the condition to apply Schauder theorem are satisfied.

Appendix B

Existence of weak solutions for the linear Maxwell's equations

We prove in this appendix the existence of weak solutions for the initial boundary problem (3.15).

Definition 92. We say a function

$$U = (\mathcal{E}, H) \in L^{\infty}(0, T; \mathcal{H}) \tag{B.1}$$

is a weak solution of the hyperbolic initial boundary-value problem (3.15) provided

$$\begin{aligned} &\int_0^T [-(U(t), \frac{\partial \phi(t)}{\partial t})_{\mathcal{H}_t} - (U(t), \mathcal{A}(t)\phi(t))_{\mathcal{H}_t} - (\mathcal{B}(t)U(t), \phi(t))_{\mathcal{H}_t} \\ &+ (\mathcal{M}(t)U(t), \phi(t))_{\mathcal{H}_t}]dt = \int_0^T (\mathbf{G}(t), \phi(t))_{\mathcal{H}_t} dt + (U_0, \phi(0))_{\mathcal{H}_0} \end{aligned} \tag{B.2}$$

for each

$$\phi \in L^2(0, T; D(\mathcal{A})) \text{ such that } \frac{d\phi}{dt} \in L^2(0, T; \mathcal{H}) \text{ and } \phi(T) = 0, \tag{B.3}$$

where

$$\mathcal{B}(t)U(t) = \{\hat{\epsilon}(\cdot, z(\cdot, t))\partial_z \epsilon(\cdot, z(\cdot, t))\partial_t z(\cdot, t)\mathcal{E}, 0\}, \tag{B.4}$$

$\mathcal{A}(t)$, $\mathcal{M}(t)$ and $\mathbf{G}(t)$ are respectively given by (3.24), (3.25) and (3.17).

$$U_0 = (E_0, H_0) \in \mathcal{H}. \tag{B.5}$$

Remark 16. $-\mathcal{B}(t)U(t) + \mathcal{M}(t)U(t) = \{\hat{\epsilon}(\cdot, z(\cdot, t))\sigma\mathcal{E}, 0\}$

We will start first by constructing solutions of certain finite-dimensional approximations to (3.15) and then passing to limits. More precisely, since $D(\mathcal{A})$ is separable, let $\{\phi_i\}_{i=1}^{\infty}$ be an orthonormal basis in $D(\mathcal{A})$ endowed with the graph norm. For arbitrary but fixed $m \in \mathbb{N}$, we now determine approximations $U_m := U_m(x, t)$ through the ansatz

$$U_m(x, t) = \sum_{i=1}^{m} d_m^i(t)\phi_i(x) \tag{B.6}$$

with unkonwn functions $d_i^m : [0,T] \to \mathbb{C}$, $i = 1, \cdots, m$ such that

$$d_m^i(0) = (U_0, \phi_i)_{\mathcal{H}_0} \quad (i = 1, \cdots, m) \tag{B.7}$$

and

$$\begin{aligned}(U_m'(t), \phi_i)_{\mathcal{H}_t} + (\mathcal{A}(t)U_m(t), \phi_i)_{\mathcal{H}_t} + (\mathcal{M}(t)U_m(t), \phi_i)_{\mathcal{H}_t} \\ = (\mathbf{G}(t), \phi_i)_{\mathcal{H}_t} \quad (0 \le t \le T,\ i = 1, \cdots, m)\end{aligned} \tag{B.8}$$

Theorem 93 (Construction of approximate solutions)**.** *For each integer $m = 1, 2, \cdots$ there exists a unique function U_m of the form (B.6) satisfying (B.7), (B.8).*

Proof. Assuming U_m has the structure (B.6), we first note that

$$(U_m'(t), \phi_i)_{\mathcal{H}_t} = \sum_{l=1}^{m} (\phi_i, \phi_l)_{\mathcal{H}_t} d_m^l{}'(t) = A(t).(d_m^1{}'(t), \cdots, d_m^{m\prime}(t))^\intercal, \tag{B.9}$$

where $A(t) := ((\phi_i, \phi_l)_{\mathcal{H}_t})_{il}$ $(i, l = 1, \cdots, m)$, positive-definite matrix due to assumptions (H1) and Remark 2 .

Furthermore

$$\begin{aligned}(\mathcal{A}(t)U_m(t), \phi_i)_{\mathcal{H}_t} + (\mathcal{M}(t)U_m(t), \phi_i)_{\mathcal{H}_t} &= \sum_{l=1}^{m} [(\mathcal{A}(t)\phi_i, \phi_l)_{\mathcal{H}_t} + (\mathcal{M}(t)\phi_i, \phi_l)_{\mathcal{H}_t}] d_m^l(t) \\ &= B(t).(d_m^1(t), \cdots, d_m^m(t))^\intercal,\end{aligned} \tag{B.10}$$

for $B(t) := ((\mathcal{A}(t)\phi_i, \phi_l)_{\mathcal{H}_t} + (\mathcal{M}(t)\phi_i, \phi_l)_{\mathcal{H}_t})_{il}$ $(i, l = 1, \cdots, m)$. Let us further write $G^i(t) := (\mathbf{G}(t), \phi_i)_{\mathcal{H}_t}$ $(i = 1, \cdots, m)$. Then (B.8) becomes the linear system of ODEs

$$(d_m^1{}'(t), \cdots, d_m^{m\prime}(t))^\intercal + A(t)^{-1}B(t)(d_m^1(t), \cdots, d_m^m(t))^\intercal = A(t)^{-1}(\mathbf{G}^1(t), \cdots, \mathbf{G}^m(t))^\intercal \tag{B.11}$$

subject to the initial condition (B.7). Thus, there exists a unique absolutely continuous function $t \mapsto (d_m^1(t), \cdots, d_m^m(t))$ satisfying (B.8) and (B.7) for a.e. $0 \le t \le T$. And then U_m defined by (B.6)-(B.7) solves (B.8) for a.e. $0 \le t \le T$. □

We propose now to send m to infinity and to show a subsequence of our solutions U_m of the approximate problem (B.7), (B.8) converges to a weak solution of (3.15).

Theorem 94 (Existence of weak solution)**.** *There exists a weak solution of (3.15).*

Proof. First, we need some energy estimates. We multiply equation (B.8) by $d_m^i(t)$ and sum over $i = 1, \cdots, m$; it follows according to (B.6) that

$$(U_m'(t), U_m(t))_{\mathcal{H}_t} + (\mathcal{A}(t)U_m(t), U_m(t))_{\mathcal{H}_t} + (\mathcal{M}(t)U_m(t), U_m(t))_{\mathcal{H}_t} = (\mathbf{G}(t), U_m(t))_{\mathcal{H}_t} \tag{B.12}$$

for a.e $0 \le t \le T$.
We consider the real part of (B.12); taking account of

$$\mathcal{R}(\mathcal{A}(t)U_m(t), U_m(t))_{\mathcal{H}_t} = 0 \quad \text{for a.e } 0 \le t \le T$$

and the fact that

$$\mathcal{R}(U_m'(t), U_m(t))_{\mathcal{H}_t} = \frac{1}{2}\frac{d}{dt} \parallel U_m(t) \parallel^2_{\mathcal{H}_t} - \frac{1}{2}\int_\Omega \frac{\partial \epsilon}{\partial z}(x, z(x,t))\frac{dz}{dt}(x,t)U_m(x,t)^2 dx, \quad \text{(B.13)}$$

for a.e $0 \leq t \leq T$, according to assumptions (H1)-(H2) and (3.8) there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\parallel U_m(t) \parallel^2_{\mathcal{H}_t}) &\leq C_1 \parallel U_m(t) \parallel^2_{\mathcal{H}_t} + \parallel \mathbf{G}(t) \parallel_{\mathcal{H}_t} \parallel U_m(t) \parallel_{\mathcal{H}_t} \\ &\leq C_1 \parallel U_m(t) \parallel^2_{\mathcal{H}_t} + \frac{1}{2} \parallel \mathbf{G}(t) \parallel^2_{\mathcal{H}_t} + \frac{1}{2} \parallel U_m(t) \parallel^2_{\mathcal{H}_t} \\ &\leq C_1 \parallel U_m(t) \parallel^2_{\mathcal{H}_t} + C_2 \parallel \mathbf{G}(t) \parallel^2_{\mathcal{H}_t}. \end{aligned} \quad \text{(B.14)}$$

Now we write

$$\eta(t) := \parallel U_m(t) \parallel^2_{\mathcal{H}_t} \quad \text{(B.15)}$$

and

$$\xi(t) := \parallel \mathbf{G}(t) \parallel^2_{\mathcal{H}_t}. \quad \text{(B.16)}$$

Then (B.14) implies

$$\eta'(t) \leq C_1\eta(t) + C_2\xi(t) \quad \text{(B.17)}$$

for a.e. $0 \leq t \leq T$. Thus the differential form of Gronwall's inequality yields the estimate

$$\eta(t) \leq e^{C_1 t}\left(\eta(0) + C_2\int_0^t \xi(s)ds\right) \quad \text{for a.e. } 0 \leq t \leq T. \quad \text{(B.18)}$$

Since $\eta(0) = \parallel U_m(0) \parallel^2_{\mathcal{H}_0} \leq C_3 \parallel U_0 \parallel^2_{\mathcal{H}}$ by (B.7), we obtain from (B.15)-(B.16)-(B.18) and Remark 11 :

$$\parallel U_m(t) \parallel^2_{\mathcal{H}} \leq \text{constant, independent of } m. \quad \text{(B.19)}$$

According to (B.19), U_m is bounded in $L^\infty(0,T;\mathcal{H}) =$ dual of $L^1(0,T;\mathcal{H})$. Thus, there exists a subsequence $(U_{m_k})_{k\in\mathbb{N}^*}$ such that

$$U_{m_k} \overset{*}{\rightharpoonup} U \quad \text{weakly star in } L^\infty(0,T;\mathcal{H}). \quad \text{(B.20)}$$

Equivalently

$$< U_{m_k}, \Psi > \to < U, \Psi > \quad \text{for a.e } \Psi \in L^1(0,T;\mathcal{H}), \quad \text{(B.21)}$$

$< \cdot, \cdot >$ being the pairing of $L^\infty(0,T;\mathcal{H})$ and $L^1(0,T;\mathcal{H})$. We next fix an integer N and choose a function $V \in C^1([0,T], D(\mathcal{A}))$ having the form

$$V(t) = \sum_{i=1}^N d^i(t)\phi_i \quad \text{(B.22)}$$

where $\{d^i(t)\}_{i=1}^N$ are given smooth functions such that $d^i(T) = 0$.

We choose $m \geq N$, multiply (B.8) by $d^i(t)$, sum $i = 1, \cdots, N$, and then integrate with respect to t to find

$$\begin{aligned}&\int_0^T [(U_m'(t), V(t))_{\mathcal{H}_t} + (\mathcal{A}(t)U_m(t), V(t))_{\mathcal{H}_t} + (\mathcal{M}(t)U_m(t), V(t))_{\mathcal{H}_t}]dt \\ &= \int_0^T (\mathbf{G}(t), V(t))_{\mathcal{H}_t}\, dt.\end{aligned} \tag{B.23}$$

We set $m = m_k$ and we integrate over $[0, T]$ by parts in (B.23), to find

$$\begin{aligned}&\int_0^T [-(U_{m_k}(t), \frac{dV(t)}{dt})_{\mathcal{H}_t} - (U_{m_k}(t), \mathcal{A}(t)V(t))_{\mathcal{H}_t} + (\mathcal{M}(t)U_{m_k}(t), V(t))_{\mathcal{H}_t}]dt \\ &= (\mathcal{B}(t)U_{m_k}(t), \phi(t))_{\mathcal{H}_t} + \int_0^T (\mathbf{G}(t), V(t))_{\mathcal{H}_t} dt + (U_{m_k}(0), V(0))_{\mathcal{H}_0}.\end{aligned} \tag{B.24}$$

Due to (B.20), since $U_{m_k}(0) \to U_0$ in $L^2(0, T; \mathcal{H})$, we can pass to the limit in (B.24). Thus we obtain the existence of a U which satisfies (B.1) and (B.2) for every $\phi = V$ of the form (B.22).

But, due to [36, Theorem 2.1 p.11] if ϕ is given with (B.3), we can find a sequence $\{V_j\}_{j\in\mathbb{N}}$ of functions of the form (B.22) such that

$$V_j \to \phi \text{ in } L^2(0, T; D(\mathcal{A})),$$

and

$$\frac{dV_j}{dt} \to \frac{d\phi}{dt} \text{ in } L^2(0, T; \mathcal{H}).$$

We conclude that (B.2) holds for every ϕ satisfying (B.3). □

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